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CONSTRUCTION OF ARITHMETIC AUTOMORPHIC FUNCTIONS FOR SPECIAL CLIFFORD GROUPS

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An important problem in the theory of arithmetic automorphic functions is to construct, for a reductive algebraic group over $\boldsymbol{Q}$ which defines a bounded symmetric domain, a system of canonical models [2], [6], [18]. For the similitude group of a hermitian form over a quaternion algebra whose center is a totally real field, this is solved by Shimura [17], and for the similitude group of a hermitian form with respect to an involution of the second kind of a central division algebra over a *CM*-field, by Miyake [8]. In this paper, we show that this also can be done for the special Clifford group of a quadratic form Q over a totally real algebraic number field. (We have to impose certain conditions on the signature of Q in order that G defines a bounded symmetric domain, see 1.1.)

That this is possible is suggested by Satake's works [11], [12]. Instead of his symplectic embeddings, we introduce in § 3 an embedding of G into a reductive group $G^{\cdot}$ of Shimura type. We then show that (§ 4) the system of canonical models constructed by Shimura for $G^{\cdot}$ gives rise to a system of canonical models for G. Here we adopt the technique employed by Shimura in [17, § 6] (see also [2, § 5]).

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Notation

We refer to [1], [3], [5] and [9] for general information concerning quadratic forms. For the definition of the Clifford algebra C of a quadratic form Q on a vector space V over a field F of characteristic $\neq 2$, see Chapter II of [1]. The subalgebra E of C consisting of all even

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elements is called the *even Clifford algebra*. By the *main involution* ι on E, we mean the one induced by the identity mapping on V. This is called the main anti-automorphism in Chevalley's book. Let M be the matrix of Q with respect to a basis of V. We call $(-1)^{n(n-1)/2}\det M$ a *signed discriminant* of Q, where $n = \dim V$. All signed discriminants of Q form a square class in $F^\times$, the multiplicative group of F.

For a number field F, $F_A^\times$ denotes the idele group of F, and F_{ab} the abelian closure of F. For $c \in F_A^\times$, let $[c, F]$ be the image of c in $\mathrm{Gal}(F_{ab}/F)$ under the Artin map. We use $F_\infty^\times$ and $F_0^\times$ to denote the infinite and finite part of $F_A^\times$ respectively. The identity component of $F_\infty^\times$ is denoted by $F_{\infty+}^\times$, and the closure of $F^\times F_{\infty+}^\times$ in $F_A^\times$ is denoted by F_c.

For an algebraic group G over $\boldsymbol{Q}$, G_A denotes the adelization of G. We use $G_\infty\,(=G_{\boldsymbol{R}})$, G_0 to denote the infinite and finite part of G_A respectively. The identity component of G_∞ is denoted by $G_{\infty+}$.

1. Preliminaries

The purpose of this section is to introduce the notions those are needed in the subsequent discussions.

1.1. Let F be a totally real algebraic number field of degree g, V a $(p+2)$-dimensional vector space over F, where $p \geq 1$, and Q a non-degenerate quadratic form on V. Denote by E the even Clifford algebra of Q and ι the main involution on E (see Notation). Define an algebraic group G over $\boldsymbol{Q}$ whose $\boldsymbol{Q}$-rational points are

$$G_{\boldsymbol{Q}} = \{g \in E^\times \mid gVg^{-1} = V\}\,.$$

In Chevalley's terminology [1], $G_{\boldsymbol{Q}}$ is the *special Clifford group* of Q. For $g \in G_{\boldsymbol{Q}}$ put $\nu(g) = gg^\iota$. Then $\nu(g) \in F^\times$, see [1, II.3.5]. The semi-simple part of G is

$$G^u = \{g \in G \mid \nu(g) = 1\}\,,$$

which is simply connected. The $\boldsymbol{Q}$-rational points of G^u form the *spin group* (or the "reduced Clifford group" in Chevalley's terminology) of Q over F.

Let $\tau_1, \cdots, \tau_g$ be the g distinct embeddings of F into $\boldsymbol{R}$. Denote the completion of F at τ_ν by F_ν, $V_\nu = V \otimes_F F_\nu$, and Q_ν the extension of Q to V_ν. We assume the signature of Q_ν is either $(p, 2)$ or $(p+2, 0)$, so that the quotient of $G_{\boldsymbol{R}}^u$ modulo a maximal compact subgroup has the structure of a bounded symmetric domain. By rearranging the τ_ν's, we shall assume that the signature of Q_ν is $(p, 2)$ when $\nu \leq r$ and $(p+2, 0)$ otherwise. We exclude the case $r = 0$, i.e. the case where G_R^u is a compact group, from our consideration. By [9, 101: 8], the image of G_Q under ν is the set of all $x \in F^\times$ which is positive at $\tau_{r+1}, \cdots, \tau_g$.

1.2. Throughout this subsection, let V be a $(p+2)$-dimensional vector space over $\boldsymbol{R}$, and Q a quadratic form of signature $(p, 2)$ on V. Take an orthogonal basis $e_1, e_2, \cdots, e_{p+2}$ of V so that

(1.2.1) $$Q(e_\nu) = \begin{cases} 1 & \text{if } \nu = 1, \cdots, p \\ -1 & \text{if } \nu = p+1,\ p+2\,. \end{cases}$$

A basis of the even Clifford algebra E of Q is given by

$$e_{\nu_1} e_{\nu_2} \cdots e_{\nu_{2k}} \qquad \left(\nu_1 < \nu_2 < \cdots < \nu_{2k},\ k = 0, 1, \cdots, \left[\frac{p}{2}\right] + 1\right).$$

Let Gpin (Q) (resp. Spin (Q)) be the special Clifford group (resp. spin group) of Q over $\boldsymbol{R}$. Put $j = e_{p+1} e_{p+2} \in E$, and let

$$K = \{g \in \mathrm{Spin}\,(Q) | gj = jg\}\,.$$

Then K is a maximal compact subgroup of Spin (Q). Furthermore, every maximal compact subgroup of Spin (Q) is obtained this way. Now fix an orthogonal basis $e_1, e_2, \cdots, e_{p+2}$ of V satisfying (1.2.1) and let K be the corresponding maximal compact subgroup of Spin (Q). It is possible to introduce two complex structures on the quotient Spin $(Q)/K$. We fix one as follows.

