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# THE PRINCIPLE OF LIMITING ABSORPTION FOR UNIFORMLY PROPAGATIVE SYSTEMS WITH PERTURBATIONS OF LONG-RANGE CLASS

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## §1. Introduction

The aim of this paper is to establish the principle of limiting absorption for uniformly propagative systems $\Lambda(x, D_x) = E(x)^{-1} \sum_{j=1}^{n} A_j D_j$, $D_j = -i\partial/\partial x_j$, with perturbations of long-range class, where the perturbation of long-range class, roughly speaking, means that $E(x)$ approaches to $E_0$, $E_0$ being the $N \times N$ identity matrix, as $|x| \to \infty$ with order $O(|x|^{-\delta})$, $0 < \delta \leqq 1$. (The more precise assumptions will be stated below and we require some additional assumptions on the derivatives of $E(x)$.) The spectral and scattering problem for uniformly propagative systems was first formulated by Wilcox [10]. Since then, the principle of limiting absorption has been proved by many authors ([5], [7], [8], [11] etc.). The perturbations discussed in their works belong to the short-range class with $\delta > 1$.

On the other hand, for the Schrödinger operators with long-range potentials, this principle has been already verified by many authors ([2], [3], [6] etc.). Especially, S. Agmon has extended their results to general elliptic operators of higher order, using the localization theory in the momentum space, $\xi$-space (lecture given at the Kyoto University, 1977).

In this paper, we also use the localization theory, so we owe much to Agmon's idea. However, his method cannot be directly applied to our problem. In particular, when the characteristic equation for the unperturbed system $\Lambda_0(D_x) = \sum_{j=1}^{n} A_j D_j$ has multiple roots, a few difficulties occur and we need some modifications.

**1.1.** Notations. We first list up the notations which will be used throughout this paper. (1) $\boldsymbol{R}_x^n$ and $\boldsymbol{R}_\xi^n$ denote the $n$-dimensional euclidean space with generic points $x = (x_1, \cdots, x_n)$ and $\xi = (\xi_1, \cdots, \xi_n)$, respectively.

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We often write $\boldsymbol{R}^n$ instead of $\boldsymbol{R}_x^n$ or $\boldsymbol{R}_\xi^n$ without subscript $x$ or $\xi$. We denote by $x\cdot\xi$ the scalar product between $x$ and $\xi$; $x\cdot\xi = \sum_{j=1}^n x_j\xi_j$, and by $|x|$ the length of $x$. (2) $C^n$ denotes the $n$-dimensional unitary space with the usual scalar product ( , ). (3) For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_j$ being a non-negative integer, we denote by $|\alpha|$ the length of $\alpha$; $|\alpha| = \sum_{j=1}^n |\alpha_j|$. We write $\partial_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $D_x = (D_1, \cdots, D_n)$, $D_j = -i\partial/\partial x_j$, and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$. We occasionally use the symbol $\alpha$ to denote real numbers but there will be no fear of confusions.

**1.2.** Assumptions. We shall formulate the problem to be discussed here with several assumptions. The operators to be considered are given in the following form:

$$\Lambda = E(x)^{-1} \sum_{j=1}^n A_j D_j \, , \tag{1.1}$$

$$\Lambda_0 = E_0^{-1} \sum_{j=1}^n A_j D_j = \sum_{j=1}^n A_j D_j \, , \tag{1.2}$$

where $E_0$ is the identity matrix of size $N \times N$. We make the following assumptions:

(A.1) $A_j, j = 1, \cdots, n$, is a symmetric constant matrix of size $N \times N$;
(A.2) The unperturbed system $\Lambda_0$ is uniformly propagative in the sense of Wilcox ([10]).

We do not give the definition of uniformly propagative system here but some important properties which are necessary to the later argument will be summarized in § 2.

(A.3) $E(x) = \{e_{jk}(x)\}_{j,k=1,N}$ is symmetric and positive definite uniformly in $x$. Furthermore, it belongs to the long-range class in the following sense:

$$|e_{jk}(x) - \delta_{jk}| \leqq C(1 + |x|)^{-\delta} \, , \qquad \delta > 0 \, ; \tag{1.3}$$

$$|\partial_x^\beta e_{jk}(x)| \leqq C_\beta(1 + |x|)^{-(1+\delta)} \, , \qquad |\beta| \geqq 1 \, , \tag{1.4}$$

where $\delta_{jk}$ is Kronecker's delta.

**1.3.** Functional spaces. We shall introduce the various functional spaces in which we work. We denote by $H_m(H_0 = L_2)$ the usual Sobolev space of order $m$ over the whole space $\boldsymbol{R}^n$ and the norm is denoted by $\| \ \|_m$. We introduce the Sobolev space $H_{m,\alpha}$ with weight $\alpha$ by $H_{m,\alpha} = \{\phi; (1 + |x|^2)^{\alpha/2}\phi \in H_m\}$ and define the norm $\| \ \|_{m,\alpha}$ by $\|\phi\|_{m,\alpha} = \|(1 + |x|^2)^{\alpha/2}\phi\|_m$.

We further define the space $H^{(\ell)}_{m,\alpha}$ as $H^{(\ell)}_{m,\alpha} = \sum \oplus H_{m,\alpha}$, $\ell$ summands, and denote by $|\ \ |^{(\ell)}_{m,\alpha}$ the norm in $H^{(\ell)}_{m,\alpha}$. When $m = 0$, we write $H^{(\ell)}_{0,\alpha} = L^{(\ell)}_{2,\alpha}$. In the future argument, the spaces of $N$ summands are most frequently used, so we simply write $|\ \ |_{m,\alpha}$ instead of $|\ \ |^{(N)}_{m,\alpha}$ for the norms in these spaces.

**1.4.** Results. We shall state the main results obtained in this paper. As is easily shown, the operator $\Lambda$ defined by (1.1) has a natural self-adjoint realization (denoted by the same symbol $\Lambda$) in $L^{(N)}_{2,0}$ with the energy scalar product

$$((\phi, \psi))_0 = \int_{R^n} (\phi, E(x)\psi)dx$$

and the domain $\mathscr{D}(\Lambda)$ is given by $\mathscr{D}(\Lambda) = \{u; u \in L^{(N)}_{2,0}, \Lambda u \in L^{(N)}_{2,0}\}$. Similarly we denote by the same symbol $\Lambda_0$ a self-adjoint realization of $\Lambda_0$ defined by (1.2) with domain $\mathscr{D}(\Lambda_0)$.

With the above notations, we are now able to state the first result. We always assume that (A.1) $\sim$ (A.3) are satisfied.

THEOREM 1.1. *The eigenvalues of $\Lambda$ are discrete with possible accumulating points* 0 *and* $\pm\infty$.

Next, we consider the equation

$$\Lambda u - (\lambda \pm i\kappa)u = f\,, \qquad 0 < \kappa \leqq 1\,, \tag{1.5}$$

with $f \in L^{(N)}_{2,\alpha}$, $\alpha > \frac{1}{2}$. Clearly, for $\kappa > 0$ there exists a unique solution $u = R(\lambda \pm i\kappa)f = (\Lambda - (\lambda \pm i\kappa))^{-1}f$ such that $u \in L^{(N)}_{2,0}$. Then, the second result is stated as follows:

THEOREM 1.2. *Assume that $\lambda, \lambda \neq 0$, is not an eigenvalue of $\Lambda$. Let $u = R(\lambda \pm i\kappa)f$ be a solution of equation* (1.5) *with $f \in L^{(N)}_{2,\alpha}$, $\alpha > \frac{1}{2}$. Then, the following statements hold*: (i) *There exists a constant $C_\alpha$ independent of $\kappa$, $0 < \kappa \leqq 1$, such that $|R(\lambda \pm i\kappa)f|_{0,-\alpha} \leqq C_\alpha |f|_{0,\alpha}$*; (ii) *There exist bounded operators $R(\lambda \pm i0)$ from $L^{(N)}_{2,\alpha}$ to $L^{(N)}_{2,-\alpha}$ defined by $R(\lambda \pm i0)f = \lim_{\kappa \downarrow 0} R(\lambda \pm i\kappa)f$ strongly in $L^{(N)}_{2,-\alpha}$.*

*Remark.* We can show that the convergence in (ii) is uniform in $\lambda$ when $\lambda$ ranges over a compact interval, not containing the origin and eigenvalues of $\Lambda$. Hence, we can prove that $R(\lambda \pm i0)f$ is locally continuous in $\lambda$ under the norm in $L^{(N)}_{2,-\alpha}$.

We prove Theorem 1.2 only for the "+" case; the proof for the "−" case is done without any essential changes.

1.5. Remarks. (i) The assumption that $E_0$ is the identity matrix loses no generality. The general case in which $E_0$ is symmetric and positive definite can be reduced to this case by a simple transformation. (ii) The assumption in (A.3) seems to be rather restrictive. However, our results cover the case in which $E(x)$ is of the following form: $E(x) = E_1(x) + E_2(x)$, where $E_1(x)$ satisfies (A.3) and $E_2(x) = O(|x|^{-(1+\delta)})$ as $|x| \to \infty$. Hence, by use of mollifier technique, $E(x)$ for which (1.4) with $|\beta| = 1$ only is valid can be decomposed into the above form.

Finally we note the following fact: We use the symbols $C, C_1, \cdots$ to denote positive constants which are not necessarily the same. In particular, when we specify the dependence of such a constant on a parameter, say $\varepsilon$, we denote it by $C_\varepsilon$.

## § 2. Preliminaries

**2.1.** Systems in homogeneous media. We summarize the results derived from assumption (A.2) which are necessary to the later argument. The proof of these results can be found in [10]. Let $\Lambda_0(\xi)$ be defined by

$$\Lambda_0(\xi) = \sum_{j=1}^{n} A_j \xi_j \,. \tag{2.1}$$

Then, by the definition of uniformly propagative system, $\Lambda_0(\xi)$ has $r$ distinct real eigenvalues with constant multiplicity.

PROPOSITION 2.1. (i) *One of the following alternative holds for a system of the above eigenvalues*: (a) *when* $r = 2\rho + 1$ *is odd,*

$$\lambda_1(\xi) > \cdots > \lambda_\rho(\xi) > \lambda_0(\xi) = 0 > \lambda_{-\rho}(\xi) > \cdots > \lambda_{-1}(\xi) \tag{2.2}$$

*with the relation* $\lambda_{-j}(-\xi) = -\lambda_j(\xi)$, $j = 1, \cdots, \rho$; (b) *when* $r = 2\rho$ *is even,*

$$\lambda_1(\xi) > \cdots > \lambda_\rho(\xi) > 0 > \lambda_{-\rho}(\xi) > \cdots > \lambda_{-1}(\xi) \tag{2.3}$$

*with the same relation as above.* (ii) *Each* $\lambda_j(\xi)$ *is smooth in* $\boldsymbol{R}^n - \{0\}$ *and positively homogeneous of degree one*; $\lambda_j(\mu\xi) = \mu\lambda_j(\xi)$, $\mu > 0$. *Hence, when* $j \geqq 1$, *there exists a constant* $C_j$ *such that* $\lambda_j(\xi) \geqq C_j |\xi|$.

We now define the bounded surface $\Xi_j$, $j = 1, \cdots, \rho$, as

$$\Xi_j = \{\xi ; \lambda_j(\xi) = 1\} \,. \tag{2.4}$$

PROPOSITION 2.2. *$\Xi_j$ has the following properties*: (a) *$\Xi_j$ is smooth*; (b) *$\Xi_j$ does not intersect with each other and is a closed hypersurface, enclosing the origin.*

For brevity, we restrict our attention to the case (a) in Proposition 2.1 throughout the entire discussion.

Let $d_j$, $j = 0, \pm 1, \cdots, \pm\rho$, be the multiplicity of $\lambda_j(\xi)$, $d_{-j} = d_j$, and hence $N = \sum_{j=-\rho}^{\rho} d_j$. Let

$$\Gamma_j(\xi) = \text{projection on the eigenspace corresponding to } \lambda_j(\xi),\ j = 0, \pm 1, \cdots, \pm\rho\ . \tag{2.5}$$

Then, it is easy to see that $\Gamma_j(\xi)$ is smooth in $\boldsymbol{R}^n - \{0\}$ and homogeneous of degree zero. Moreover, $\Gamma_j(\xi)$ has the following properties: (a) $\Gamma_j(-\xi) = \Gamma_{-j}(\xi)$ for $j \neq 0$; (b) $\Gamma_j(\xi)\Gamma_k(\xi) = \delta_{jk}\Gamma_j(\xi)$; (c) $\sum_{j=-\rho}^{\rho} \Gamma_j(\xi) = E_0$, $E_0$ being the $N \times N$ identity matrix.

We denote by $I_j$ the $d_j \times d_j$ identity matrix and define $\mathscr{D}_0^{(\pm)}(\xi)$ as follows:

$$\mathscr{D}_0^{(\pm)}(\xi) = \begin{pmatrix} \lambda_{\pm 1}(\xi) I_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{\pm\rho}(\xi) I_\rho \end{pmatrix}, \qquad I_{-j} = I_j\ .$$

We further define $\mathscr{D}_0(\xi)$ as

$$\mathscr{D}_0(\xi) = \begin{pmatrix} \mathscr{D}_0^{(+)}(\xi) & & 0 \\ & \lambda_0(\xi) I_0 & \\ 0 & & \mathscr{D}_0^{(-)}(\xi) \end{pmatrix}. \tag{2.6}$$

For brevity, we assume that there exists a $N \times N$ unitary matrix $U_0(\xi)$ such that $U_0(\xi)$ is smooth globally in $\boldsymbol{R}^n - \{0\}$ and that

$$U_0(\xi) A_0(\xi) U_0(\xi)^{-1} = \mathscr{D}_0(\xi)\ , \qquad \xi \neq 0\ . \tag{2.7}$$

If $U_0(\xi)$ exists, then $U_0(\xi)$ is homogeneous of degree zero. In the later argument, we have only to assume that such an $U_0(\xi)$ exists in a small neighborhood of each $\xi_0$ ($\xi_0 \neq 0$) fixed arbitrarily, not containing the origin and this assumption is always satisfied for uniformly propagative systems.

**2.2.** Symbol class of weighted pseudo-differential operators. In this subsection, we introduce a special class of pseudo-differential operators and state some fundamental properties without proofs.

DEFINITION 2.1. We say that $P(x, \xi) = \{p_{jk}(x, \xi)\}_{j,k=1,\ell}$, $(x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$, belongs to $A_{\theta,\sigma}^{(m)}(\ell)$, $\sigma \geqq \theta \geqq 0$, when the following conditions are satisfied:

(a) $p_{jk}(x, \xi)$ is smooth in $\boldsymbol{R}^n \times \boldsymbol{R}^n$,

(b) $|\partial_\xi^\gamma p_{jk}(x, \xi)| \leqq C_\gamma(1 + |x|)^{-\theta}(1 + |\xi|)^{m-|\gamma|}$,

(c) $|\partial_x^\beta \partial_\xi^\gamma p_{jk}(x, \xi)| \leqq C_{\beta,\gamma}(1 + |x|)^{-\sigma}(1 + |\xi|)^{m-|\gamma|}$, $|\beta| \geqq 1$.

We say that a family of $P(x, \xi; \varepsilon)$ with parameter $\varepsilon$ belongs to $A_{\theta,\sigma}^{(m)}(\ell)$ uniformly in $\varepsilon$, if the above constants $C_\gamma$ and $C_{\beta,\gamma}$ are independent of $\varepsilon$.