Let $\mathfrak{g}$ be the linear span of $\{e_{\nu_1} e_{\nu_2} | \nu_1 < \nu_2\}$ in E. For $x, y \in \mathfrak{g}$, $[x, y] = xy - yx \in \mathfrak{g}$. Therefore, with this bracket operation $\mathfrak{g}$ becomes a Lie algebra. This is the Lie algebra of Spin (Q), see [1, 2.9]. Let $\mathfrak{k}$ be the linear span of $\{e_{p+1} e_{p+2}\} \cup \{e_{\nu_1} e_{\nu_2} | \nu_1 < \nu_2 \leq p\}$, and $\mathfrak{p}$ the linear span of $\{e_\nu e_{p+1} | \nu \leq p\} \cup \{e_\nu e_{p+2} | \nu \leq p\}$. Then

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

is the Cartan decomposition of $\mathfrak{g}$ corresponding to the maximal compact subgroup K. Now $j = e_{p+1} e_{p+2}$ is in the center of $\mathfrak{k}$, and the restriction J of $\frac{1}{2}\,\mathrm{ad}\,(j)$ to $\mathfrak{p}$ is a linear transformation with $J^2 = -\mathrm{id}$. Identifying the tangent space of Spin $(Q)/K$ at K with $\mathfrak{p}$, we use J to define a complex structure on $\mathrm{Spin}(Q)/K$. (Another structure is given by $-J$.) The complex manifold $\mathrm{Spin}(Q)/K$ can be realized as a bounded domain X_p in $\boldsymbol{C}^p$:

$$X_p = \{(z_1, \cdots, z_p) \in \boldsymbol{C}^p \,|\, \textstyle\sum_{\nu=1}^{p} |z_\nu|^2 < \frac{1}{2}(1 + |\sum_{\nu=1}^{p} z_\nu^2|^2) < 1\}\,,$$

see for example [10, 3.5].

Let

$$\mathrm{Gpin}^+(Q) = \{g \in \mathrm{Gpin}(Q) \,|\, \nu(g) > 0\}$$

be the identity component of $\mathrm{Gpin}(Q)$. For $g \in \mathrm{Gpin}^+(Q)$, define the action of g on $X_p = \mathrm{Spin}(Q)/K$ to be that of $(\nu(g))^{-1/2} g \in \mathrm{Spin}(Q)$.

1.3. Let z be a point of X_p. Then there is an orthogonal basis $e_1, e_2, \cdots, e_{p+2}$ of V satisfying (1.2.1) so that z corresponds to the maximal compact subgroup

$$K_z = \{g \in \mathrm{Spin}(Q) \,|\, gj = jg\}\,,$$

where $j = e_{p+1}e_{p+2}$, and so that j (instead of $-j$) determines the given complex structure of X_p. This element j of E is uniquely determined by these properties. We shall refer to it as the *complex structure of* X_p *at* z. We have $j^\iota = -j$ and $j^2 = -1$.

The $\boldsymbol{R}$-linear span of K_z in E is

$$Y_z = \{x \in E \,|\, xj = jx\}\,.$$

By [11, Proposition 2], ι induces a positive involution on Y_z. It is obvious that $\boldsymbol{R}[j]^\times$ is contained in $\mathrm{Gpin}^+(Q)$, hence in the center of Y_z. Also it can be verified in a straightforward way that z is the only fixed point of $\boldsymbol{R}[j]^\times$ on X_p.

1.4. Let $V, Q, E, G, G^u, V_\nu, Q_\nu$ etc. be as in 1.1. Denote the completion of E, G and G^u at τ_ν by E_ν, G_ν and G_ν^u respectively. For $\nu > r$, the signature of Q_ν is $(p+2, 0)$ and G_ν^u is compact. For $\nu \leq r$, the signature of Q_ν is $(p, 2)$ and $G_\nu \cong \mathrm{Gpin}(Q_\nu)$, $G_\nu^u \cong \mathrm{Spin}(Q_\nu)$. For each $\nu \leq r$, we fix once and for all an orthogonal basis of V_ν with respect to Q_ν so that (1.2.1) holds for Q_ν. Such a (ordered) basis determines uniquely a maximal compact subgroup K_ν of G_ν^u and a complex structure on G_ν^u/K_ν as described in 1.2.

We have an isomorphism

(1.4.1) $$G_R \cong \prod_{\nu=1}^{g} G_\nu \,.$$

Let K be the maximal compact subgroup of G_R^u corresponding to $\prod_{\nu=1}^{r} K_\nu \times \prod_{\nu=r+1}^{g} G_\nu^u$ under the above isomorphism. We then fix a complex structure on G_R^u/K via the homeomorphism

$$G_R^u/K \cong \prod_{\nu=1}^{r} G_\nu^u/K_\nu$$

induced by (1.4.1). We denote the bounded symmetric domain G_R^u/K by X. This domain is equivalent to the product of r copies of X_p.

The identity component of G_R is

$$G_R^+ = \{g \in G_R | \nu(g) \text{ is totally positive}\} ,$$

which is isomorphic to $\prod_{\nu=1}^{r} \mathrm{Gpin}^+(Q_\nu) \times \prod_{\nu=r+1}^{g} G_\nu$ under (1.4.1). We define the action of G_R^+ on $X \cong X_p^r$ component-wise.