We now define the pseudo-differential operator $P = P(x, D_x)$ with symbol $P(x, \xi) \in A_{\theta,\sigma}^{(m)}(\ell)$ as follows:

$$Pu = (2\pi)^{-n} \int e^{ix\cdot\xi} P(x, \xi)\hat{u}(\xi)d\xi$$

for $u(x) = {}^t(u_1(x), \cdots, u_\ell(x)) \in \mathscr{S}$, $\mathscr{S}$ being the Schwartz space of rapidly decreasing smooth functions, where $\hat{u}(\xi)$ is the Fourier transform of $u$;

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)dx$$

and the integration with no domain attached is taken over $\boldsymbol{R}^n$.

DEFINITION 2.2. We say that $P(x, D_x) \in OPA_{\theta,\sigma}^{(m)}(\ell)$, when it is a pseudo-differential operator with symbol $P(x, \xi) \in A_{\theta,\sigma}^{(m)}(\ell)$.

For the calculus of pseudo-differential operators of class $OPA_{\theta,\sigma}^{(m)}(\ell)$, we can obtain formulas similar to those in the standard Hörmander class $S_{\rho,\delta}^{(m)}$ (Hörmander [1]). We state these formulas below without proofs.

PROPOSITION 2.3. (i) *Let $P_j(x, D_x)$, $j = 1, 2$, be pseudo-differential operators of class $OPA_{\theta_j,\sigma_j}^{(m_j)}(\ell)$. Then, the product $P = P_1 P_2$ is also a pseudo-differential operator of class $OPA_{\theta,\sigma}^{(m)}(\ell)$, where $m = m_1 + m_2$, $\theta = \theta_1 + \theta_2$ and $\sigma = \min(\theta_1 + \sigma_2, \theta_2 + \sigma_1)$, and the symbol $P(x, \xi)$ is expressed as $P(x, \xi) = P_1(x, \xi)P_2(x, \xi) + Q(x, \xi)$ with $Q(x, \xi) \in A_{\sigma,\sigma}^{(\tilde{m})}(\ell)$, $\tilde{m} = m_1 + m_2 - 1$.* (ii) *Let $P(x, D_x)$ be a pseudo-differential operator of class $OPA_{\theta,\sigma}^{(m)}(\ell)$. Then, $P^*$, $P^*$ being the adjoint of $P$ in $L_{2,0}^{(N)}$, is also a pseudo-differential operator of class $OPA_{\theta,\sigma}^{(m)}(\ell)$, and the symbol $\sigma(P^*)(x, \xi)$ is expressed as $\sigma(P^*)(x, \xi) = P^*(x, \xi) + Q(x, \xi)$ with $Q(x, \xi) \in A_{\sigma,\sigma}^{(\tilde{m})}(\ell)$, $\tilde{m} = m - 1$, where $P^*(x, \xi)$ is the adjoint matrix of $P(x, \xi)$.*

PROPOSITION 2.4. *Let $P(x, D_x)$ be of class $OPA_{\theta,\sigma}^{(m)}(\ell)$. Then, $P(x, D_x)$ is a bounded operator from $H_{k+m,\gamma}^{(\ell)}$ to $H_{k,\gamma+\theta}^{(\ell)}$ for any $k$ and $\gamma$. Furthermore, if $P(x, D_x; \varepsilon)$ belongs to $OPA_{\theta,\sigma}^{(m)}(\ell)$ uniformly in $\varepsilon$, then $P(x, D_x; \varepsilon)$ is uniformly bounded.*

In the later discussion, the class $A^{(0)}_{\theta,\sigma}(\ell)$ is most frequently used[1], so we simply write $A_{\theta,\sigma}(\ell)$ and $OPA_{\theta,\sigma}(\ell)$ instead of $A^{(0)}_{\theta,\sigma}(\ell)$ and $OPA^{(0)}_{\theta,\sigma}(\ell)$, respectively.

**2.3.** Preliminary lemmas. We conclude this section by stating some simple *a priori* estimates for solutions to the equation

$$\Lambda_0 u - (\lambda + i\kappa)E(x)u = f\,, \qquad 0 \leqq \kappa \leqq 1\,. \tag{2.8}$$

LEMMA 2.1. *Assume that* $\kappa > 0$. *Let* $u \in L^{(N)}_{2,0}$ *be a solution to equation* (2.8) *with* $f \in L^{(N)}_{2,\alpha}$, $\alpha > 0$. *Then,* $u \in L^{(N)}_{2,\alpha}$ *and* $\kappa\,|u|_{0,\alpha} \leqq C(|f|_{0,\alpha} + |u|_{0,\alpha-1})$ *with* $C$ *independent of* $\kappa$.

LEMMA 2.2. *Under the same assumptions as in Lemma* 2.1,

$$\kappa\,|u|^2_{0,0} \leqq C\,|f|_{0,\alpha}\,|u|_{0,-\alpha}\,.$$

The proof of Lemma 2.1 and 2.2 is easy, so we omit it.

Let $\Gamma_j(D_x)$, $j = 0, \pm 1, \cdots, \pm\rho$, be the pseudo-differential operator with symbol $\Gamma_j(\xi)$ defined by (2.5). We define $\Gamma(D_x)$ as $\Gamma(D_x) = E_0 - \Gamma_0(D_x)$. Then, we have the following results.

LEMMA 2.3. *Let* $\Gamma(D_x)$ *be as above. Assume that* $0 \leqq \kappa \leqq 1$. *Moreover, assume that* $u$ *is a solution to equation* (2.8) *with* $f \in L^{(N)}_{2,0}$ *such that* $u \in L^{(N)}_{2,0}$. *Then,* $\Gamma(D_x)u \in H^{(N)}_{1,0}$ *and*

$$|\Gamma(D_x)u|_{1,0} \leqq C(|f|_{0,0} + |u|_{0,0})\,.$$

*Proof.* We obtain from equation (2.8) that

$$\Lambda_0\Gamma(D_x)u = (\lambda + i\kappa)\Gamma(D_x)E(x)u + \Gamma(D_x)f\,.$$

Since $\Lambda_0(\xi)\Gamma(\xi) = \sum_{j=1}^{\rho} \lambda_j(\xi)\Gamma_j(\xi) + \sum_{j=-\rho}^{-1} \lambda_j(\xi)\Gamma_j(\xi)$, it follows from (ii) of Proposition 2.1 that

$$|\Gamma(D_x)u|_{1,0} \leqq C(|\Lambda_0\Gamma(D_x)u|_{0,0} + |\Gamma(D_x)u|_{0,0})\,.$$

This completes the proof.

LEMMA 2.4. *Let* $\Gamma_0(D_x)$ *and* $\Gamma(D_x)$ *be as above. Let* $\phi(\xi)$ *be a smooth function with compact support such that* $\phi(\xi) = 1$ *in a neighborhood of the origin and let* $\chi(\xi) = 1 - \phi(\xi)$. *Assume that* $0 \leqq \kappa \leqq 1$. *Moreover, assume that* $u$ *is a solution to equation* (2.8) *with* $f \in L^{(N)}_{2,\gamma}$, $-\infty < \gamma < \infty$, *such that* $u \in L^{(N)}_{2,\gamma}$. *Then, the following estimates hold*:

1) More precisely, symbols with compact support in $\xi$ are used and such symbols belong to $A_{\theta,\sigma}(\ell)$ or $A^{(m)}_{\theta,\sigma}(\ell)$ for any $m$.

(i) $|\phi(D_x)u|_{1,\gamma} \leqq C|u|_{0,\gamma}$;

(ii) $|\chi(D_x)\Gamma(D_x)u|_{1,\gamma} \leqq C(|f|_{0,\gamma} + |u|_{0,\gamma})$.

*Proof.* (i) is obvious. For the proof of (ii), we set $v = \chi(D_x)(\rho(x)u)$ with $\rho(x) = (1 + |x|^2)^{\gamma/2}$ and hence $v \in L_{2,0}^{(N)}$ by assumption. Then, $v$ obeys the equation $\Lambda_0 v = \chi(D_x)g$, where $g = (\lambda + i\kappa)\rho(x)E(x)u + \rho(x)f + [\Lambda_0, \rho(x)]u$ and [ , ] denotes the usual commutator notation. Since $g \in L_{2,0}^{(N)}$ and since $|g|_{0,0} \leqq C(|f|_{0,\gamma} + |u|_{0,\gamma})$, we obtain by Lemma 2.3 that

$$|\Gamma(D_x)v|_{1,0} \leqq C(|f|_{0,\gamma} + |u|_{0,\gamma}) . \tag{2.9}$$

We can write $\Gamma(D_x)v$ as $\Gamma(D_x)v = \rho(x)\chi(D_x)\Gamma(D_x)u + R(x, D_x)\rho(x)u$ with $R(x, D_x) = [\chi(D_x)\Gamma(D_x), \rho]\rho^{-1}$. As is easily seen, $R(x, D_x)$ belongs to $OPA_{0,0}^{(-1)}(N)$ and hence $|R(x, D_x)\rho u|_{1,0} \leqq C|u|_{0,\gamma}$. This, together with (2.9), proves (ii).

## § 3. Fundamental proposition

In this section we consider an equation of the following form:

$$\partial_t v + \kappa b(D_t, D_y)v + iA(t, y, D_y)v = \kappa g + f , \qquad 0 \leqq \kappa \leqq 1 , \tag{3.1}$$

where $\partial_t = \partial/\partial t$, $D_t = -i\partial_t$ and $D_y = -i(\partial/\partial y_1, \cdots, \partial/\partial y_m)$, $m = n - 1$. We write $z = (t, y)$, $y = (y_1, \cdots, y_m)$ and denote by $\zeta = (\tau, \eta)$, $\eta = (\eta_1, \cdots, \eta_m)$, *the coordinate system dual to z. The following hypotheses are made.*

(H.1) $A(t, y, D_y)$ is a pseudo-differential operator with symmetric matrix symbol $A(t, y, \eta) = \{a_{jk}(t, y, \eta)\}_{j,k=1,\ell}$ and each component $a_{jk}(t, y, \eta)$ satisfies the estimates;

$$|\partial_\eta^\gamma a_{jk}(t, y, \eta)| \leqq C_\gamma(1 + |\eta|)^{-|\gamma|} ;$$
$$|\partial_y^\beta \partial_\eta^\gamma a_{jk}(t, y, \eta)| \leqq C_{\beta,\gamma}(1 + |t|)^{-(1+\theta)}(1 + |\eta|)^{-|\gamma|} , \qquad |\beta| \geqq 1 ,$$

for some $\theta, 0 < \theta < 1$.

(H.2) $b(D_t, D_y)$ is a pseudo-differential operator with non-negative symbol $b(\zeta) = b(\tau, \eta)$ and $b(\zeta)$ is expressed as $b(\zeta) = c(\zeta)^2$ for some smooth function $c(\zeta)$, $c(\zeta) \geqq 0$, with compact support.

We further assume that

$$f \in L_{2,\alpha}^{(\ell)} , \qquad \alpha > \tfrac{1}{2} , \tag{3.2}$$

$$g \in L_{2,\theta}^{(\ell)} \qquad \text{for } \theta \text{ introduced in (H.1)} . \tag{3.3}$$

We use the following notations throughout this section: (a) We work exclusively in the spaces of $\ell$ summands, so we drop the superscript $\ell$ to denote the norm in $L_{2,\alpha}^{(\ell)}$; $|\ \ |_{0,\alpha} = |\ \ |_{0,\alpha}^{(\ell)}$; (b) We denote by $\langle\ ,\ \rangle$ and $\|\ \ \|$ the scalar product and norm, respectively, in the space $\mathscr{L} = L_{2,0}^{(\ell)}(R_y^m) = \sum \oplus L_2(R_y^m)$, $\ell$ summands. The next lemma is easily verified.

LEMMA 3.1. (i) *Assume that $A(t, y, \eta)$ satisfies* (H.1). *Then, there exists a constant $C$ independent of $t$ such that*

$$|\operatorname{Im} \langle A(t, y, D_y)\psi, \psi\rangle| \leqq C(1 + |t|)^{-(1+\theta)} \|\psi\|^2$$

*for any $\psi \in \mathscr{L}$.* (ii) *Assume that $b(\zeta)$ satisfies* (H.2) *and that $v = v(t, y) \in L_{2,\sigma}^{(\ell)}$, $\sigma \geqq 0$. Let $\phi(t)$ be a non-negative smooth function such that $\phi(t) = 0$ for $t \geqq 1$ and $\phi(t) = (1 + t^2)^{\sigma/2}$ for $t \leqq 0$. Then,*

$$\operatorname{Re} \int_{-\infty}^{\infty} \langle b(D_t, D_y)v, \phi^2 v\rangle dt \geqq -C \int_{-\infty}^{\infty} (1 + t^2)^{\nu} \|v(t)\|^2 dt , \qquad \nu = \sigma - \tfrac{1}{2} .$$

Now, we prove a series of propositions which will be applied to derive *a priori* estimates in § 6.

PROPOSITION 3.1. *Assume* (H.1), (H.2), (3.2) *and* (3.3). *Let $v = v(t, y)$ be a solution to equation* (3.1) *such that $v \in L_{2,0}^{(\ell)}$. Then, there exists a constant $C$ independent of $t$ such that*

$$\|v(t)\|^2 \leqq C\{|f|_{0,\alpha}^2 + |v|_{0,-\alpha}^2 + \kappa(|g|_{0,0}^2 + |v|_{0,0}^2)\} .$$

*Proof.* Since $v \in L_{2,0}^{(\ell)}$, $\partial_t v \in L_{2,0}^{(\ell)}$ by equation (3.1). According to the trace theory ([4]), we see that $v(t, \cdot)$ is continuous in $t$ as an $\mathscr{L}$-valued function and therefore $\|v(t)\|$ is well-defined for all $t$. We take the scalar product $\langle\ ,\ \rangle$ between equation (3.1) and $v$ and integrate the resulting equality with respect to $t$ over the interval $(a, T)$, $-\infty < a < T < \infty$. Furthermore, taking the real part, we have

$$\tfrac{1}{2} \|v(T)\|^2 = \operatorname{Im} \int_a^T \langle A(t, y, D_y)v, v\rangle dt + J(a, T) + \tfrac{1}{2} \|v(a)\|^2 , \tag{3.4}$$

where

$$J(a, T) = \operatorname{Re} \int_a^T \{\langle f, v\rangle + \kappa\langle g, v\rangle - \kappa\langle b(D_t, D_y)v, v\rangle\} dt .$$

Since $b(D_t, D_y)$ is a bounded operator in $L_{2,0}^{(\ell)}$, $J(a, T)$ is estimated as $|J(a, T)| \leqq C\{|f|_{0,\alpha}^2 + |v|_{0,-\alpha}^2 + \kappa(|g|_{0,0}^2 + |v|_{0,0}^2)\}$ with $C$ independent of $a$ and

$T$. We apply (i) of Lemma 3.1 to the first term on the right side of (3.4), so that

$$\|v(T)\|^2 \leqq C\left\{K(a) + \int_a^T (1+|t|)^{-(1+\theta)} \|v(t)\|^2 \, dt\right\}, \tag{3.5}$$

where $K(a) = \|v(a)\|^2 + \{|f|^2_{0,\alpha} + |v|^2_{0,-\alpha} + \kappa(|g|^2_{0,0} + |v|^2_{0,0})\}$. We now apply the well-known Gronwall inequality to (3.5), so that $\|v(T)\|^2 \leqq CK(a)$ with $C$ independent of $a$ and $T$. Since $\liminf_{a\to-\infty} \|v(a)\|^2 = 0$ by $v \in L^{(\ell)}_{2,0}$, the desired result follows at once.

PROPOSITION 3.2. *Under the same assumptions as in Proposition* 3.1, *it holds that for any* $\sigma$, $\sigma > \frac{1}{2}$,

$$\begin{aligned}|v|^2_{0,-\sigma} &\leqq \int_{-\infty}^{\infty} (1+t^2)^{-\sigma}(\|v(t)\|^2 + \|\partial_t v(t)\|^2) dt \\ &\leqq C_\sigma\{|f|^2_{0,\alpha} + |v|^2_{0,-\alpha} + \kappa(|g|^2_{0,0} + |v|^2_{0,0})\}.\end{aligned}$$

*Proof.* The first inequality is evident by definition. The second one follows immediately from equation (3.1) and Proposition 3.1.

PROPOSITION 3.3. *Assume* (H.1), (H.2), (3.2) *and* (3.3). *Moreover, assume that* $\frac{1}{2} < \alpha < \frac{1}{2}(1+\theta)$ *for* $\theta$ *in* (H.1). *Let* $\mu$ *be a constant such that* $0 < \mu < 2\alpha - 1$ $(< \theta < 1)$. *Let* $v = v(t, y)$ *be a solution to equation* (3.1) *satisfying* $v \in L^{(N)}_{2,\mu/2}$. *Then, there exists a constant* $C = C_\mu$ *independent of* $\kappa$ *such that*

$$\int_{-\infty}^{-1} (1+t^2)^{(\mu-1)/2} \|v(t)\|^2 \, dt \leqq C\{|f|^2_{0,\alpha} + |v|^2_{0,-\alpha} + \kappa(|g|^2_{0,\theta} + |v|^2_{0,0})\}.$$

*Proof.* We first introduce a smooth function $\phi(t)$ with the following properties: (a) $\phi(t) \geqq 0$; (b) $\phi'(t) \leqq 0$; (c) $\phi(t) = 0$ for $t \geqq 1$; (d) $\phi(t) = (1+t^2)^{\mu/4}$ for $t \leqq 0$. We take the scalar product $\langle\ ,\ \rangle$ between equation (3.1) and $\psi(t)v$, $\psi(t) = \phi(t)^2$, and integrate the resulting equality with respect to $t$ over $(-\infty, \infty)$, noting the fact that $\liminf_{t\to\pm\infty} |t|^\mu \|v(t)\|^2 = 0$, $\mu < 1$, which follows from $v \in L^{(\ell)}_{2,\mu/2}$. Furthermore, taking the real part and making use of (ii) in Lemma 3.1, we obtain

$$\begin{aligned}&\int_{-\infty}^{-1} (1+t^2)^{(\mu-1)/2} \|v(t)\|^2 \, dt - C_1\kappa \int_{-\infty}^{\infty} (1+t^2)^{(\mu-1)/2} \|v(t)\|^2 \, dt \\ &\leqq C_2 \int_{-\infty}^{\infty} \psi(t) \{\operatorname{Im} \langle A(t, y, D_y)v, v\rangle + \operatorname{Re}(\langle f, v\rangle + \kappa\langle g, v\rangle)\} dt \qquad (3.6) \\ &= \sum_{j=1}^{3} I_j.\end{aligned}$$

We estimate each term on the right side. Since $f \in L^{(\ell)}_{2,\alpha}$ and since $0 < \mu < 2\alpha - 1$, $I_2$ is estimated as

$$|I_2| \leqq \varepsilon \int_{-\infty}^{-1} (1 + t^2)^{(\mu-1)/2} \|v(t)\|^2 \, dt + C_\varepsilon(|f|^2_{0,\alpha} + |v|^2_{0,-\alpha})$$

for any $\varepsilon > 0$ small enough and hence the first term is absorbed in the left side of (3.6). Similarly we have $|I_3| \leqq C\kappa(|g|^2_{0,\theta} + |v|^2_{0,0})$ since $g \in L^{(\ell)}_{2,\theta}$ and $\mu < \theta$. To estimate $I_1$, we use (i) in Lemma 3.1 and obtain

$$|I_1| \leqq C \int_{-\infty}^{\infty} (1 + t^2)^{-\sigma} \|v(t)\|^2 \, dt \, , \qquad \sigma = \tfrac{1}{2}(1 + \theta - \mu) > \tfrac{1}{2} \, ,$$

which, together with Proposition 3.2, implies that

$$|I_1| \leqq C\{|f|^2_{0,\alpha} + |v|^2_{0,-\alpha} + \kappa(|g|^2_{0,0} + |v|^2_{0,0})\} \, .$$

Thus, we have only to combine all the above estimates to conclude the proof.

Next, we consider the equation (3.1) with $\kappa = 0$;

$$\partial_t v + iA(t, y, D_y)v = f \, . \tag{3.7}$$

The next proposition plays an important role in the proof of the main theorems.

PROPOSITION 3.4. *Assume that* (H.1) *and* (3.2) *are satisfied. Moreover, assume that* $v = v(t, y)$ *is a solution to equation* (3.7) *such that* $(1 + t^2)^{-\sigma} \|v(t)\|^2$ *is integrable for any* $\sigma, \sigma > \frac{1}{2}$, *and that*

$$\liminf_{t \to \pm\infty} \|v(t)\|^2 = 0 \, . \tag{3.8}$$

*Then,*

$$\int_{-\infty}^{\infty} (1 + t^2)^{\beta} \|v(t)\|^2 \, dt < \infty \qquad \textit{for } \beta, \; -\tfrac{1}{2} < \beta < \alpha - 1 \, .$$

*Proof.* We fix $\varepsilon$ arbitrarily so that $0 < \varepsilon < \min(\theta, 2\alpha - 1)$. Let $\phi(t) = -1 + (1 + t^2)^{-\varepsilon/2}$ for $t \geqq 1$, so that $\phi(t) \leqq 0$ and $-\phi'(t) \geqq C_\varepsilon(1 + t^2)^{-(1+\varepsilon)/2}$. As before, we take the scalar product $\langle \, , \rangle$ between equation (3.7) and $\phi(t)v$, and integrate the obtained equality with respect to $t$ over $(s, \infty)$, $s > 1$. Then, making use of Lemma 3.1 and (3.8), we have

$$\begin{aligned} -\phi(s) \|v(s)\|^2 &+ C_1 \int_s^{\infty} (1 + t^2)^{-(1+\varepsilon)/2} \|v(t)\|^2 \, dt \\ &\leqq C_2 \int_s^{\infty} (1 + t^2)^{-(1+\theta)/2} \|v(t)\|^2 \, dt + \mathrm{Re} \int_s^{\infty} \phi(t)\langle f, v\rangle dt \, . \end{aligned}$$

Since $\varepsilon < \theta$, we can choose $s$ so large that the first term on the right side is absorbed in the left side and we therefore obtain

$$\|v(s)\|^2 \leqq C\int_s^\infty (1+t^2)^{(1+\varepsilon)/2}\|f(t)\|^2\,dt\,, \qquad 1+\varepsilon < 2\alpha\,, \tag{3.9}$$

with $C$ independent of $s$, $s$ being large enough. Next, we multiply both sides of (3.9) by $s^{2\gamma}$, $\gamma = \alpha - 1 - \varepsilon/2 > -1/2$, and integrate with respect to $s$ over $(T, S)$;

$$\int_T^S s^{2\gamma}\|v(s)\|^2\,ds \leqq C\int_T^\infty (1+t^2)^\alpha\|f(t)\|^2\,dt\,.$$

Letting $S \to \infty$ yields that $(1+t^2)^\gamma\|v(t)\|^2$ is integrable over $(0, \infty)$. If we use $\psi(t) = 1 - (1-t)^{-\varepsilon}$, $t \leqq -1$, instead of $\phi(t)$, a similar argument gives the integrability over $(-\infty, 0)$. Thus, the proof is completed.

## § 4. Diagonalization

**4.1.** Decomposition. By Assumption (A.3), we can choose a constant $\theta$, $0 < \theta < \delta$, $\delta$ being as in (A.3), so that the following decomposition is made for each component $e_{jk}(x)$ of $E(x)$, $j, k = 1, \cdots, N$: For any $\varepsilon > 0$ small enough, there exists a constant $R = R(\theta, \varepsilon)$ such that (a) $e_{jk}(x) = e_{jk}(x;\varepsilon) + \tilde{e}_{jk}(x;\varepsilon)$, (b) $e_{jk}(x;\varepsilon) = e_{jk}(x)$ for $|x| \geqq R$ (and hence $\tilde{e}_{jk}(x;\varepsilon)$ is of compact support); (c) for all $x$, $|e_{jk}(x;\varepsilon) - \delta_{jk}| \leqq \varepsilon C(1+|x|)^{-\theta}$ and $|\partial_x^\beta e_{jk}(x;\varepsilon)| \leqq \varepsilon C_\beta(1+|x|)^{-\sigma}$, $\sigma = 1+\theta$, $|\beta| \geqq 1$.

We may assume that $\theta < 1$. We denote by $E(x;\varepsilon)$ the matrix with components $e_{jk}(x;\varepsilon)$ defined above. From now on, we fix the constants $\theta$ and $\sigma$, $\sigma = 1+\theta$, with the meaning ascribed here throughout the later argument.

**4.2.** Diagonalization. We now consider the equation

$$\Lambda_0 u - (\lambda + i\kappa)E(x;\varepsilon)u = f\,, \qquad 0 \leqq \kappa \leqq 1\,, \tag{4.1}$$

with $f \in L_{2,\alpha}^{(N)}$, $\alpha > 1/2$. It is evident that for $\kappa > 0$, there exists a unique solution $u = Q(\lambda + i\kappa;\varepsilon)f = (\Lambda_0 - (\lambda + i\kappa)E(x;\varepsilon))^{-1}f$ such that $u \in L_{2,0}^{(N)}$ and therefore $u \in L_{2,\alpha}^{(N)}$ by Lemma 2.1. Let $\psi(\xi)$ be a smooth function with compact support such that $\psi(\xi) = 0$ in a neighborhood of the origin. Then, we have

$$\Lambda_0\psi(D_x)u - (\lambda + i\kappa)E(x;\varepsilon)\psi(D_x)u = \psi(D_x)f + r(x;\varepsilon)\,, \tag{4.2}$$

where $r(x;\varepsilon) = (\lambda + i\kappa)[\psi(D_x), E(x;\varepsilon)]u$ and satisfies the estimate $|r|_{0,\nu} \leqq$

$\varepsilon C\,|u|_{0,-\nu}$, $\nu = \sigma/2 = (1/2)(1+\theta)$, for $C$ independent of $\kappa$ and $\varepsilon$ (small enough).

The aim of this subsection is to transform (4.2) into an equation of the diagonalized form. To do this, it is convenient to the later argument to introduce the following notations: (a) For a vector-valued function $v(x)$ with component $v_{jk}(x)$, $j = 0, \pm 1, \cdots, \pm\rho$, $k = 1, \cdots, d_j$, $d_j$ being the multiplicity, we write

$$v(x) = {}^t(v_1(x), \cdots, v_\rho(x), v_0(x), v_{-1}(x), \cdots, v_{-\rho}(x)) ,$$

where $v_j(x) = {}^t(v_{j1}(x), \cdots, v_{jd_j}(x))$; (b) For the solution $u$ of equation (4.1) such that $u \in L^{(N)}_{2,\gamma-\nu}$ with some $\gamma$, $\nu = \sigma/2$, we denote by $r(\varepsilon) = r(x;\varepsilon) = {}^t(r_1(\varepsilon), \cdots, r_{-\rho}(\varepsilon))$ all terms satisfying an estimate of the following type:

$$|r(\varepsilon)|_{0,\gamma+\nu} \leqq \varepsilon C\,|u|_{0,\gamma-\nu} \tag{4.3}$$

with $C$ independent of $\kappa$, $0 \leqq \kappa \leqq 1$ and $\varepsilon$ (small enough).

*Remark.* In the later argument, such an $r(\varepsilon)$ always appears in a form of $r(\varepsilon) = R(x, D_x;\varepsilon)u$ with some $R(x, D_x;\varepsilon) \in OPA_{\sigma,\sigma}(N)$ for which $\varepsilon^{-1}R(x,\xi;\varepsilon)$ belongs to $A_{\sigma,\sigma}(N)$ uniformly in $\varepsilon$.

The transformation is made with the aid of the following lemmas.

LEMMA 4.1. *Let $\mathscr{D}_0(\xi)$ be defined by (2.6) and let $U_0(\xi)$ be the unitary matrix given in (2.7). Let $\psi(\xi)$ be the function introduced at the beginning of this subsection. Set $\Lambda(\varepsilon) = \Lambda(x,\xi;\lambda,\varepsilon) = \Lambda_0(\xi) - \lambda E(x;\varepsilon)$. Then, for $\varepsilon$ small enough, there exists a $N \times N$ matrix $U(\varepsilon) = U(x,\xi;\lambda,\varepsilon)$ such that*

$$U(\varepsilon)\Lambda(\varepsilon)U(\varepsilon)^{-1} = \mathscr{D}_0(\xi) - \lambda E_0 + \mathscr{X}(\varepsilon) + \mathscr{Z}(\varepsilon) , \qquad \xi \neq 0 .$$

*Here $U(\varepsilon)$, $\mathscr{X}(\varepsilon) = \mathscr{X}(x,\xi;\lambda,\varepsilon)$ and $\mathscr{Z}(\varepsilon) = \mathscr{Z}(x,\xi;\lambda,\varepsilon)$ have the following properties:* (a) *$\psi(\xi)\mathscr{Z}(\varepsilon)$ belongs to $A_{\sigma,\sigma}(N)$;* (b) *$\mathscr{X}(\varepsilon)$ is of the following form:*

$$\mathscr{X}(\varepsilon) = \begin{pmatrix} X^{(+)}(\varepsilon) & & 0 \\ & X_0(\varepsilon) & \\ 0 & & X^{(-)}(\varepsilon) \end{pmatrix}, \qquad X^{(\pm)}(\varepsilon) = \begin{pmatrix} X_{\pm 1}(\varepsilon) & & 0 \\ & \ddots & \\ 0 & & X_{\pm\rho}(\varepsilon) \end{pmatrix},$$

*where $X_{\pm j}(\varepsilon) = X_{\pm j}(x,\xi;\lambda,\varepsilon)$, $j = 0, 1, \cdots, \rho$, is a $d_j \times d_j$ symmetric matrix and $\psi(\xi)X_{\pm j}(\varepsilon)$ belongs to $A_{\theta,\sigma}(d_j)$;* (c) *$U(\varepsilon)$ is represented as*

$$U(\varepsilon) = U_0(\varepsilon) + U_1(x,\xi;\lambda,\varepsilon) \tag{4.4}$$

*with $U_0(\xi)$ given in (2.7), where $\psi(\xi)U_1(x,\xi;\lambda,\varepsilon)$ belongs to $A_{\theta,\sigma}(N)$. Furthermore, $U(\varepsilon)$ satisfies*

$$U^*(\varepsilon)U(\varepsilon) = E_0 - \mathscr{Z}_1(\varepsilon) \tag{4.5}$$

*with some* $\mathscr{Z}_1(\varepsilon) = \mathscr{Z}_1(x, \xi; \lambda, \varepsilon)$ *such that* $\psi(\xi)\mathscr{Z}_1(\varepsilon)$ *belongs to* $A_{\sigma,\sigma}(N)$, *where* $U^*(\varepsilon)$ *is the adjoint of* $U(\varepsilon)$; (d) $\varepsilon^{-1}\psi(\xi)\mathscr{Z}(\varepsilon)$, $\varepsilon^{-1}\psi(\xi)X_{\pm j}(\varepsilon)$, $\varepsilon^{-1}\psi(\xi)U_1(\varepsilon)$ *and* $\varepsilon^{-1}\psi(\xi)\mathscr{Z}_1(\varepsilon)$ *belong to the corresponding symbol classes uniformly in* $\varepsilon$.

A result similar to this lemma has been already verified in Appendix of [9], so we omit the proof. The next result follows from Lemma 4.1 at once.