1.5. Let Θ be a representation of F equivalent to $\sum_{\nu=1}^{r} \tau_\nu$. Define the reflex (F', Θ') of (F, Θ) as in [17I, 1.1]. Put $\lambda = \det \Theta'$. Then λ is a homomorphism of $F'^\times$ to $F^\times$. Extend λ to a homomorphism of $F_A'^\times$ to $F_A^\times$, still denoted by λ. Denote by λ^* the composite of $\lambda: F_A'^\times \to \lambda(F_A'^\times)F_c$ with the natural mapping $\lambda(F_A'^\times)F_c \to \lambda(F_A'^\times)F_c/F_c$. Then λ^* is a surjective continuous open homomorphism [17 II, Lemma 2.5]. Denote by $\mathfrak{k}^*$ the infinite abelian extension of F' corresponding to the kernel of λ^*. Then

(1.5.1) $$\mathrm{Gal}(\mathfrak{k}^*/F') \cong \lambda(F_A'^\times)F_c/F_c = \lambda^*(F_A'^\times) \,.$$

Let $\nu^*: G_A \to F_A^\times/F_c$ be the composite of $\nu: G_A \to F_A^\times$ with the natural homomorphism $F_A^\times \to F_A^\times/F_c$. We put

$$\mathscr{G}_+ = \{g \in G_{A+} | \nu^*(g) \in \lambda^*(F_A'^\times)\} \,.$$

For $g \in \mathscr{G}_+$, define $\rho(g)$ to be the element of $\mathrm{Gal}(\mathfrak{k}^*/F')$ corresponding to $\nu^*(g^{-1}) \in \lambda^*(F_A'^\times)$ under the isomorphism (1.5.1). Then ρ is a continuous homomorphism of $\mathscr{G}_+$ to $\mathrm{Gal}(\mathfrak{k}^*/F')$. We shall see that ρ is surjective and open (Proposition 7).

1.6. For $z \in X$, put

$$G_z = \{\alpha \in G_{Q+} | \alpha(z) = z\}$$

and let Y be the F-linear span of G_z in E. Identify X with r copies of X_p, and let $z_1, \cdots, z_r$ be the components of z. For each $\nu \le r$, let $j_\nu \in E_\nu$ be the complex structure of X_p at z_ν, see 1.3. Then $Y_R = Y \otimes_Q \boldsymbol{R}$ can be identified with an R-subalgebra of $Y_{z_1} \oplus \cdots \oplus Y_{z_r}$, where

$$Y_{z_\nu} = \{x \in E_\nu | xj_\nu = j_\nu x\} .$$

Hence $Y \cap G_{Q^+}$ fixes z. Therefore $G_z = Y \cap G_{Q^+}$.

Consider the centralizer H_z of G_z in G_{Q^+}. First note that for $\beta \in H_z$, $\beta(z)$ is fixed by G_z. Therefore, if z is the only fixed point of G_z, then $H_z \subset G_z$. On the other hand, since $R[j_1]^\times \times \cdots \times R[j_r]^\times \subset H_{zR}$, z is the only fixed point of H_z. (See the remark at the end of § 1.3.) Hence, if $H_z \subset G_z$, then z is the only fixed point of G_z. This shows:

PROPOSITION 1. *Let the notation be as above. Then z is the only fixed point of G_z if and only if G_z contains its centralizer H_z. When this is the case, z is the only fixed point of H_z.*

We call z an *isolated fixed point* of G_{Q^+} on X if it is the only fixed point of G_z.

1.7. Assume z is an isolated fixed point of G_z. Let P be the F-linear span of H_z. Then $H_z = P \cap G_{Q^+}$. Obviously P is contained in Y, and contains the center of E. Now P is semi-simple because it has a positive involution. Write $P = P_1 \oplus \cdots \oplus P_t$ with algebraic number fields $P_1, \cdots, P_t$. Then each P_k is either a totally real field or a CM-field. Since P_R contains $j_1, \cdots, j_r$ ($r > 0$), we see that every P_k is a CM-field.

1.8. Fix $\nu \leq r$. We introduce a complex structure on the real vector space E_ν by defining $\sqrt{-1}x$ to be $j_\nu x$ for $x \in E_\nu$. Since every element of Y commutes with j_ν, the left multiplication on E_ν by Y defines a 2^p-dimensional complex representation Ψ_ν of Y. The restriction of Ψ_ν to P_k together with its complex conjugation contains all the embeddings of P_k into C extending τ_ν with the same multiplicity. Actually, we can use j_ν to define a complex structure on P_R. Then modulo a zero representation, the restriction of Ψ_ν to P_k is equivalent to a multiple of the representation $\Psi_{k\nu}$ of P_k in the complex vector space P_R. Put $m_k = [P_k : F]/2$. Then it is easy to see that there are embeddings $\chi_{k\nu}^{(i)}$, $i = 1, \cdots, m_k$, of P_k into C so that $\{\chi_{k\nu}^{(i)}, \bar{\chi}_{k\nu}^{(i)} | i = 1, \cdots, m_k\}$ coincides with the set of all embeddings of P_k into C extending τ_ν, and

$$\Psi_{k\nu} \sim \textstyle\sum_{i=1}^{m_k} \chi_{k\nu}^{(i)} + \text{(zero representation)} .$$

Now let Φ_k be a representation of P_k equivalent to

(1.8.1) $$\sum_{\nu=1}^{r} \sum_{i=1}^{m_k} \chi_{k\nu}^{(i)} .$$

Let (P'_k, Φ'_k) be the reflex of (P_k, Φ_k) in the sense of Shimura [17I, 1.1]. Then each P'_k contains F'. Denote by P' the composite of $P'_1, \cdots, P'_k$. We define a homomorphism $\eta: P'^{\times} \to P^{\times}$ by

$$\eta(v) = (\Phi'_1(N_{P'/P'_1}(v)), \cdots, \Phi'_t(N_{P'/P'_t}(v))) \qquad (v \in P'^{\times}) .$$

It can be shown that η is a $\boldsymbol{Q}$-homomorphism of $P'^{\times}$ into $H_z \subset G_{\boldsymbol{Q}+}$ [2, 3.9]. Furthermore, by [16, (4.10.4)], we have

(1.8.2) $$\nu(\eta(v)) = \lambda(N_{P'/F'}(v)) \qquad (v \in P'^{\times}) .$$

Therefore $\eta(P'^{\times}_A) \subset \bar{\mathscr{G}}_+$.