LEMMA 4.2. *Let* $\psi(\xi)$, $U(\varepsilon)$, $\mathscr{X}(\varepsilon)$ *and* $\mathscr{Z}(\varepsilon)$ *be as in Lemma* 4.1. *Set* $\Lambda(\kappa, \varepsilon) = \Lambda(x, \xi; \lambda, \kappa, \varepsilon) = \Lambda_0(\xi) - (\lambda + i\kappa)E(x; \varepsilon)$. *Then, for* $\varepsilon$ *small enough, it holds that*

$$U(\varepsilon)\Lambda(\kappa, \varepsilon)U(\varepsilon)^{-1} = \mathscr{D}_0(\xi) - (\lambda + i\kappa)E_0 + \mathscr{X}(\varepsilon) + \mathscr{Z}(\varepsilon) + \kappa\mathscr{T}(\varepsilon)\,,$$

*where* $\mathscr{T}(\varepsilon) = \mathscr{T}(x, \xi; \lambda, \varepsilon)$ *is expressed as*

$$\mathscr{T}(\varepsilon) = -iU(\varepsilon)\{E(x; \varepsilon) - E_0\}U(\varepsilon)^{-1}\,. \tag{4.6}$$

*Moreover,* $\varepsilon^{-1}\psi(\xi)\mathscr{T}(\varepsilon)$ *belongs to* $A_{\theta,\sigma}(N)$ *uniformly in* $\varepsilon$.

Let $\chi(\xi)$ be a smooth function with compact support such that $\chi(\xi) = 1$ in a small neighborhood of the support of $\psi(\xi)$ (not containing the origin), $\psi(\xi)$ being the function introduced at the beginning of this subsection, so that $\chi(\xi)\psi(\xi) = \psi(\xi)$. We define $\tilde{U}(x, \xi; \lambda, \varepsilon)$ as $\tilde{U}(x, \xi; \lambda, \varepsilon) = \chi(\xi)U(x, \xi; \lambda, \varepsilon)$ for $U(x, \xi; \lambda, \varepsilon)$ introduced in Lemma 4.1. Similarly we define $\tilde{X}_j(x, \xi; \lambda, \varepsilon)$ and $\tilde{\mathscr{T}}(x, \xi; \lambda, \varepsilon)$. Furthermore, we set

$$v(x) = \tilde{U}(x, D_x; \lambda, \varepsilon)\psi(D_x)u$$

for the solution $u$ of equation (4.1). With the above preparations, we now transform equation (4.2).

LEMMA 4.3. *Let* $v(x) = {}^t(v_1(x), \cdots, v_{-\rho}(x))$ *be as above. Then, each* $v_j(x)$ *obeys the equation*

$$\{\lambda_j(D_x) - (\lambda + i\kappa)\}\chi(D_x)v_j + \tilde{X}_j(x, D_x; \lambda, \varepsilon)v_j = h_j + \kappa g_j + r_j(\varepsilon) \tag{4.7}$$

*with some* $r_j(\varepsilon)$, *where* $\lambda_j(\xi)$ *is the eigenvalue of* $\Lambda_0(\xi)$ *(given by* (2.2)), *while* $h(x)$ *and* $g(x)$ *are defined by*

$$h(x) = {}^t(h_1(x), \cdots, h_{-\rho}(x)) = \tilde{U}(x, D_x; \lambda, \varepsilon)\psi(D_x)f\,,$$
$$g(x) = {}^t(g_1(x), \cdots, g_{-\rho}(x)) = -\tilde{\mathscr{T}}(x, D_x; \lambda, \varepsilon)v\,.$$

For the proof, we give only a sketch. Equation (4.7) is derived by using Lemma 4.2 and by making a simple calculation based on Proposition 2.3. Indeed, using the relation $\chi(\xi)\psi(\xi) = \psi(\xi)$ and taking account of (d) in Lemma 4.1 and of the property of $\tilde{\mathscr{T}}(x, D_x; \lambda, \varepsilon)$, we see that all remainder terms appearing in commutator calculations can be written as $r(\varepsilon)$. (4.7) is the desired equation.

## § 5. A priori estimates

In this section, we continue to consider the equation (4.1) and derive various *a priori* estimates for solutions of this equation, which are valid uniformly for $\kappa$, $0 < \kappa \leqq 1$. The main result obtained here is stated as follows:

THEOREM 5.1. *Let $u = Q(\lambda + i\kappa; \varepsilon)f$, $\kappa > 0$, be a solution of equation (4.1) with $f \in L^{(N)}_{2,\alpha}$. Assume that $\frac{1}{2} < \alpha < \frac{1}{2}(1+\theta)$. Then, the following estimate holds*: For any $\nu$, $\frac{1}{2} < \nu < \alpha$,

$$|u|_{0,-\nu} \leqq C_\nu(|f|_{0,\alpha} + |u|_{0,-\alpha}) .$$

This theorem will be proved in § 6. From now on, we always fix $\alpha$ so that $\frac{1}{2} < \alpha < \frac{1}{2}(1+\theta)$ and assume that $\lambda > 0$. These assumptions loses no generality; the case of $\lambda < 0$ is dealt with similarly.

**5.1.** Partition of unity. Let $\Xi_j(\lambda) = \{\xi; \lambda_j(\xi) = \lambda\}$, $\lambda > 0$, $j = 1, 2, \cdots, \rho$. Then, by the homogeneity of $\lambda_j(\xi)$, $\Xi_j(\lambda) = \lambda\Xi_j = \{\lambda\xi; \xi \in \Xi_j\}$ with $\Xi_j$ defined by (2.4). In view of Proposition 2.2, we see that; (a) $\Xi_j(\lambda)$ are smooth and bounded; (b) $\Xi_j(\lambda)$ are non-intersecting closed hypersurfaces, enclosing the origin.

We now introduce the partition of unity

$$\begin{aligned} &\Phi = \{\phi_k\}\,, \qquad k = 0, 1, \cdots, K\,; \\ &\Psi_j = \{\psi_{jk}\}\,, \qquad j = 1, \cdots, \rho\,, \quad k = 1, \cdots, K_j\,, \end{aligned} \tag{5.1}$$

with the following properties: (a) $\phi_k(\xi)$ and $\psi_{jk}(\xi)$ are non-negative and smooth; (b) $\phi_k(\xi)$ and $\psi_{jk}(\xi)$ are of compact support except for $\phi_K(\xi)$; (c) for all $\xi$,

$$\sum_{k=0}^{K} \phi_k(\xi)^2 + \sum_{j=1}^{\rho} \sum_{k=1}^{Kj} \psi_{jk}(\xi)^2 = 1\,;$$

(d) $\phi_0(\xi) = 1$ in a small neighborhood of the origin and hence $\phi_k(\xi)$, $k \neq 0$, and $\psi_{jk}(\xi)$ vanish there; (e) the supports of $\phi_k(\xi)$, $k = 0, 1, \cdots, K$, do not intersect with $\Xi_j(\lambda)$, $j = 1, \cdots, \rho$; (f) the supports of $\psi_{jk}(\xi)$ belonging to $\Psi_j$ only intersect with $\Xi_j(\lambda)$; (g) for $j \neq i$, the supports of $\psi_{jk}(\xi)$ and $\psi_{i\ell}(\xi)$ do not intersect with each other.

The existence of such a partition of unity with properties mentioned above is guaranteed by the geometrical property of $\Xi_j(\lambda)$.

We use the localization theory in the $\xi$-space to derive *a priori* estimates, which are divided into the following two types: (I) estimate in an outside of $\Xi_j(\lambda)$; (II) estimate in a neighborhood of $\Xi_j(\lambda)$.

**5.2.** Estimate of type (I). The estimates of type (I) are rather easy to derive. The supports of $\phi_k(\xi)$, $k = 0, 1, \cdots, K$, do not intersect with $\Xi_j(\lambda)$, so that if $\varepsilon$ is taken small enough, then the matrix $\Lambda_0(\xi) - (\lambda + i\kappa)E(x; \varepsilon)$ is invertible uniformly in $\kappa$, $0 \leqq \kappa < 1$, and $\varepsilon$ for $\xi \in \operatorname{supp} \phi_k$. We define $P_k(x, \xi; \lambda, \kappa, \varepsilon)$ as

$$P_k(x, \xi; \lambda, \kappa, \varepsilon) = \phi_k(\xi)(\Lambda_0(\xi) - (\lambda + i\kappa)E(x; \varepsilon))^{-1}, \qquad k = 0, 1, \cdots, K, \tag{5.2}$$

which belongs to $A_{0,\sigma}(N)$. Letting $P_k(x, D_x; \lambda, \kappa, \varepsilon)$ operate on equation (4.1), we obtain

$$\phi_k(D_x)u = P_k(x, D_x; \lambda, \kappa, \varepsilon)f + r(\varepsilon) \tag{5.3}$$

with some $r(\varepsilon)$. Thus, we have the following result.

LEMMA 5.1. *Let $u = Q(\lambda + i\kappa; \varepsilon)f$, $\kappa > 0$, be a solution of equation* (4.1) *with $f \in L^{(N)}_{2,\alpha}$. Let $P_k(x, D_x; \lambda, \kappa, \varepsilon)$ be as above. Then, $\phi_k(D_x)u$ is represented as* (5.3) *and satisfies*

$$|\phi_k(D_x)u|_{0,\alpha} \leqq C(|f|_{0,\alpha} + |u|_{0,-\alpha})$$

*with $C$ independent of $\kappa$ and $\varepsilon$ (small enough). Furthermore, the above result is still valid for $\kappa = 0$ if it is assumed that there exists a solution $u$ of equation* (4.1) *with $\kappa = 0$ such that $u \in L^{(N)}_{2,-\alpha}$.*

We proceed to the estimate of $\psi_{jk}(D_x)u$. We fix one pair $(p, q)$, $1 \leqq p \leqq \rho$, $1 \leqq q \leqq K_p$. Let $\chi_{pq}(\xi)$ be a smooth function with compact support such that $\chi_{pq}(\xi) = 1$ in a small neighborhood of the support of $\psi_{pq}(\xi)$ and hence $\chi_{pq}(\xi)\psi_{pq}(\xi) = \psi_{pq}(\xi)$. For notational convenience, we drop the subscripts $p$ and $q$ to denote $\psi_{pq}(\xi)$ and $\chi_{pq}(\xi)$; $\psi(\xi) = \psi_{pq}(\xi)$, etc. We define the symbol $\hat{U}(x, \xi; \lambda, \varepsilon) \in A_{0,\sigma}(N)$ as

$$\hat{U}(x, \xi; \lambda, \varepsilon) = \chi(\xi)U(x, \xi; \lambda, \varepsilon) \tag{5.4}$$

with $U(x, \xi; \lambda, \varepsilon)$ introduced in Lemma 4.1. Similarly we define

$\hat{X}_j(x, \xi; \lambda, \varepsilon) \in A_{\theta,\sigma}(d_j)$, $j = 0, \pm 1, \cdots, \pm \rho$, and $\hat{\mathscr{T}}(x, \xi; \lambda, \varepsilon) \in A_{\theta,\sigma}(N)$. We further set

$$v(x) = {}^t(v_1(x), \cdots, v_{-\rho}(x)) = \hat{U}(x, D_x; \lambda, \varepsilon)\psi(D_x)u \;. \tag{5.5}$$

Then, in virtue of Lemma 4.3, it follows that each $v_j(x)$ obeys the equation

$$\{\lambda_j(D_x) - (\lambda + i\kappa)\}\chi(D_x)v_j + \hat{X}_j(x, D_x; \lambda, \varepsilon)v_j = h_j + \kappa g_j + r_j(\varepsilon) \;, \tag{5.6}$$

where $h_j$ and $g_j$ are given by

$$h(x) = {}^t(h_1(x), \cdots, h_{-\rho}(x)) = \hat{U}(x, D_x; \lambda, \varepsilon)\psi(D_x)f \;, \tag{5.7}$$

$$g(x) = {}^t(g_1(x), \cdots, g_{-\rho}(x)) = -\hat{\mathscr{T}}(x, D_x; \lambda, \varepsilon)v \;. \tag{5.8}$$

When $j \neq p$, we can choose the support of $\chi(\xi)$ so small that it does not intersect with $\Xi_j(\lambda)$. Therefore, if $\varepsilon$ is taken small enough, the $d_j \times d_j$ matrix $\lambda_j(\xi) - (\lambda + i\kappa) + X_j(x, \xi; \lambda, \varepsilon)$ is invertible uniformly in $\kappa$ and $\varepsilon$ for $\xi \in \operatorname{supp} \chi$. We define the symbol $Q_{jp}(x, \xi; \lambda, \kappa, \varepsilon) \in A_{0,\sigma}(d_j)$ as

$$Q_{jp}(x, \xi; \lambda, \kappa, \varepsilon) = \chi(\xi)(\lambda_j(\xi) - (\lambda + i\kappa) + X_j(x, \xi; \lambda, \varepsilon))^{-1} \tag{5.9}$$

for $j \neq p$. As in the derivation of (5.3), we let $Q_{jp}(x, D_x; \lambda, \kappa, \varepsilon)$ operate on equation (5.6) to obtain

$$\chi(D_x)^2 v_j = Q_{jp}(x, D_x; \lambda, \kappa, \varepsilon)\{h_j + \kappa g_j\} + r_j(\varepsilon) \tag{5.10}$$

with some $r_j(\varepsilon)$. Thus, we have the following result.

LEMMA 5.2. *Let $u = Q(\lambda + i\kappa; \varepsilon)f$, $\kappa > 0$, be a solution of equation* (4.1) *with $f \in L^{(N)}_{2,\alpha}$. Let $v_j$, $h_j$ and $g_j$ be defined by* (5.5), (5.7) *and* (5.8), *respectively. Let $Q_{jp}(x, D_x; \lambda, \kappa, \varepsilon)$ be as above. If $\varepsilon$ is taken small enough and if $j \neq p$, then*

$$v_j = Q_{jp}(x, D_x; \lambda, \kappa, \varepsilon)\{h_j + \kappa g_j\} + r_j(\varepsilon) \tag{5.11}$$

*and the norm of $v_j$ in $L^{(d_j)}_{2,\alpha}$ is dominated by $C(|f|_{0,\alpha} + |u|_{0,-\alpha})$ with $C$ independent of $\kappa$ and $\varepsilon$. Furthermore, the above result is still valid for $\kappa = 0$ under the same hypothesis as in Lemma* 5.1.

*Proof.* (5.11) follows from (5.10) at once by use of the relation $\chi(\xi)\psi(\xi) = \psi(\xi)$. Since $Q_{jp}(x, D_x; \lambda, \kappa, \varepsilon) \in OPA_{0,\sigma}(d_j)$, it is a bounded operator from $L^{(d_j)}_{2,\alpha}$ to itself uniformly in $\kappa$ and $\varepsilon$. Hence, we have

$$|v_j|^{(d_j)}_{0,\alpha} \leqq C(|h_j|^{(d_j)}_{0,\alpha} + \kappa |g_j|^{(d_j)}_{0,\alpha} + |u|_{0,-\alpha}) \;,$$

where $|\ \ |_{0,\alpha}^{(d_j)}$ denotes the norm in $L_{2,\alpha}^{(d_j)}$. By the definition of $h_j$, the norm of $h_j$ is dominated by $C|f|_{0,\alpha}$. Moreover, since $\hat{\mathscr{T}}(x, D_x; \lambda, \varepsilon) \in OPA_{\theta,\sigma}(N)$, we see, recalling the definition of $g_j$, that the norm of $g_j$ is dominated by $C|u|_{0,\mu}$, $\mu = \alpha - \theta$. This together with Lemma 2.1, implies that

$$\kappa |g_j|_{0,\alpha}^{(d_j)} \leqq C(|f|_{0,\alpha} + |u|_{0,\nu}) \leqq C(|f|_{0,\alpha} + |u|_{0,-\alpha})$$

since $\nu = \alpha - \theta - 1 < -\alpha$ by assumption. Thus, the desired result is obtained.

## § 6. Continuation of a priori estimates

In this section we consider the equation (5.6) with $j = p$;

$$\text{(6.1)} \quad \{\lambda_p(D_x) - (\lambda + i\kappa)\}\chi(D_x)v_p + \hat{X}_p(x, D_x; \lambda, \varepsilon)v_p = h_p + \kappa g_p + r_p(\varepsilon)$$

and derive a priori estimates of type (II) by applying the results obtained in § 3. To do this, we have to transform (6.1) into an equation of the form like (3.1). The transformation is made through two steps. As in the preceding section, we drop the subscripts $p$ and $q$ to denote $\psi_{pq}(\xi)$ and $\chi_{pq}(\xi)$.

**6.1.** Preparation. Let $\mathcal{O}$ be a small neighborhood of the support of $\chi(\xi)$ (not containing the origin) and let $\Xi = \Xi_p(\lambda) \cap \mathcal{O}$. Let $\tau$ be a vector transversal to $\Xi$. We denote by $\Sigma$ the $m$-dimensional linear space (hyperplane), $m = n - 1$, orthogonal to $\tau$ and by $\eta = (\eta_1, \cdots, \eta_m)$ a system of orthogonal bases generating $\Sigma$. We take $\zeta = (\tau, \eta)$ as an orthogonal coordinate system in $\boldsymbol{R}_\xi^n$ and therefore $\zeta = \Pi\xi$ for some unitary matrix $\Pi$. We denote by $z = (t, y)$, $y = (y_1, \cdots, y_m)$ the orthogonal coordinate system dual to $\zeta$; $z = \Pi^* x$. The unitary matrix $\Pi$ induces naturally the one to one transformation denoted by the same symbol $\Pi$; $(\Pi\phi)(z, \zeta) = \phi(\Pi z, \Pi^*\zeta)$. For notational convenience, we denote a representation in terms of the $(z, \zeta)$-coordinates by the same symbol as an original function which is represented in terms of the $(x, \xi)$-coordinates; $\phi(z, \zeta) = (\Pi\phi)(z, \zeta) = \phi(\Pi z, \Pi^*\zeta)$ for $\phi = \phi(x, \xi)$. Here we note that this transformation is unitary in $L_{2,0}^{(\ell)}$ and that the Fourier transformation is invariant with respect to this transformation.

**6.2.** The first step. Let $\mathcal{O}_1$ be an open set such that $\bar{\mathcal{O}} \subset \mathcal{O}_1$, $\bar{\mathcal{O}}$ being the closure of $\mathcal{O}$. Since $\tau$ is a transversal vector, we can take $\mathcal{O}_1$ small enough, if necessary, so that for $\lambda_p(\zeta) = \lambda_p(\xi)$

$$\lambda_p(\zeta) - \lambda = (\tau - a_1(\eta;\lambda))\sigma_1(\zeta;\lambda) \tag{6.2}$$

in $\mathcal{O}_1$ with some real functions $a_1(\eta;\lambda)$ and $\sigma_1(\zeta;\lambda)$. Here we choose a direction of $\tau$ so that $\partial\lambda_p/\partial\tau > 0$ in $\mathcal{O}_1$. Hence, $\sigma_1(\zeta;\lambda) > 0$ and we may write

$$\sigma_1(\zeta;\lambda) = s_1(\zeta;\lambda)^2 \tag{6.3}$$

in $\mathcal{O}_1$ with some $s_1(\zeta;\lambda) > 0$. Let $\Omega$ and $\Omega_1$ be the projections of $\mathcal{O}$ and $\mathcal{O}_1$ to $\Sigma$, respectively. Let $\omega(\eta)$ be a smooth function of $\eta$ only such that $\omega(\eta) = 1$ in $\Omega$ and that the support of $\omega(\eta)$ is contained in $\Omega_1$. When $\omega(\eta)$ is regarded as a function of $\zeta = (\tau, \eta)$, $\omega(\eta)\chi(\zeta) = \chi(\zeta)$ and hence $\omega(\eta)\psi(\zeta) = \psi(\zeta)$, where $\psi(\zeta)$ and $\chi(\zeta)$ are the representations of $\psi(\xi)$ and $\chi(\xi)$ in terms of the $\zeta$-coordinates, respectively.

Now we define the symbol $a(\eta;\lambda)$ as

$$a(\eta;\lambda) = \omega(\eta)a_1(\eta;\lambda) \tag{6.4}$$

with $a_1(\eta;\lambda)$ introduced in (6.2). We further define $s(\zeta;\lambda)$ as

$$s(\zeta;\lambda) = \chi(\zeta)s_1(\zeta;\lambda) \tag{6.5}$$

with $s_1(\zeta;\lambda)$ given in (6.3). We set

$$\tilde{v}_p = \tilde{v}_p(t, y) = s(D_z;\lambda)v_p \tag{6.6}$$

for $v_p$ of equation (6.1). Then, we see from (6.1) that $\tilde{v}_p$ satisfies the equation

$$\{D_t - a(D_y;\lambda)\}\tilde{v}_p - i\kappa b(D_z;\lambda)\tilde{v}_p + Y_p(z, D_z;\lambda,\varepsilon)\tilde{v}_p = \tilde{h}_p + \kappa\tilde{g}_p + r_p(\varepsilon) \tag{6.7}$$

with another $r_p(\varepsilon) = r_p(z;\varepsilon)$, where $Y_p(z, D_z;\lambda,\varepsilon) \in OPA_{\theta,\sigma}(d_p)$ is the pseudo-differential operator with symmetric matrix symbol given by $Y_p(z,\zeta;\lambda,\varepsilon) = \chi(\zeta)s_1(\zeta;\lambda)^{-2}\hat{X}_p(z,\zeta;\lambda,\varepsilon)$ and $b(\zeta;\lambda)$ is the symbol defined by

$$b(\zeta;\lambda) = c(\zeta;\lambda)^2 \tag{6.8}$$

with $c(\zeta;\lambda) = \chi(\zeta)s_1(\zeta;\lambda)^{-1}$, while $\tilde{h}_p$ and $\tilde{g}_p$ are given by

$$\tilde{h}_p = \tilde{h}_p(t, y) = c(D_z;\lambda)h_p \tag{6.9}$$

$$\tilde{g}_p = \tilde{g}_p(t, y) = c(D_z;\lambda)g_p\ , \tag{6.10}$$

respectively. Equation (6.7) is easily derived by letting $c(D_z;\lambda)$ operate on (6.1) and by making a simple calculation using Proposition 2.3. This is the equation transformed through the first step.

**6.3.** The second step. The second step is based on the next lemma.

LEMMA 6.1. *Let* $a(\eta;\lambda)$ *and* $Y_p(z,\zeta;\lambda,\varepsilon)$ *be as above. If* $\varepsilon$ *is taken small enough, then there exist* $d_p\times d_p$ *matrices* $L_p=L_p(z,\zeta;\lambda,\varepsilon)$ *and* $N_p=N_p(z,\eta;\lambda,\varepsilon)$ *such that*

$$(I_p+L_p)\{(\tau-a(\eta;\lambda))+Y_p\}(I_p+L_p)=(\tau-a(\eta;\lambda))+N_p+Z\,,$$

*where* $I_p$ *is the* $d_p\times d_p$ *identity matrix. Furthermore,* $L_p, N_p$ *and* $Z=Z(z,\zeta;\lambda,\varepsilon)$ *have the following properties:* (a) $\varepsilon^{-1}\chi(\zeta)Z(z,\zeta;\lambda,\varepsilon)$ *belongs to* $A_{\sigma,\sigma}(d_p)$ *uniformly in* $\varepsilon$; (b) $L_p$ *is symmetric and* $\varepsilon^{-1}\chi(\zeta)L_p(z,\zeta;\lambda,\varepsilon)$ *belongs to* $A_{\theta,\sigma}(d_p)$ *uniformly in* $\varepsilon$. *Moreover,* $(I_p+L_p)$ *is invertible uniformly in* $\varepsilon$; (c) $N_p$ *is also symmetric and each component* $n_{jk}(z,\eta;\lambda,\varepsilon)=n_{jk}(t,y,\eta;\lambda,\varepsilon)$ *satisfies*

$$|\partial_\eta^\gamma n_{jk}(t,y,\eta;\lambda,\varepsilon)|\leqq\varepsilon C_\gamma(1+|z|)^{-\theta}\,,$$
$$|\partial_y^\beta\partial_\eta^\gamma n_{jk}(t,y,\eta;\lambda,\varepsilon)|\leqq\varepsilon C_{\beta,\gamma}(1+|z|)^{-\sigma}\,,\qquad|\beta|\geqq1\,.$$

The proof will be given in Appendix. Before making a transformation, we need to introduce several new symbols and functions. We first define the symbols $V_p$ and $W_p$ of class $A_{\theta,\sigma}(d_p)$ as

$$V_p(z,\zeta;\lambda,\varepsilon)=\chi(\zeta)(I_p+L_p(z,\zeta;\lambda,\varepsilon))^{-1}\,,\tag{6.11}$$

$$W_p(z,\zeta;\lambda,\varepsilon)=\chi(\zeta)(I_p+L_p(z,\zeta;\lambda,\varepsilon))\,,\tag{6.12}$$

respectively, with $L_p$ introduced in Lemma 6.1. For later use, we note here that since $L_p$ is symmetric, $V_p^*=V_p$, $V_p^*$ being the adjoint of $V_p$, and therefore

$$V_p^*(z,D_z;\lambda,\varepsilon)W_p(z,D_z;\lambda,\varepsilon)=\chi(D_z)^2+R(z,D_z;\lambda,\varepsilon)\tag{6.13}$$

with some $R(z,D_z;\lambda,\varepsilon)$ such that the symbol $\varepsilon^{-1}R(z,\zeta;\lambda,\varepsilon)$ belongs to $A_{\sigma,\sigma}(d_p)$ uniformly in $\varepsilon$, where $V_p^*(z,D_z;\lambda,\varepsilon)$ is the adjoint operator in $L_{2,0}^{(\ell)}$, $\ell=d_p$. (6.13) follows immediately from property (b) in Lemma 6.1. We set

$$w_p=w_p(t,y)=V_p(z,D_z;\lambda,\varepsilon)\tilde v_p\tag{6.14}$$

for $\tilde v_p$ of equation (6.7) and

$$F_p=F_p(t,y)=W_p(z,D_z;\lambda,\varepsilon)\tilde h_p\tag{6.15}$$

for $\tilde h_p$ defined by (6.9). We further set

$$G_p=G_p(t,y)=W_p(z,D_z;\lambda,\varepsilon)\tilde g_p+K_p(z,D_z;\lambda,\varepsilon)w_p\tag{6.16}$$

for $\tilde{g}_p$ defined by (6.10), where $K_p(z, D_z; \lambda, \varepsilon) \in OPA_{\theta,\sigma}(d_p)$ is the pseudo-differential operator with symbol given by $K_p(z, \zeta; \lambda, \varepsilon) = ib(\zeta; \lambda)L_p(2I_p + L_p)$, $b(\eta; \lambda)$ being the symbol defined by (6.8).

LEMMA 6.2. *Let $F_p$ and $G_p$ be as above. Then, the following estimates hold*: (i) $|F_p|_{0,\alpha}^{(\ell)} \leqq C|f|_{0,\alpha}$, $\ell = d_p$; (ii) $|G_p|_{0,\theta}^{(\ell)} \leqq C|u|_{0,0}$.

*Proof.* (i) is proved by combining (6.15), (6.9) and (5.7). Since $|w_p|_{0,0}^{(\ell)} \leqq C|u|_{0,0}$, (ii) is proved if we note that $\hat{\mathscr{T}}(z, D_z; \lambda, \varepsilon)$ in (5.8) belongs to $OPA_{\theta,\sigma}(N)$ and that $K_p(z, D_z; \lambda, \varepsilon)$ belongs to $OPA_{\theta,\sigma}(d_p)$.

Let $\omega(\eta)$ be the function introduced at the beginning of this subsection. We define the symbol $A(t, y, \eta; \lambda, \varepsilon)$ as

$$A(t, y, \eta; \lambda, \varepsilon) = \omega(\eta)\{-a(\eta; \lambda) + N_p(t, y, \eta; \lambda, \varepsilon)\}, \tag{6.17}$$

where $a(\eta; \lambda)$ is given by (6.4) and $N_p$ is introduced in Lemma 6.1. This definition, together with property (c) of Lemma 6.1, implies that $A(t, y, \eta; \lambda, \varepsilon)$ satisfies (H.1) in § 3 with $\ell = d_p$ uniformly in $\varepsilon$. Under these preparations, we can now transform (6.7) into an equation of the form discussed in § 3.

LEMMA 6.3. *Let $w_p$, $F_p$ and $G_p$ be defined by* (6.14), (6.15) *and* (6.16), *respectively. Let $A(t, y, \eta; \lambda, \varepsilon)$ be as above and let $b(\eta; \lambda)$ be defined by* (6.8). *Then, $w_p$ satisfies the equation*

$$\partial_t w_p + \kappa b(D_z; \lambda)w_p + iA(t, y, D_y; \lambda, \varepsilon)w_p = iF_p + i\kappa G_p + r_p(\varepsilon). \tag{6.18}$$

Equation (6.18) is derived from (6.7) by use of Lemma 6.1 and the derivation is similar to that of (4.7) in Lemma 4.3, so we omit the proof. This is the desired equation and the second step is completed.

**6.4.** Proof of Theorem 5.1. We are now in a position to apply Propositions 3.1 ~ 3.3 to equation (6.18). To do this, we have to check that all the assumptions in these propositions are satisfied. First we have stated above that (H.1) is satisfied. (3.2) and (3.3) follow immediately from Lemma 6.2 and (H.2) follows from the definition (6.8) of $b(\zeta; \lambda)$. Furthermore, for $\kappa > 0$, $u \in L_{2,\alpha}^{(N)}$ by Lemma 2.1 and hence $w_p \in L_{2,\alpha}^{(\ell)}$, $\ell = d_p$. (We use the symbol $\ell$ in this sense throughout the remainder.). Thus, we have the following result.

LEMMA 6.4. (i) *For any $\nu$, $\nu > \frac{1}{2}$*,

$$|w_p|_{0,-\nu}^{(\ell)} \leqq C(|f|_{0,\alpha} + |u|_{0,-\alpha}); \tag{a}$$

(b) $$\int_{-\infty}^{\infty} (1+t^2)^{-\nu}(\|w_p\|^2 + \|\partial_t w_p\|^2)dt \leqq C(|f|_{0,\alpha}^2 + |u|_{0,-\alpha}^2) .$$

(ii) *For* $\mu$, $0 < \mu < 2\alpha - 1$ $(< \theta < 1)$,

$$\int_{-\infty}^{-1} (1+t^2)^{(\mu-1)/2} \|w_p\|^2 \, dt \leqq C(|f|_{0,\alpha}^2 + |u|_{0,-\alpha}^2) .$$

*Here* $\| \ \|$ *denotes the norm in* $L_{2,0}^{(\ell)}(R_y^m)$, $m = n-1$.

*Proof.* Proposition 3.2 combined with Lemma 6.2 yields

$$|w_p|_{0,-\nu}^{(\ell)} \leqq C(|f|_{0,\alpha} + |u|_{0,-\alpha} + \sqrt{\kappa}\,|u|_{0,0})$$

which, together with Lemma 2.2, proves (a). The proof for (b) and (ii) is done in a similar way.