1.9. Let V_+ be a p-dimensional F-linear subspace of V so that the restriction of Q to V_+ is positive definite at every infinite places. Denote by V_- the orthogonal complement of V_+. Then Q restricted to V_- is negative definite at $\tau_1, \cdots, \tau_r$ and positive definite at $\tau_{r+1}, \cdots, \tau_g$. The orthogonal decomposition $V = V_+ \perp V_-$ determines uniquely a point z of X (see 1.2). Take an orthogonal basis $\{e_{p+1}, e_{p+2}\}$ of V_- and put $e = e_{p+1}e_{p+2} \in E$. Then e^2 is a totally negative number in F. With the notation of 1.6 and 1.7, we have

$$Y = \{\alpha \in E \mid \alpha \text{ commutes with } e\}$$

and

$$P = Z[e] ,$$

where Z is the center of E. The structure of Z is well-known, see for example [3, Satz 4.1].

Let $K = F[e]$. Then we can identify K with the even Clifford algebra of the restriction of Q to V_-. Note that K is a totally imaginary quadratic extension of F. Let $\delta \in F^{\times}$ be a signed discriminant of Q (see Notation). Then from the structure of Z, we derive the following

PROPOSITION 2. *Let the notation be as above.*

(i) *If p is odd, then $P \cong K$.*

(ii) *If p is even, and δ is not a square in K, then $P \cong K[\sqrt{\delta}]$.*

(iii) *If p is even, and δ is a square in K, then $P \cong K \oplus K$.*

Let $j_\nu \in E_\nu$, $\nu = 1, 2, \cdots, r$, be the complex structures determined by z. Then j_ν belongs to the completion K_ν of K at τ_ν. Use j_ν to define a complex structure on K_ν. The multiplication by K on K_ν from the left gives rise to an embedding σ_ν of K into C extending τ_ν. Let Φ be a representation of K equivalent to $\sum_{\nu=1}^{r}\sigma_\nu$. Denote by P' the field determined by the isolated fixed point z as in 1.8.

PROPOSITION 3. *Let (K', Φ') be the reflex of (K, Φ). Then K' coincides with P'.*

This can be proved case by case according to the classification given in Proposition 2.

1.10. Assume $p = 1$. In this case E is a quaternion algebra over F which is indefinite at $\tau_1, \cdots, \tau_r$ and definite at $\tau_{r+1}, \cdots, \tau_g$, see [9, 57: 9]. The involution ι coincides with the main involution of the quaternion algebra E, and

$$G_Q = \{\alpha \in E^\times | \alpha\alpha^\iota \in F^\times\} ,$$

see [3, 5.2]. So G belongs to the type of groups investigated by Shimura in [14], [17]. The symmetric domain X can be identified with r copies of the upper half plane $\mathfrak{H} = \{z = x + iy \in C | y > 0\}$.

A decomposition $V = V_+ \perp V_-$ with a totally positive line V_+ determines an isolated fixed point of G_{Q+} on X. Conversely, every isolated fixed point comes from such a decomposition. In fact, given an isolated fixed point z, there is a totally imaginary quadratic extension K of F and a F-linear embedding f of K into E so that $G_z = f(K^\times)$ [14, 2.6]. But K is embeddable in E if and only if $E \otimes_F K$ is isomorphic to $M_2(K)$, [14, 2.3], i.e., if and only if Q becomes isotropic over K. And this is the case if and only if there is a F-rational decomposition $V = V_+ \perp V_-$ so that K is isomorphic to the even Clifford algebra of the restriction of Q to V_-. This follows from [5, Lemma 3.1] if Q is anisotropic over F. For Q is isotropic, this can be proved in a straightforward way.

1.11. Now come back to the general case where $p \geq 1$.

PROPOSITION 4. *Given any algebraic extension R of F', there is an isolated fixed point $z \in X$ so that P', the field associated with z, is linearly disjoint with R over F'.*

This can be proved in a general fashion [2, Théorème 5.1]. Here we reduce the proposition to a corresponding assertion for Shimura's groups [16, 7.5]. We start with a 3-dimensional F-linear subspace W of V so that the restriction Q' of Q to W has signature $(1, 2)$ at $\tau_1, \cdots, \tau_r$ and $(3, 0)$ at $\tau_{r+1}, \cdots, \tau_g$. Let G' be the spin group of Q'. Then there is a natural embedding i of G' into G rational over $\boldsymbol{Q}$. Let X' be the quotient of $G_R'^u$ modulo a maximal compact subgroup. We can give X' a complex structure in such a way that i induces a holomorphic embedding of X' into X. By [16, 7.5] there is an isolated fixed point z' of G'_{Q+} on X' so that the reflex field K' associated with z' is linearly disjoint with R over F'. From the discussion of 1.10, z' corresponds to a decomposition $W = W_+ \perp W_-$. Let $V_- = W_-$ and $V_+ = W_+ \perp (W)^\perp$. The decomposition $V = V_+ \perp V_-$ determines an isolated fixed point z of G_{Q+} on X. In view of our choice of the complex structure on X' and Proposition 3, we see that the field P' determined by z coincides with the field K' above.