*Proof of Theorem* 5.1. We first recall the definitions of $v_p$, $\tilde{v}_n$ and $w_p$ given by (5.5), (6.6) and (6.14), respectively. And we note that all the symbols of pseudo-differential operators in these definitions are invertible in a small neighborhood of the support of $\psi(\xi)$. Thus, taking account of this fact, we see from (6.14) and (6.6) that $\chi(D_z)^4 v_p$ is expressed as $\chi(D_z)^4 v_p = c(D_z; \lambda) W_p(z, D_z; \lambda, \varepsilon) w_p + r_p(\varepsilon)$ with some $r_p(\varepsilon)$. Therefore, by Lemma 6.4, $|\chi(D_z)^4 v_p|_{0,-\nu}^{(\ell)}$, $\nu > \frac{1}{2}$, is majorized by $C(|f|_{0,\alpha} + |u|_{0,-\alpha})$. This, together with Lemma 5.2, implies that $|\chi(D_x)^4 v|_{0,-\nu}$ is also majorized by the same bound as above with another $C$. Furthermore, by use of the relation $\chi(\xi)\psi(\xi) = \psi(\xi)$, it follows from (5.5) that

$$\psi(D_x)u = V(x, D_x; \lambda, \varepsilon)\chi(D_x)^4 v + r(\varepsilon) ,$$

where $V(x, D_x; \lambda, \varepsilon) \in OPA_{0,\sigma}(N)$ is the pseudo-differential operator with symbol $\chi(\xi)U(x, \xi; \lambda, \varepsilon)^{-1}$. Thus, we have

$$|\psi_{pq}(D_x)u|_{0,-\nu} \leqq C(|f|_{0,\alpha} + |u|_{0,-\alpha}) , \qquad \psi_{pq}(\xi) = \psi(\xi) .$$

This estimate holds for any pair $(p, q)$. Hence, combining this estimate with Lemma 5.1, we obtain the desired result and the proof is completed.

## § 7. Discreteness of eigenvalues

In this section, we shall prove Theorem 1.1 stated in § 1.

*Proof of Theorem* 1.1. Let $I \subset R^+ = (0, \infty)$ be a compact interval fixed arbitrarily. To prove this theorem, it is sufficient to show that there is a finite number of eigenvalues with finite multiplicity in $I$. We assume the contrary; there is an infinite sequence of eigenvalues with repetition according to multiplicity; $\{a_n\}_{n=1}^{\infty}$, $a_n \in I\ (a_n > 0)$. Let $u^{(n)}$ be the eigenfunction corresponding to $a_n$; $\Lambda_0 u^{(n)} = a_n E(x) u^{(n)}$, $|u^{(n)}|_{0,0} = 1$. We assert that $\{u^{(n)}\}_{n=1}^{\infty}$ forms a precompact set in $L_{2,0}^{(N)}$. If this assertion is verified, the proof is completed. In fact, $\{u^{(n)}\}_{n=1}^{\infty}$ is an orthogonal system with respect to the energy scalar product $((\quad))_0$ defined in subsection 1.4, which contradicts the assumption made above. To prove this assertion, we write

$$\Lambda_0 u^{(n)} - a_n E(x; \varepsilon) u^{(n)} = f^{(n)} \tag{7.1}$$

for $\varepsilon$ fixed small enough, where $f^{(n)} = a_n\{E(x) - E(x; \varepsilon)\} u^{(n)}$. Throughout the proof of this theorem, we fix $\varepsilon$ small enough and positive constants $C_\varepsilon$ appearing below depend on $\varepsilon$ but are independent of $n$.

We first recall the definition of partition of unity introduced by(5.1). Since $f^{(n)}$ is of compact support and since $|f^{(n)}|_{0,\alpha} \leqq C_\varepsilon$, it follows from Lemma 5.1 that

$$|\phi_k(D_x) u^{(n)}|_{0,\alpha} \leqq C_\varepsilon\,, \qquad k = 0, 1, \cdots, K\,. \tag{7.2}$$

Next, we proceed to the estimate for $\psi_{jk}(D_x) u^{(n)}$. We again fix one pair $(p, q)$ and use $\psi(\xi)$ and $\chi(\xi)$ with the same meaning as before. For $u^{(n)}$, we denote by $v^{(n)} = {}^t(v_1^{(n)}, \cdots, v_p^{(n)}, \cdots, v_{-\rho}^{(n)})$ and $w_p^{(n)}$ the functions defined through transformations (5.5) and (6.14) with $\lambda = a_n$, respectively. Similarly, for $f^{(n)}$, we denote by $F_p^{(n)}$ the function defined through a series of transformations (5.7), (6.9) and (6.15). If $j \neq p$, it then follows from Lemma 5.2 that

$$|v_j^{(n)}|_{0,\alpha}^{(d_j)} \leqq C_\varepsilon\,. \tag{7.3}$$

We shall make an estimate of $w_p^{(n)}$. $w_p^{(n)}$ obeys the equation (6.18) with $\kappa = 0$ and $\lambda = a_n$;

$$\partial_t w_p^{(n)} + iA(t, y, D_y; a_n, \varepsilon) w_p^{(n)} = iF_p^{(n)} + r_p(\varepsilon)\,. \tag{7.4}$$

Since $f^{(n)}$ is of compact support, we see easily that $F_p^{(n)} \in L_{2,\beta}^{(\ell)}$, $\ell = d_p$, for any $\beta$ and that its norm in this space is bounded uniformly in $n$. Furthermore, according to (4.3), $r_p(\varepsilon) \in L_{2,\sigma}^{(\ell)}$ by $u^{(n)} \in L_{2,0}^{(N)}$ and $|r_p(\varepsilon)|_{0,\sigma}^{(\ell)} \leqq C$. Thus, the terms on the right side of (7.4) belong to $L_{2,\sigma}^{(\ell)}$ uniformly in $n$. We now want to apply Proposition 3.4 to equation (7.4). Since $u^{(n)} \in L_{2,0}^{(N)}$, all the assumptions in this proposition are satisfied. Thus, for $\nu$, $0 < \nu < \theta$, we have

(7.5) $$\int_{-\infty}^{\infty}(1+t^2)^{\nu}\,\|w_p^{(n)}(t)\|^2\,dt \leqq C_\epsilon\,.$$

As was noted in the proof of Theorem 5.1, the symbols of pseudo-differential operators in (6.6) and (6.14) are invertible in a small neighborhood of the support of $\psi(\xi)$. Therefore, by the same argument as in the proof of Theorem 5.1, we see that $v_p^{(n)}(t, y)$, which is the representation of $v_p^{(n)}(x)$ in terms of the $z$-coordinates ($z = (t, y)$), also satisfies (7.5). An estimate similar to (7.5) is still valid for another transversal vector $\tilde{\tau}$ in subsection 6.1. Thus, taking $n$ linearly independent transversal vectors ($n$ being the dimension of the basic space $\boldsymbol{R}^n$), we may conclude that $v_p^{(n)}(x) \in L_{2,\nu}^{(\ell)}$ for $\nu$ above, $\ell = d_p$, and that $|v_p^{(n)}|_{0,\nu}^{(\ell)} \leqq C_\epsilon$. We may assume that $\nu < \alpha$. Then, we obtain from (7.3) that $|v^{(n)}|_{0,\nu} \leqq C_\epsilon$ and hence $|\psi(D_x)u^{(n)}|_{0,\nu} \leqq C_\epsilon$ with another $C_\epsilon$. This, together with (7.2), yields that

(7.6) $$|u^{(n)}|_{0,\nu} \leqq C_\epsilon\,.$$

The norm in the Sobolev space $H_{1,0}^{(N)}$ is estimated by use of Lemma 2.4. We set $\chi_0(\xi) = 1 - \phi_0(\xi)$, where $\phi_0(\xi)$ is a member of the partition of unity introduced by (5.1) such that $\phi_0(\xi) = 1$ in a neighborhood of the origin. Let $\Gamma_0(\xi)$ be the projection on the eigenspace corresponding to the zero eigenvalue (defined by (2.5)) and let $\Gamma(\xi) = E_0 - \Gamma_0(\xi)$. We write $u^{(n)}$ as

(7.7) $$u^{(n)} = \tilde{u}^{(n)} + \hat{u}^{(n)}\,,$$

where $\tilde{u}^{(n)} = \phi_0(D_x)u^{(n)} + \chi_0(D_x)\Gamma(D_x)u^{(n)}$ and $\hat{u}^{(n)} = \chi_0(D_x)\Gamma_0(D_x)u^{(n)}$. Then, Lemma 2.4 combined with (7.6) shows that $\{\tilde{u}^{(n)}\}_{n=1}^{\infty}$ forms a precompact set in $L_{2,0}^{(N)}$. Thus, to complete the proof, we have only to show that $\{\hat{u}^{(n)}\}_{n=1}^{\infty}$ forms a precompact set. To see this, we introduce the subspace $\mathcal{N}$ of $L_{2,0}^{(N)}$ as $\mathcal{N} = \Gamma_0(D_x)L_{2,0}^{(N)} = \{\Gamma_0(D_x)u;\, u \in L_{2,0}^{(N)}\}$ and consider the operator $B = \Gamma_0(D_x)E(x)$ acting on $\mathcal{N}$. Clearly $B$ is bounded and invertible. Letting $\Gamma_0(D_x)$ operate on the equation $\Lambda_0 u^{(n)} = a_n E(x)u^{(n)}$, we have $B\hat{u}^{(n)} = -\Gamma_0(D_x)E(x)\tilde{u}^{(n)}$ and hence $\hat{u}^{(n)} = -B^{-1}\Gamma_0(D_x)E(x)\tilde{u}^{(n)}$. This shows that $\{\hat{u}^{(n)}\}_{n=1}^{\infty}$ forms a precompact set and the proof is completed.

## § 8. The principle of limiting absorption

In this section we shall prove Theorem 1.2.

*Proof of Theorem* 1.2. First we make the following simple reduction. Let $Q(\lambda + i\kappa) = (\Lambda_0 - (\lambda + i\kappa)E(x))^{-1}$. Then, $R(\lambda + i\kappa) = Q(\lambda + i\kappa)E(x)$. To prove statement (i), it is sufficient to show that there exists a constant $C$ independent of $\kappa$ such that

$$|Q(\lambda + i\kappa)f|_{0,-\alpha} \leqq C|f|_{0,\alpha} . \tag{8.1}$$

We prove this by contradiction. Assume that there exist sequences $\{f^{(n)}\}_{n=1}^{\infty}$ and $\{\kappa_n\}_{n=1}^{\infty}$, $0 < \kappa_n \leqq 1$, such that $f^{(n)}$ converges to 0 strongly in $L_{2,\alpha}^{(N)}$ and that $|u^{(n)}|_{0,-\alpha} = |Q(\lambda + i\kappa_n)f^{(n)}|_{0,-\alpha} = 1$. We may assume that $\kappa_n \to 0$ as $n \to \infty$. We write

$$\Lambda_0 u^{(n)} - (\lambda + i\kappa_n)E(x;\varepsilon)u^{(n)} = f^{(n)} + g^{(n)} \tag{8.2}$$

for $\varepsilon$ small enough, where $g^{(n)} = (\lambda + i\kappa_n)\{E(x) - E(x;\varepsilon)\}u^{(n)}$. Clearly $g^{(n)}$ is of compact support and $|g^{(n)}|_{0,\alpha} \leqq C_\varepsilon |u^{(n)}|_{0,-\alpha}$. Throughout the proof of this theorem, we regard $\varepsilon$ as a parameter and denote by $C$ positive constants independent of $\varepsilon$, $\kappa$ and $n$. ($C_\varepsilon$ denotes positive constants depending only on $\varepsilon$.)

As the first step toward the proof, we shall show that $\{u^{(n)}\}_{n=1}^{\infty}$ forms a precompact set in $L_{2,-\alpha}^{(N)}$. To see this, we write $u^{(n)}$ in the form of (7.7); $u^{(n)} = \tilde{u}^{(n)} + \hat{u}^{(n)}$. $\hat{u}^{(n)}$ satisfies the equation

$$\Gamma_0(D_x)E(x)\hat{u}^{(n)} = -\Gamma_0(D_x)E(x)\tilde{u}^{(n)} - (\lambda + i\kappa_n)^{-1}\Gamma_0(D_x)f^{(n)}. \tag{8.3}$$

Applying Theorem 5.1 to equation (8.2) and using Lemma 2.4, we may conclude that $\{\tilde{u}^{(n)}\}_{n=1}^{\infty}$ forms a precompact set in $L_{2,-\alpha}^{(N)}$. Therefore, we can choose a subsequence denoted by the same symbol $\{u^{(n)}\}_{n=1}^{\infty}$ such that $u^{(n)}$ converges to some $u$ weakly in $L_{2,-\alpha}^{(N)}$ and $\tilde{u}^{(n)}$ converges strongly in $L_{2,-\alpha}^{(N)}$. The strong limit of $\{\tilde{u}^{(n)}\}_{n=1}^{\infty}$ is given by $\tilde{u} = \phi_0(D_x)u + \chi_0(D_x)\Gamma(D_x)u$. We write $\hat{u} = \chi_0(D_x)\Gamma_0(D_x)u$, which is well-defined since $\chi_0(D_x)\Gamma_0(D_x)$ is a bounded operator from $L_{2,-\alpha}^{(N)}$ to itself. Hence, $u = \tilde{u} + \hat{u}$ and $\hat{u}^{(n)}$ converges to $\hat{u}$ weakly in $L_{2,-\alpha}^{(N)}$. Furthermore, $u$ satisfies the equation

$$\Lambda_0 u - \lambda E(x)u = 0 \tag{8.4}$$

and hence $\hat{u}$ satisfies

$$\chi_0(D_x)\Gamma_0(D_x)E(x)\hat{u} = -\chi_0(D_x)\Gamma_0(D_x)E(x)\tilde{u} . \tag{8.5}$$

We combine equations (8.3) and (8.5) to obtain

$$\begin{aligned}\chi_0(D_x)\Gamma_0(D_x)E(x)(\hat{u}^{(n)} - \hat{u})\\ = -\chi_0(D_x)\Gamma_0(D_x)E(x)(\tilde{u}^{(n)} - \tilde{u}) - (\lambda + i\kappa_n)^{-1}\chi_0(D_x)\Gamma_0(D_x)f^{(n)} .\end{aligned} \tag{8.6}$$

Here we note that the terms on the right side converge to 0 strongly in

$L^{(N)}_{2,-\alpha}$. We assert that $\hat{u}^{(n)}$ converges to $\hat{u}$ strongly in $L^{(N)}_{2,-\alpha}$. To prove this, we put $\rho(x)=(1+|x|^2)^{-\alpha/2}$ and denote by ( , ) the usual scalar product in $L^{(N)}_{2,0}$. We write

$$|\hat{u}^{(n)}-\hat{u}|^2_{0,-\alpha}\leqq C(E(x)(\hat{u}^{(n)}-\hat{u}),\ \rho(x)^2(\hat{u}^{(n)}-\hat{u}))=C(I_1+I_2)\ ,$$

where

$$I_1=(\chi_0(D_x)\Gamma_0(D_x)E(x)(\hat{u}^{(n)}-\hat{u}),\ \rho(x)^2(u^{(n)}-u))$$

and

$$I_2=(\rho(x)^2E(x)(\hat{u}^{(n)}-\hat{u}),\ \rho(x)^{-2}[\rho(x)^2,\ \chi_0(D_x)\Gamma_0(D_x)](u^{(n)}-u))\ .$$

Making use of equation (8.6), we see that $I_1=I_1(n)$ converges to 0 as $n\to\infty$. For the second term $I_2=I_2(n)$, we note that the pseudo-differential operator $\rho(x)^{-2}[\rho(x)^2,\chi_0(D_x)\Gamma_0(D_x)]$ belongs to $OPA^{(-1)}_{1,1}(N)$, which implies that it is a compact operator in $L^{(N)}_{2,-\alpha}$ and hence $I_2(n)$ also converges to 0. Thus, we can prove that $\hat{u}^{(n)}$ converges to $\hat{u}$ strongly and this shows that $\{u^{(n)}\}^\infty_{n=1}$ forms a precompact set.