2. Main Theorem

2.1. Let ρ be the homomorphism of $\overline{\mathscr{G}}_+$ to $\mathrm{Gal}(\mathfrak{k}_*/F')$ defined in 1.4. Denote by G_{c+} the kernel of ρ. Since the strong approximation theorem holds for G^u [4], the argument of [17II], §§ 3.2, 3.4, can be used to prove the following propositions.

PROPOSITION 5. *We have*

$$G_{c+} = F_c G_{Q+} G_A^u = \text{the closure of } G_{Q+} G_{R+} \text{ in } G_{A+}.$$

PROPOSITION 6. *Let* $D_+ = \{x \in G_{A+} | \nu(x) \in \lambda(F_A'^\times)\}$.

Then

$$\overline{\mathscr{G}}_+ = G_{c+} D_+ = F_c G_{Q+} D_+ = F_c D_+ G_{Q+} .$$

2.2.

PROPOSITION 7. *The homomorphism* $\rho: \overline{\mathscr{G}}_+ \to \mathrm{Gal}(\mathfrak{k}^*/F')$ *is a surjective open mapping. Especially* ρ *induces an isomorphism of* $\overline{\mathscr{G}}_+/G_{c+}$ *onto* $\mathrm{Gal}(\mathfrak{k}^*/F')$.

Proof. We follow Miyake's argument [8, Prop. 15]. Take an isolated fixed point z. Let P, P' and $\eta: P_A'^\times \to \overline{\mathscr{G}}_+$ be as in 1.7, 1.8. To show that ρ is open, it suffices to show that the restriction of ρ to $\eta(P_A'^\times)$ is open. From (1.8.2) we have

$$\nu^*(\eta(v)) = \lambda^*(N_{P'/F'}(v)) \qquad (v \in P_A'^\times) . \tag{2.2.1}$$

Since η is continuous, and both λ^* and $N_{P'/F'}$ are open, this shows the restriction of ν^* to $\eta(P_A'^\times)$ is an open mapping from $\eta(P_A'^\times)$ to $\lambda^*(F_A'^\times) \cong \mathrm{Gal}(\mathfrak{k}^*/F')$. Hence ρ is open.

To show that ρ is surjective, take another isolated fixed point w so that the reflex field Q' associated with it is linearly disjoint with P' over F'. This is possible in view of Proposition 4. Let $\xi: Q_A'^\times \to \bar{\mathscr{G}}_+$ be the homomorphism determined by w. Then

$$\nu^*(\xi(v)) = \lambda^*(N_{Q'/F'}(v)) \qquad (v \in D_A'^\times) .$$

Together with (2.2.1), this shows $\nu^*(\eta(P_A'^\times)\cdot\xi(Q_A'^\times))$ contains $\lambda^*(N_{P'/F'}(P_A'^\times) \cdot N_{Q'/F'}(Q_A'^\times))$, which is $\lambda^*(F_A'^\times)$ because P' and Q' are linearly disjoint over F'.

2.3. Let $\mathscr{Z}^*$ be the set of all the subgroups S of $\bar{\mathscr{G}}_+$ containing F_cG_{R+} such that S/F_cG_{R+} is open and compact in $\bar{\mathscr{G}}_+/F_cG_{R+}$. For $S \in \mathscr{Z}^*$, $\rho(S)$ is open in $\mathrm{Gal}(\mathfrak{k}^*/F')$ in view of Proposition 7. We denote by k_S the finite abelian extension of F' corresponding to $\rho(S)$. Put $\Gamma_S = S \cap G_{Q+}$. Then Γ_S acts on X discontinuously and $\Gamma_S\backslash X$ has finite volume. Recall that a *model* (V, φ) of $\Gamma_S\backslash X$ consists of a Zariski open subset V of an absolutely irreducible projective variety, and a Γ_S-invariant holomorphic map φ of X into V which induces a biregular isomorphism of $\Gamma_S\backslash X$ to V [17I, 0.6].

2.4.

MAIN THEOREM. *There exists a system*

$$\{V_S, \varphi_S, J_{TS}(x), (S, T \in \mathscr{Z}^*, x \in \bar{\mathscr{G}}_+)\}$$

formed by the objects satisfying the following conditions:

(2.4.1) *For each $S \in \mathscr{Z}^*$, (V_S, φ_S) is a model of $\Gamma_S\backslash X$.*

(2.4.2) *V_S is defined over k_S.*

(2.4.3) *$J_{TS}(x)$, defined if and only if $xSx^{-1} \subset T$, is a morphism of V_S onto $V_T^{\rho(x)}$ rational over k_S, and has the following properties*:

(2.4.3$_a$) *$J_{SS}(x)$ is the identity map if $x \in S$;*

(2.4.3$_b$) *$J_{TS}(x)^{\rho(y)} \circ J_{SR}(y) = J_{TR}(xy)$;*

(2.4.3$_c$) *$J_{TS}(\alpha)[\varphi_S(z)] = \varphi_T(\alpha(z))$ if $\alpha \in G_{Q+}$ (and $\alpha S\alpha^{-1} \subset T$).*

(2.4.4) *Let z be an isolated fixed point of G_{Q+} on X, and let P' and η be as in* 1.8. *Then for every $S \in \mathscr{Z}^*$, the point $\varphi_S(z)$ is rational over P'_{ab}. Furthermore, for every $v \in P_A'^{\times}$, one has $\varphi_T(z)^\tau = J_{TS}(\eta(v)^{-1})[\varphi_S(z)]$, where $\tau = [v, P']$ and $T = \eta(v)^{-1}S\eta(v)$.*

This system is unique in the sense that if $\{V'_S, \varphi'_S, J'_{TS}(x)\}$ is another canonical system for G, then there exists, for each $S \in \mathscr{Z}^*$, a biregular isomorphism M_S of V_S onto V'_S rational over k_S such that $\varphi'_S = M_S \circ \varphi_S$ and $M_T^{\rho(x)} \circ J_{TS}(x) = J'_{TS}(x) \circ M_S$ for any $x \in \mathscr{G}_+$ satisfying $xSx^{-1} \subset T$. See [17I, 3.9] for proof.

2.5. We let $G \subset E$ act on E from the right in the natural way. Consider E as a $\boldsymbol{Q}$-vector space. Let $\mathfrak{m}$ be a $\boldsymbol{Z}$-lattice in E. For a rational prime p, put $E_p = E \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$ and $\mathfrak{m}_p = \mathfrak{m} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$. For $x \in G_A$, we can define a $\boldsymbol{Z}$-lattice $\mathfrak{m}x$ as usual: if x_p denotes the p-component of x, then $(\mathfrak{m}x)_p = \mathfrak{m}_p x_p$. For a positive integer c, we write $x \equiv 1 \bmod_0 (\mathfrak{m}, c)$ if $\mathfrak{m}x = \mathfrak{m}$ and $\mathfrak{m}_p(x_p - 1) \subset c\mathfrak{m}_p$ for all p [17I, 0.5].

Put

$$S(\mathfrak{m}, c) = F_c \cdot \{x \in \mathscr{G}_+ | x \equiv 1 \bmod_0 (\mathfrak{m}, c)\} .$$

Then $S(\mathfrak{m}, c) \in \mathscr{Z}^*$, and every member of $\mathscr{Z}^*$ contains some $S(\mathfrak{m}, c)$. We have

$$S(\mathfrak{m}, c) \cap G_{Q+} = F^\times \cdot \{x \in G_{Q+} | \mathfrak{m}x = \mathfrak{m} \text{ and } \mathfrak{m}(x-1) \subset c\mathfrak{m}\} .$$

2.6. We can extend $\mathscr{G}_+$ to a bigger group $\mathfrak{A}$ as in [17II, § 4], [8, § 3], and investigate a larger system of canonical models for G. These discussions are rather formal, and will be skipped here.

2.7. For $S \in \mathscr{Z}^*$, let L_S be the k_S-rational function field of V_S, and put

$$\mathfrak{L}_S = \{f \circ \varphi_S | f \in L_S\} .$$

The union $\mathfrak{L}$ of $\mathfrak{L}_S$ for all $S \in \mathscr{Z}^*$ is a field containing $\mathfrak{k}^*$. We call it *the field of arithmetic automorphic functions on X* with respect to G. For $x \in \mathscr{G}_+$ and $f \in L_S$, $f^{\rho(x)}$ is a function on $V_S^{\rho(x)}$ rational over $k_S^{\rho(x)}$. Define

$$(f \circ \varphi_S)^{\tau(x)} = f^{\rho(x)} \circ J_{ST}(x) \circ \varphi_T \qquad (T = x^{-1}Sx) .$$

Then τ is a homomorphism of $\mathscr{G}_+$ into $\mathrm{Aut}(\mathfrak{L}/F')$. This fact is equivalent to (2.4.3_b). Properties (2.4.3_c) and (2.4.4) can also be translated into statements about the field $\mathfrak{K}$. For details see [17II, 6.2], [8, 4.2]. We have $\tau(x) = \rho(x)$ on $\mathfrak{k}^*$.

2.8. From the system of canonical models for G we can obtain a system of canonical models for the special orthogonal group of Q. This can be done as in [17I, 2.11]. Let G' be the algebraic group over $\boldsymbol{Q}$ so that the $\boldsymbol{Q}$-rational points of G' form the special orthogonal group of Q over F. There is a $\boldsymbol{Q}$-homomorphism φ of G to G' given by $v\varphi(g) = gvg^{-1}$ for $v \in V$. The sequence

$$1 \longrightarrow F^\times \longrightarrow G \xrightarrow{\varphi} G' \longrightarrow 1$$

is exact. The action of $G_{\boldsymbol{R}+}$ on X factors through $G'_{\boldsymbol{R}+}$, and defines a natural action of $G'_{\boldsymbol{R}+}$ on X.

Put $F_A^{\times 2} = \{a^2 | a \in F_A^\times\}$ and let $\pi: F_A^\times \to F_A^\times / F_A^{\times 2}$ be the natural homomorphism. Define $\nu': G'_A \to F_A^\times / F_A^{\times 2}$ so that $\nu' \circ \varphi = \pi \circ \nu$. For $g \in G'_{\boldsymbol{Q}}$, $\nu'(g) \in F^\times / F^{\times 2}$ is the spinor norm of g. Let $\lambda' = \pi \circ \lambda: F_A'^\times \to F_A^\times / F_A^{\times 2}$. Define

$$D'_+ = \{x \in G'_{A+} | \nu'(x) \in \lambda'(F_A'^\times)\}$$

and

$$\mathscr{G}'_+ = G'_{\boldsymbol{R}+} D'_+ G'_{\boldsymbol{Q}+} = D'_+ G'_{\boldsymbol{Q}+} .$$

Now consider the set $\mathscr{X}'$ of all subgroups S of $\mathscr{G}'_+$ satisfying the following two conditions:

(2.8.1) S contains $G'_{\boldsymbol{R}+}$ and $S/G'_{\boldsymbol{R}+}$ is compact in $\mathscr{G}'_+/G'_{\boldsymbol{R}+}$.

(2.8.2) S contains the image of some member of $\mathscr{X}^*$ under φ.