The second step is to prove that the limit $u$ ($u\neq 0$) belongs to $L^{(N)}_{2,0}$. If this is verified, it then follows that $u$ must be equal to 0 since by assumption $\lambda$ is not an eigenvalue. This contradicts the assumption made above and statement (i) is proved.

To use the results obtained in sections $3\sim 6$ for the proof of the above statement, we rewrite (8.4) as

$$\Lambda_0u-\lambda E(x;\varepsilon)u=f(\varepsilon) \tag{8.7}$$

for $\varepsilon$ small enough, where

$$f(\varepsilon)=\lambda\{E(x)-E(x;\varepsilon)\}u\ . \tag{8.8}$$

We recall the definition of partition of unity introduced by (5.1). First, applying Lemma 5.1 to equation (8.7), we see that $\phi_k(D_x)u\in L^{(N)}_{2,\alpha}$, $k=0,1,\cdots,K$. Next, to estimate $\psi_{jk}(D_x)u$, we fix, as usual, one pair $(p,q)$ and use $\psi(\xi)$ and $\chi(\xi)$ with the same meaning as before. We denote by $v$ the function defined through transformation (5.5) for $u$ above;

$$v={}^t(v_1,\cdots,v_p,\cdots,v_{-\rho})=\hat{U}(x,D_x;\lambda,\varepsilon)\psi(D)_xu\ . \tag{8.9}$$

Then, applying Lemma 5.2 to equation (8.7), we know that $v_j\in L^{(d_j)}_{2,\alpha}$ for $j\neq p$. Our next task is to show that $v_p\in L^{(\ell)}_{2,0}$, $\ell=d_p$. If this is verified, it then follows that $\psi(D_x)u\in L^{(\ell)}_{2,0}$ and hence $u\in L^{(N)}_{2,0}$.

We denote by $w_p(\varepsilon) = w_p(t, y; \varepsilon)$, $z = (t, y)$, the function defined by transformations (6.6) and (6.14) for $v_p$ above;

$$w_p(\varepsilon) = V_p(z, D_z; \lambda, \varepsilon)s(D_z; \lambda)v_p . \tag{8.10}$$

Similarly, we denote by $F_p(\varepsilon) = F_p(t, y; \varepsilon)$ the function defined by transformations (5.7), (6.9) and (6.15) for $f(\varepsilon)$ given by (8.8);

$$F_p(\varepsilon) = W_p(z, D_z; \lambda, \varepsilon)c(D_z; \lambda)h_p(\varepsilon) , \tag{8.11}$$

where $h_p(\varepsilon) = h_p(x; \varepsilon)$ is defined by

$$h(\varepsilon) = {}^t(h_1(\varepsilon), \cdots, h_p(\varepsilon), \cdots, h_{-\rho}(\varepsilon)) = \hat{U}(x, D_x; \lambda, \varepsilon)\psi(D_x)f(\varepsilon) . \tag{8.12}$$

Then, $w_p(\varepsilon)$ obeys the equation (6.18) with $\kappa = 0$;

$$\partial_t w_p(\varepsilon) + iA(t, y, D_y; \lambda, \varepsilon)w_p(\varepsilon) = iF_p(\varepsilon) + r_p(\varepsilon) \tag{8.13}$$

with some $r_p(\varepsilon) \in L^{(\ell)}_{2,\alpha}$. Since $f(\varepsilon)$ is of compact support, $F_p(\varepsilon) \in L^{(\ell)}_{2,\nu}$ for any $\nu$ and hence the term on the right side of (8.13) belongs to $L^{(\ell)}_{2,\alpha}$. To prove that $v_p \in L^{(\ell)}_{2,0}$, it is sufficient to show that $w_p(\varepsilon) \in L^{(\ell)}_{2,0}$ since the symbols of pseudo-differential operators in (8.10) are invertible in a small neighborhood of the support of $\psi(\xi)$. We apply Proposition 3.4 to equation (8.13) to prove this fact.

LEMMA 8.1. *If $\varepsilon$ is taken small enough, then $w_p(\varepsilon) = w_p(t; \varepsilon) \in L^{(\ell)}_{2,0}(R^m_y)$, $m = n - 1$, for all $t$ and $\liminf_{t\to\pm\infty} \|w_p(t; \varepsilon)\| = 0$, where $\| \;\|$ denotes the norm in $L^{(\ell)}_{2,0}(R^m_y)$.*

The proof of this lemma will be given after the completion of the proof of this theorem. If we admit the validity of Lemma 8.1, we see that all the assumptions in Proposition 3.4 are satisfied. Indeed, the fact that $(1 + t^2)^{-\nu}\|w_p(t; \varepsilon)\|^2$ is integrable for any $\nu$, $\nu > \frac{1}{2}$, follows from Lemma 6.4 by a limit procedure ($n \to \infty$). Hence, this proposition enables us to obtain that $(1 + t^2)^{\beta}\|w_p(t; \varepsilon)\|^2$ is integrable for $\beta = \alpha - 1 - \mu$, $0 < \mu < \theta$. If $\beta < 0$, this implies that $v_p \in L^{(\ell)}_{2,\beta}$ with $\beta$ above (and hence $u \in L^{(N)}_{2,\beta}$) since $w_p(\varepsilon) \in L^{(\ell)}_{2,\beta}$. (If $\beta \geqq 0$, $w_p(\varepsilon) \in L^{(\ell)}_{2,0}$ and hence $v_p \in L^{(\ell)}_{2,0}$.) Hence, according to (4.3), $r_p(\varepsilon) \in L^{(\ell)}_{2,\gamma}$ with $\gamma = \alpha + (\theta - \mu)$ since $u \in L^{(N)}_{2,\beta}$. Thus, the term on the right side of (8.13) belongs to $L^{(\ell)}_{2,\gamma}$ with $\gamma$ above, $\gamma > \alpha$. We repeat the above argument (boot-strap argument) until we obtain that $r_p(\varepsilon) \in L^{(\ell)}_{2,\nu}$ for some $\nu$, $\nu > 1$, and then apply Proposition 3.4 once again to conclude that $w_p(\varepsilon) \in L^{(\ell)}_{2,0}$. Thus, the second step is completed and statement (i) is proved.

For the proof of statement (ii), it is sufficient to show that the limit

$Q(\lambda + i0)$ of $Q(\lambda + i\kappa)$ as $\kappa \to 0$ exists in the topology of strong convergence in $L_{2,-\alpha}^{(N)}$. As is easily seen from the proof of statement (i), there exists a subsequence $\{\kappa_n\}_{n=1}^{\infty}$ such that $Q(\lambda + i\kappa_n)f$ converges strongly in $L_{2,-\alpha}^{(N)}$. Hence, to show that $Q(\lambda + i0)$ is well-defined, we have only to prove that the above limit is independent of the choice of a subsequence. To see this, assume that there exist two limits $u^{(1)}$ and $u^{(2)}$ and put $u = u^{(1)} - u^{(2)}$. Then, by an argument similar to the proof of statement (i), we may conclude that $u = 0$. Thus, statement (ii) is proved and the proof of this theorem is completed.

Finally we must prove Lemma 8.1. The proof is rather long and is divided into several steps.

LEMMA 8.2. *For $\varepsilon$ small enough, $w_p(\varepsilon) = w_p(t; \varepsilon) \in L_{2,0}^{(\ell)}(R_y^m)$ for all $t$ and* $\liminf_{t\to-\infty} \|w_p(t; \varepsilon)\| = 0$.

*Proof.* All the estimates in Lemma 6.4 are still valid for $w_p(\varepsilon)$ by a limit procedure. The first assertion follows from (b) in Lemma 6.4 by use of the trace theory and the second one follows from (ii) in Lemma 6.4 at once.

Thus, to complete the proof of Lemma 8.1, we have only to show that $\liminf_{t\to\infty} \|w_p(t; \varepsilon)\| = 0$. Here we introduce new notations; $(\ ,\ )_j$, $j = 0, \pm 1, \cdots, \pm\rho$, denotes the scalar product in $L_{2,0}^{(d_j)}$ and $(\ ,\ )$ the scalar product in $L_{2,0}^{(N)}$.

LEMMA 8.3. *Let $F_p(\varepsilon)$ be as in equation* (8.13) (*defined by* (8.11)). *If $\varepsilon$ is small enough, then*

$$\lim_{t\to\infty} \|w_p(t; \varepsilon)\|^2 = -2\,\mathrm{Im}\,(F_p(\varepsilon), w_p(\varepsilon))_p + O(\varepsilon)\ .$$

*Proof.* We simply write $w_p$ and $F_p$ instead of $w_p(\varepsilon)$ and $F_p(\varepsilon)$, respectively. We take the scalar product $\langle\ ,\ \rangle$ in $L_{2,0}^{(\ell)}(R_y^m)$ between equation (8.13) and $w_p$ and then real part;

$$\begin{aligned} \tfrac{1}{2}\partial_t\langle w_p, w_p\rangle = {} & \mathrm{Im}\,\langle A(t, y, D_y; \lambda, \varepsilon)w_p, w_p\rangle \\ & - \mathrm{Im}\,\langle F_p, w_p\rangle + \mathrm{Re} < r_p(\varepsilon), w_p\rangle\ . \end{aligned} \tag{8.14}$$

Here we recall the definition of the symbol $A(t, y, \eta; \lambda, \varepsilon)$ given by (6.17) and note, in particular, that $A(t, y, \eta; \lambda, \varepsilon)$ is symmetric. Taking account of this fact and making use of property (c) in Lemma 6.1, we can estimate the first term on the right side of (8.14) as $\int \mathrm{Im}\,\langle A(t, y, D_y; \lambda, \varepsilon)w_p, w_p\rangle dt = O(\varepsilon)\,|u|_{0,-\alpha}^2$. Using this fact and Lemma 8.2, we integrate (8.14) with respect to $t$ over $(-\infty, T)$ and then let $T \to \infty$. Then, we have

$$\lim_{t\to\infty} \|w_p(t)\|^2 = -2\,\mathrm{Im}\,(F_p, w_p)_p + 2\,\mathrm{Re}\,(r_p(\varepsilon), w_p)_p + O(\varepsilon)\,.$$

According to (4.3), $|(r_p(\varepsilon), w_p)_p| \leqq |r_p(\varepsilon)|_{0,\alpha}^{(\ell)}\, |w_p|_{0,-\alpha}^{(\ell)} = O(\varepsilon)$. Thus, the desired result is obtained.

LEMMA 8.4. *Let $v_p = v_p(\varepsilon)$ and $h_p = h_p(\varepsilon)$ be defined by* (8.9) *and* (8.12), *respectively. Then,* $(F_p(\varepsilon), w_p(\varepsilon))_p = (h_p(\varepsilon), v_p(\varepsilon))_p + O(\varepsilon)$.

*Proof.* First, recall the expression (4.4) for $U(x, \xi; \lambda, \varepsilon)$ and the definition (8.8) of $f(\varepsilon)$ and note that $\varepsilon^{-1}\chi(\xi)U_1(x, \xi; \lambda, \varepsilon)$ belongs to $A_{\theta,\sigma}(N)$ uniformly in $\varepsilon$. Then, it follows from the relation $\chi(\xi)\psi(\xi) = \psi(\xi)$ that $\chi(D_x)h_p(\varepsilon) = h_p(\varepsilon) + r_p(\varepsilon)$. Making use of this fact and the relation (6.13), we obtain

$$(F_p(\varepsilon), w_p(\varepsilon))_p = (s(D_z; \lambda)c(D_z; \lambda)h_p(\varepsilon), w_p(\varepsilon))_p + O(\varepsilon)\,.$$

Since $s(\zeta; \lambda)c(\zeta; \lambda) = \chi(\zeta)^2$ by definition, the desired result follows at once.

We combine Lemmas 8.3 and 8.4 to obtain

$$\mathrm{Im}\,(h_p(\varepsilon), w_p(\varepsilon))_p \leqq \varepsilon C\,. \tag{8.15}$$

We assert that

$$\mathrm{Im}\,(h_p(\varepsilon), v_p(\varepsilon))_p = O(\varepsilon)\,. \tag{8.16}$$

To prove this, we shall prepare two lemmas.

LEMMA 8.5. *Let $\Phi = \{\phi_k\}$ and $\Psi_j = \{\psi_{jk}\}$ be the partition of unity introduced by* (5.1). *Let $v = v(\varepsilon) = {}^t(v_1, \cdots, v_p, \cdots, v_{-\rho})$ be defined by* (8.9) *and let $h(\varepsilon) = {}^t(h_1, \cdots, h_p, \cdots, h_{-\rho})$ be defined by* (8.12). *Then,*

$$\mathrm{Im}\,(\phi_k(D_x)f(\varepsilon), \phi_k(D_x)u) = O(\varepsilon)\,, \qquad k = 0, 1, \cdots, K\,, \tag{8.17}$$

$$\mathrm{Im}\,(h_j(\varepsilon), v_j(\varepsilon))_j = O(\varepsilon) \qquad \textit{for } j \neq p\,. \tag{8.18}$$

*Proof.* $\phi_k(D_x)u$ is expressed as (5.3) with $\kappa = 0$ and $f = f(\varepsilon)$; $\phi_k(D_x)u = P_k(x, D_x; \lambda, \varepsilon)f(\varepsilon) + r(\varepsilon)$, where $P_k(x, D_x; \lambda, \varepsilon)$ is the pseudo-differential operator with symbol defined by (5.2) with $\kappa = 0$. Here it should be noted that the symbol $P_k(x, \xi; \lambda, \varepsilon)$ is symmetric. Hence, the symbol $\sigma(P_k^*)(x, \xi; \lambda, \varepsilon)$ of the adjoint operator $P_k^*(x, D_x; \lambda, \varepsilon)$ is represented as $\sigma(P_k^*)(x, \xi; \lambda, \varepsilon) = P_k(x, \xi; \lambda, \varepsilon) + R_k(x, \xi; \lambda, \varepsilon)$ with some $R_k(x, \xi; \lambda, \varepsilon) \in A_{\sigma,\sigma}(N)$ for which $\varepsilon^{-1}R_k(x, \xi; \lambda, \varepsilon)$ belongs to $A_{\sigma,\sigma}(N)$ uniformly in $\varepsilon$. Hence, by use of this fact, (8.17) is easily verified. For the proof of (8.18), we use the relation

(5.11) with $\kappa = 0$ and $h_j = h_j(\varepsilon)$. Then, by the same argument as above, we can prove (8.18).

Let $U_{jk}(x, D_x; \lambda, \varepsilon)$ be the pseudo-differential operator with symbol defined by $U_{jk}(x, \xi; \lambda, \varepsilon) = \chi_{jk}(\xi)U(x, \xi; \lambda, \varepsilon)$ with $U(x, \xi; \lambda, \varepsilon)$ introduced in Lemma 4.1, where $\chi_{jk}(\xi)$ is a smooth function with compact support such that $\chi_{jk}(\xi) = 1$ in a small neighborhood of the support of $\psi_{jk}(\xi)$. In particular, when $(j, k) = (p, q)$, $(p, q)$ being the fixed pair, $U_{pq}(x, D_x; \lambda, \varepsilon) = \hat{U}(x, D_x; \lambda, \varepsilon)$ as defined by (5.4).