For $S \in \mathscr{X}'$, let

$$\mathfrak{X}'_S = \{c \in F_A'^\times | \lambda'(c) \in (F^\times F_A^{\times 2}/F_A^{\times 2}) \cdot \nu(S)\} .$$

By (2.8.2), $\mathfrak{X}'_S$ corresponds to a class field k'_S over F'. Let $\mathfrak{k}'$ be the composite of k'_S for all $S \in \mathscr{X}'$. Define a homomorphism

$$\rho': \mathscr{G}'_+ \to \mathrm{Gal}\,(\mathfrak{k}'/F')$$

by $\rho'(x) = [c^{-1}, F']$ on $\mathfrak{k}'$ with an element c of $F_A'^\times$ such that $\nu'(x)/\lambda'(c) \in (F^\times F_A^{\times 2}/F_A^{\times 2})$. A point z of X is an isolated fixed point of $G_{\boldsymbol{Q}+}$ if and only if it is an isolated fixed point of $G'_{\boldsymbol{Q}+}$. Let z be such a point and let P' be the reflex field associated with it (cf. 1.8). Denote by $\eta': P_A'^{\times} \to D'_+$ the composite of $\eta: P_A'^{\times} \to D_+$ with $\varphi: D_+ \to D'_+$. For $S \in \mathscr{X}'$, $\Gamma'_S = S \cap G'_{Q+}$ acts on X discontinuously, and $\Gamma'_S \backslash X$ has finite volume.

2.9.

THEOREM. *The notation being as above, there exists a system*

$$\{V'_S, \varphi'_S, J'_{TS}(x), (S, T \in \mathscr{X}'; x \in \overline{\mathscr{G}}'_+)\}$$

satisfying the conditions exactly like (2.4.1–2.4.4) *under the replacement of* $\mathscr{X}^*, \overline{\mathscr{G}}_+, G_{Q+}, V_S, \varphi_S, J_{TS}(x), \Gamma_S, \rho(x), \eta$ *by* $\mathscr{X}', \overline{\mathscr{G}}'_+, G'_{Q+}, V'_S, \varphi'_S, J'_{TS}(x), \Gamma'_S, \rho'(x), \eta'$.

2.10. Let $\mathfrak{o}$ be the ring of integers of F. Take an $\mathfrak{o}$-lattice $\mathfrak{m}$ in V. Define

$$S = \{x \in \overline{\mathscr{G}}'_+ \mid \mathfrak{m}x = \mathfrak{m}\} .$$

Then $S \in \mathscr{X}'$. Condition (2.8.1) is easy to see. To show (2.8.2), let $\mathfrak{o}_\mathfrak{m}$ be the order of E generated by $\mathfrak{m}$ [3, Satz 14.1]. Let

$$W = F_c \cdot \{g \in \overline{\mathscr{G}}_+ \mid \mathfrak{o}_\mathfrak{m} g = \mathfrak{o}_\mathfrak{m}\} .$$

This is a member of $\mathscr{X}^*$. If $g \in W$, then $\mathfrak{m}' = \mathfrak{m}\varphi(g)$ is an $\mathfrak{o}$-lattice which also generates $\mathfrak{o}_\mathfrak{m}$. In view of [3, Satz 14.2], there exists a fractional ideal $\mathfrak{a}$ of F so that $\mathfrak{m}' = \mathfrak{a}\mathfrak{m}$. But $\varphi(g)$ is an orthogonal transformation, so $\mathfrak{a} = \mathfrak{o}$. It follows that $\mathfrak{m}\varphi(g) = \mathfrak{m}$. Therefore $\varphi(W) \subset S$. This proves S is a member of $\mathscr{X}'$. Note that $\Gamma'_S = S \cap G'_{Q+}$ is the unit group of $\mathfrak{m}$.

3. A certain embedding of G

3.1. Let W be a 3-dimensional subspace of V so that the restriction Q' of Q to W has signature $(1,2)$ at $\tau_1, \cdots, \tau_r$, and signature $(3,0)$ at $\tau_{r+1}, \cdots, \tau_g$. Let B be the even Clifford algebra of Q'. Then B is a quaternion algebra which is indefinite at $\tau_1, \cdots, \tau_r$ and definite at $\tau_{r+1}, \cdots, \tau_g$. Via a natural embedding of B into E, we realize E as a left B-module. Define a symmetric bilinear form $f(x,y)$ on E by

$$f(x, y) = \operatorname{tr}_{E/F}(xy^\iota) ,$$

where $\operatorname{tr}_{E/F}$ denotes the reduced trace of E to F. By [15, 1.6], there is a unique B-valued ι-hermitian form $h(x,y)$ on E so that

$$\operatorname{tr}_{B/F} h(x, y) = f(x, y) .$$

Define an algebraic group $G^\cdot$ over $\boldsymbol{Q}$ whose $\boldsymbol{Q}$-rational points are

$$G_Q^{\cdot} = \{\alpha \in GL(E, B) \,|\, h(x\alpha, y\alpha) = \mu(\alpha)h(x, y), \mu(\alpha) \in F^{\times}\} \,.$$

Canonical models for groups of this type were constructed by Shimura [17]. The semi-simple part of $G^{\cdot}$ is

$$G^{\cdot u} = \{\alpha \in G^{\cdot} \,|\, \mu(\alpha) = 1\} \,.$$

Let $i: E \to \mathrm{End}\,(E, F)$ be the injection defined by $xi(y) = xy$ $(x, y \in E)$. Then i defines a $\boldsymbol{Q}$-rational injection of G into $G^{\cdot}$. Note that $\mu(i(g)) = \nu(g)$ for $g \in G$.