LEMMA 8.6. *Let $U_{jk}(\varepsilon) = U_{jk}(x, D_x; \lambda, \varepsilon)$ be as above. Then,*

$$\sum_{j=1}^{\rho} \sum_{k=1}^{K_j} \operatorname{Im} (U_{jk}(\varepsilon)\psi_{jk}(D_x)f(\varepsilon), U_{jk}(\varepsilon)\psi_{jk}(D_x)u) = O(\varepsilon) . \tag{8.19}$$

*Proof.* By the definition (8.8) of $f(\varepsilon)$, $\operatorname{Im}(f(\varepsilon), u) = 0$ and

$$0 = \sum_{k=0}^{K} \operatorname{Im} (\phi_k(D_x)f(\varepsilon), \phi_k(D_x)u) + \sum_{k=1}^{\rho} \sum_{k=1}^{K_j} \operatorname{Im} (\psi_{jk}(D_x)f(\varepsilon), \psi_{jk}(D_x)u) .$$

By (8.17), the first term is of order $O(\varepsilon)$ and hence the second term is also of order $O(\varepsilon)$. Furthermore, making use of relation (4.5) in Lemma 4.1, we obtain the desired result at once.

We can now prove the assertion (8.16). Combining (8.15) with (8.18), we see that $I_{pq} = \sum_{j=-\rho}^{\rho} \operatorname{Im}(h_j(\varepsilon), v_j(\varepsilon)) = \operatorname{Im}(\hat{U}\psi(D_x)f(\varepsilon), \hat{U}\psi(D_x)u) \leqq \varepsilon C$, where $\hat{U} = U_{pq}(x, D_x; \lambda, \varepsilon)$ and $\psi = \psi_{pq}$. Since this estimate holds for any pair $(p, q)$, we have by (8.19) that $I_{pq} = O(\varepsilon)$ and hence (8.16) is proved. Thus, we obtain

$$\lim_{t\to\infty} \| w_p(t; \varepsilon)\|^2 = O(\varepsilon) . \tag{8.20}$$

Since $w_p(t; \varepsilon)$ depends on $\varepsilon$, we cannot yet conclude from (8.20) that $\lim_{t\to\infty} \| w_p(t; \varepsilon)\| = 0$. So we need to introduce some new functions not depending on $\varepsilon$. It is convenient to represent these functions in terms of the $z$-coordinates and we use the notation $L_{2,\gamma}(R_t; \mathcal{M})$, $\mathcal{M}$ being a Hilbert space, to denote the functional space of square integrable $\mathcal{M}$-valued functions with weight $(1 + t^2)^{\gamma/2}$. We first introduce $v^{(0)} = v^{(0)}(t, y)$ as

$$v^{(0)} = {}^t(v_1^{(0)}, \cdots, v_p^{(0)}, \cdots, v_{-\rho}^{(0)}) = \hat{U}_0(D_z)\psi(D_z)u \tag{8.21}$$

for $u = u(t, y)$ represented in terms of the $z$-coordinates, where $\hat{U}_0(\zeta)$ is given by $\hat{U}_0(\zeta) = \chi(\zeta)U_0(\zeta)$ with $U_0(\zeta)$ defined in (2.7). We further define $w_p^{(0)} = w_p^{(0)}(t, y)$ as

(8.22) $$w_p^{(0)} = \chi(D_z)s(D_z; \lambda)v_p^{(0)} .$$

LEMMA 8.7. *Let $v(t; \varepsilon) = v(t, y; \varepsilon)$ be the representation of $v = v(\varepsilon)$ defined by (8.9) in terms of the z-coordinates. Let $v^{(0)}(t) = v^{(0)}(t, y)$ be defined by (8.21). Then, the difference $v(t; \varepsilon) - v^{(0)}(t)$ belongs to $L_{2,\mu}(R_t; \mathscr{M})$ with $\mathscr{M} = L_{2,0}^{(N)}(R_y^m)$ for some $\mu, \mu > -\frac{1}{2}$, and hence $v_p(t; \varepsilon) - v_p^{(0)}(t)$ belongs to $L_{2,\mu}(R_t; \mathscr{L})$ with $\mathscr{L} = L_{2,0}^{(\ell)}(R_y^m)$.*

*Proof.* Since $w(\varepsilon) = w(t; \varepsilon) \in L_{2,-\nu}(R_t; \mathscr{L})$ for any $\nu, \nu > \frac{1}{2}$, we have $\psi(D_z)u \in L_{2,-\nu}(R_t; \mathscr{M})$ by the same way as $\psi(D_x)u \in L_{2,0}^{(N)}$ was obtained from $w(\varepsilon) \in L_{2,0}^{(\ell)}$. We write $v(t; \varepsilon) - v^{(0)}(t) = (\hat{U}(z, D_z; \lambda, \varepsilon) - \hat{U}_0(D_z))\psi(D_z)u$. By the expression (4.4) for $U(x, \xi; \lambda, \varepsilon)$, we know that the symbol $\hat{U}(z, \zeta; \lambda, \varepsilon) - \hat{U}_0(\zeta)$ belongs to $A_{\theta,\sigma}(N)$. Thus, there exists $\mu, \mu > -\frac{1}{2}$, for which $v(t; \varepsilon) - v^{(0)}(t) \in L_{2,\mu}(R_t; \mathscr{M})$.

LEMMA 8.8. *Let $\mathscr{L}$ be as in Lemma 8.7. Then, the difference $w_p(t; \varepsilon) - w_p^{(0)}(t)$ belongs to $L_{2,\mu}(R_t; \mathscr{L})$ for some $\mu, \mu > -\frac{1}{2}$.*

*Proof.* According to the respective definitions of $w_p(t; \varepsilon)$ and $w_p^{(0)}(t)$, we write

$$\begin{aligned} w_p(t; \varepsilon) - w_p^{(0)}(t) = {} & V_p(z, D_z; \lambda, \varepsilon)s(D_z; \lambda)(v_p - v_p^{(0)}) \\ & + (V_p(z, D_z; \lambda, \varepsilon) - \chi(D_z))s(D_z; \lambda)v_p^{(0)} . \end{aligned}$$

By Lemma 8.7, the first term belongs to $L_{2,\mu}(R_t; \mathscr{L})$ and for the second term, we recall the definition (6.11) of $V_p(z, \zeta; \lambda, \varepsilon)$. Then, we see that $V_p(z, \zeta; \lambda, \varepsilon) - \chi(\zeta) \in A_{\theta,\sigma}(\ell)$, which implies that the second term also belongs to $L_{2,\mu}(R_t; \mathscr{L})$ for some $\mu, \mu > -\frac{1}{2}$. This completes the proof.

The proof of Lemma 8.1 is completed as an immediate consequence of Lemma 8.8.

*Completion of the proof of Lemma* 8.1. We use the notation $I(u)$ to denote

$$I(u) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| u(t) \|^2 \, dt .$$

Since $\lim_{t \to \infty} \| w_p(t; \varepsilon) \|^2 = O(\varepsilon)$, it follows that

$$I(w_p(\varepsilon)) = \lim_{t \to \infty} \| w_p(t; \varepsilon) \|^2 = O(\varepsilon) .$$

On the other hand, Lemma 8.8 shows that $I(w_p(\varepsilon) - w_p^{(0)}) = 0$ and hence $I(w_p^{(0)}) = I(w_p(\varepsilon)) = O(\varepsilon)$. Since $I(w_p^{(0)})$ does not depend on $\varepsilon$, we may conclude that $I(w_p^{(0)}) = 0$ and the desired result follows at once.

## Appendix; Proof of Lemma 6.1

We shall prove Lemma 6.1. Before proving this, we introduce two symbol classes and state one proposition from which the statement of Lemma 6.1 follows immediately.

DEFINITION A.1. We say that $A(z, \zeta) = \{a_{jk}(z, \zeta)\}_{j,k=1,\ell}$, $z = (t, y)$, $\zeta = (\tau, \eta)$, belongs to $\mathscr{A}_{\nu,\mu}(\ell)$, $\mu \geqq \nu \geqq 0$, when the following conditions are satisfied:

$$|\partial_\zeta^\gamma a_{jk}(z, \zeta)| \leqq C_\gamma(1 + |z|)^{-\nu}\,,$$
$$|\partial_z^\beta \partial_\zeta^\gamma a_{jk}(z, \zeta)| \leqq C_{\beta,\gamma}(1 + |z|)^{-\mu}\,, \qquad |\beta| \geqq 1\,.$$

The class $\mathscr{A}_{\nu,\mu}(\ell)$ is different from $A_{\nu,\mu}(\ell)$ in Definition 2.1 in one respect that the decay for $|\zeta| \to \infty$ is not assumed and it is easily seen that $A_{\nu,\mu}(\ell) \subset \mathscr{A}_{\nu,\mu}(\ell)$. We introduce another symbol class, which is a subclass of $\mathscr{A}_{\nu,\mu}(\ell)$.

DEFINITION A.2. We say that $A(z, \zeta) \in \mathscr{A}_{\nu,\mu}(\ell)$ belongs to $\mathscr{B}_{\nu,\mu}(\ell)$, if $A(z, \zeta)$ does not depend on $\tau$; $A(z, \zeta) = A(t, y, \eta)$.

We say that a family of symbols with parameter $\varepsilon$ belongs to $\mathscr{A}_{\nu,\mu}(\ell)$ uniformly in $\varepsilon$, if the above constants $C_\gamma$ and $C_{\beta,\gamma}$ are independent of $\varepsilon$.

PROPOSITION A.1. *Let $\theta$, $\theta > 0$, be the constant fixed in subsection* 4.1 *and $\sigma = 1 + \theta$. Let $s(\eta)$ be a real-valued smooth function with compact support. Let $Y = Y(z, \zeta) \in \mathscr{A}_{\nu,\mu}(\ell)$ and assume that $Y$ is symmetric. Then, there exist two symmetric matrices $L = L(z, \zeta)$ and $N = N(t, y, \eta)$ such that*

$$(I + L)\{(\tau - s(\eta)) + Y\}(I + L) = (\tau - s(\eta)) + N + Z$$

*with some $Z = Z(z, \zeta) \in \mathscr{A}_{\sigma,\sigma}(\ell)$, where $I$ is the $\ell \times \ell$ identity matrix. Furthermore, $L$ and $N$ have the following properties*: (a) *$L$ is symmetric and belongs to $\mathscr{A}_{\theta,\sigma}(\ell)$*; (b) *$N$ is symmetric and belongs to $\mathscr{B}_{\theta,\sigma}(\ell)$.*

The proof uses the following simple results.

LEMMA A.1. *Let $s(\eta)$ be as in Proposition* A.1 *and $A(z, \zeta) = A(t, y, \tau, \eta) \in \mathscr{A}_{\nu,\mu}(\ell)$. Then,*

(i) $B(t, y, \eta) = A(t, y, s(\eta), \eta) \in \mathscr{B}_{\nu,\mu}(\ell)$;

(ii) $(\tau - s(\eta))^{-1}(A(z, \zeta) - A(z, s(\eta), \eta)) \in \mathscr{A}_{\nu,\mu}(\ell)$.

*Proof of Proposition* A.1. We choose an integer $J$ so large that $J\theta \geqq \sigma$ and write $L$ and $N$ formally as $L = \sum_{j=1}^J L_j$ and $N = \sum_{j=1}^J N_j$, respectively. Here we determine $L_j = L_j(z,\zeta)$ and $N_j = N_j(t, y, \eta)$ to satisfy the following equation:

$$\text{(A.1)}\qquad \begin{aligned} &N_j = (\tau - s(\eta))\Big\{2L_j + \sum_{k=1}^{j-1} L_k L_{j-k}\Big\} + M_j\,, \qquad j \geqq 2\,,\\ &N_1 = 2(\tau - s(\eta))L_1 + M_1\,, \end{aligned}$$

where

$$\text{(A.2)}\qquad \begin{aligned} &M_j = M_j(z,\zeta) = L_{j-1}Y + YL_{j-1} + \sum_{k=1}^{j-2} L_k Y L_{j-k-1}\,, \qquad j \geqq 3\,,\\ &M_1 = Y\,, \qquad M_2 = L_1 Y + Y L_1\,. \end{aligned}$$

Furthermore, we require $L_j$ and $N_j$ to have the following properties: (a) $L_j$ is symmetric and belongs to $\mathscr{A}_{j\theta,\sigma}(\ell)$; (b) $N_j$ is symmetric and belongs to $\mathscr{B}_{j\theta,\sigma}(\ell)$. If $L_j$ and $N_j$ are determined to satisfy (A.1) and if $L$ and $N$ are defined as above, it then follows that

$$(I + L)\{(\tau - s(\eta)) + Y\}(I + L) = (\tau - s(\eta)) + N + Z\,,$$

where $Z = Z(z,\zeta)$ is given by

$$\text{(A.3)}\qquad Z = (\tau - s(\eta)) \sum_{p=1}^{J} \sum_{k=p}^{J} L_k L_{J+p-k} + \sum_{p=0}^{J} \sum_{k=p}^{J} L_k Y L_{J+p-k}\,, \qquad (L_0 = I)\,.$$

First, we shall show that there exist solutions $L_j$ and $N_j$ to equation (A.1) with the required properties. We consider the case $j = 1$. In this case, we easily see that $L_1$ and $N_1$ are given by

$$\begin{aligned} &N_1 = N_1(t, y, \eta) = M_1(t, y, s(\eta), \eta) = Y(t, y, s(\eta), \eta)\,, \qquad M_1 = Y\,,\\ &L_1 = L_1(z,\zeta) = \tfrac{1}{2}(\tau - s(\eta))^{-1}(M_1(z, s(\eta), \eta) - M_1(z, \tau, \eta))\,. \end{aligned}$$

According to Lemma A.1, $N_1 \in \mathscr{B}_{\theta,\sigma}(\ell)$ and $L_1 \in \mathscr{A}_{\theta,\sigma}(\ell)$ since $Y \in \mathscr{A}_{\theta,\sigma}(\ell)$. Furthermore, since $Y$ is symmetric, so are both $L_1$ and $N_1$. Thus, we can determine $L_1$ and $N_1$ with (a) and (b). Next, we consider the case $j = 2$. We put $K_2 = 2L_2 + L_1 L_1$ (and hence $L_2 = \frac{1}{2}(K_2 - L_1 L_1)$). Then, equation (A.1) with $j = 2$ becomes $N_2 = (\tau - s(\eta))K_2 + M_2$. Since $L_1 \in \mathscr{A}_{\theta,\sigma}(\ell)$ is symmetric, it follows from the expression (A.2) for $M_2$ that $M_2$ is also symmetric and belongs to $\mathscr{A}_{2\theta,\sigma}(\ell)$. We can determine $K_2$ and $N_2$ in the same way as $L_1$ and $N_1$ are constructed. Then, we see that $K_2$ is symmetric and belongs to $\mathscr{A}_{2\theta,\sigma}(\ell)$ and hence it follows that $L_2$ also has the same property as $K_2$. Therefore, we can determine $L_2$ and $N_2$ with (a) and (b) for $j = 2$. Thus, $L_j$ and $N_j$ can be determined to satisfy (A.1) with the required properties (a) and (b) inductively.

To complete the proof, we must show that $Z=Z(z,\zeta)$ given by (A.3) belongs to $\mathscr{A}_{\theta,\sigma}(\ell)$. To see this, we use (A.1) and write

$$(\tau-s(\eta))L_j=\frac{1}{2}\left\{N_j-(\tau-s(\eta))\sum_{k=1}^{j-1}L_kL_{j-k}-M_j\right\}.$$

If we use this relation, an inductive argument shows that $(\tau-s(\eta))L_j\in\mathscr{A}_{j\theta,\sigma}(\ell)$. Hence, applying this fact to (A.3), we have the desired result and the proof is completed.

*Remark.* If $Y$ in Proposition A.1 depends on a parameter $\varepsilon$ and if $Y$ belongs to $\mathscr{A}_{\theta,\sigma}(\ell)$ uniformly in $\varepsilon$, then $L$ and $N$ also belong to the corresponding classes uniformly in $\varepsilon$, which is easily seen from the construction of $L$ and $N$.

Lemma 6.1 is an immediate consequence of Proposition A.1 and Remark after it, so we omit the proof.

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