3.2. Fix $\nu \leq r$. Let $j_\nu \in E_\nu$ be the complex structure of X_p at a point z_ν. We have $j_\nu \in G_\nu^u$. Hence $j_\nu^{\cdot} = i(j_\nu)$ belongs to $G_\nu^{\cdot u}$, the completion of $G^{\cdot u}$ at τ_ν. Let $K_\nu^{\cdot}$ be the centralizer of $j_\nu^{\cdot}$ in $G_\nu^{\cdot u}$. Then $K_\nu^{\cdot}$ is a maximal compact subgroup. We fix a complex structure on $G_\nu^{\cdot u}/K_\nu^{\cdot}$ by requiring the differential of $j_\nu^{\cdot}$ on the tangent space at $K_\nu^{\cdot}$ act as the multiplication by $\sqrt{-1}$. We can identify $G_\nu^{\cdot u}/K_\nu^{\cdot}$ with Siegel's upper half space $\mathfrak{H}_n$, where $n = 2^{p-1}$. Using the isomorphism

$$G_{\boldsymbol{R}}^{\cdot u} \cong \textstyle\prod_{\nu=1}^{r} G_\nu^{\cdot u} \times (\text{compact group}) \,,$$

we introduce a complex structure on the quotient of $G_{\boldsymbol{R}}^{\cdot u}$ modulo a maximal compact subgroup. The complex manifold $\mathfrak{H}$ thus obtained can be identified with r copies of $\mathfrak{H}_n$.

By our choice of the complex structure on $\mathfrak{H}$, we see that $i: G \to G^{\cdot}$ induces a holomorphic embedding h of X into $\mathfrak{H}$.

3.3. Let $\mu^*: G_A^{\cdot} \to F_A^{\times}/F_c$ be the composite of $\mu: G_A^{\cdot} \to F_A^{\times}$ with the natural homomorphism $F_A^{\times} \to F_A^{\times}/F_c$. Put

$$\mathscr{G}_+^{\cdot} = \{\alpha \in G_{A+}^{\cdot} \,|\, \mu^*(\alpha) \in \lambda^*(F_A'^{\times})\} \,.$$

For $\alpha \in \mathscr{G}_+^{\cdot}$, define $\sigma(\alpha)$ to be the element of $\mathrm{Gal}\,(\mathfrak{k}^*/F')$ corresponding to $\mu^*(\alpha^{-1}) \in \lambda^*(F_A'^{\times})$ under (1.4.1). We see that i maps $\mathscr{G}_+$ into $\mathscr{G}_+^{\cdot}$ and $\sigma(i(g)) = \rho(g)$ for $g \in \mathscr{G}_+$.

3.4. Let $\mathscr{Z}^{\cdot *}$ be the set of all subgroups (S) of $\mathscr{G}_+^{\cdot}$ containing $F_c \cdot G_{\boldsymbol{R}+}^{\cdot}$ so that $(S)/F_c \cdot G_{\boldsymbol{R}+}^{\cdot}$ is open and compact in $\mathscr{G}_+^{\cdot}/F_c \cdot G_{\boldsymbol{R}+}^{\cdot}$. For $(S) \in \mathscr{Z}^{\cdot *}$, put $\Gamma_{(S)} = (S) \cap G_{\boldsymbol{Q}+}^{\cdot}$, and let $k_{(S)}$ be the class field over F' corresponding to the open subgroup $\sigma((S))$ of $\mathrm{Gal}\,(\mathfrak{k}^*/F')$. The main theorem of [17] states that there exists a system of canonical models $\{V_{(S)}, \varphi_{(S)}, J_{(T)(S)}(x), ((S), (T) \in \mathscr{Z}^{\cdot *}, x \in \mathscr{G}_+^{\cdot})\}$ for $G^{\cdot}$. Here $(V_{(S)}, \varphi_{(S)})$ is a model of $\Gamma_{(S)} \backslash \mathfrak{H}$, and $V_{(S)}$ is defined over $k_{(S)}$.

3.5. Let $z = (z_1, \cdots, z_r) \in X$ be an isolated fixed point of G_{Q+}. As in 1.7, denote by H_z the centralizer of $G_z = \{\alpha \in G_{Q+} | \alpha(z) = z\}$, and P the F-linear span of H_z. Let $j_\nu \in E_\nu$, $\nu = 1, \cdots, r$, be the complex structure at τ_ν. Then H_{zR} contains $(j_1, \cdots, j_r)$. Hence $h(z) \in \mathfrak{H}$ is the unique fixed point of $i(P) \cap G^{\cdot}_{Q+}$. Write P as the direct sum of CM-fields $P_1, \cdots, P_t$. Then the procedure of [16, 4.5–4.9] allows one to define a certain representation Ψ_k of P_k for each $k = 1, \cdots, t$. We see that Ψ_k is equivalent to the representation Φ_k given by (1.8.1). Therefore the field P' defined in [16, 4.9] coincides with the one defined in 1.8. Furthermore, if we let $\eta^{\cdot}: P'^{\times}_A \to \bar{\mathscr{G}}^{\cdot}_+$ be defined as in [17I, (2.4.3)], then we have

$$\eta^{\cdot}(v) = i(\eta(v)) \qquad (v \in P'^{\times}_A) . \tag{3.5.1}$$

4. Construction of models

4.1. Let $\mathfrak{m}$ be a lattice in E, and c a positive integer. Consider

$$S = S(\mathfrak{m}, c) = F_c \cdot \{\alpha \in \bar{\mathscr{G}}_+ | \alpha \equiv 1 \bmod_0 (\mathfrak{m}, c)\} . \tag{4.1.1}$$

Let $\mathscr{W}_c = \{p^{-1}Sp | p \in G_A\}$ and $\mathscr{W} = \bigcup_{c=1}^{\infty} \mathscr{W}_c$. Then $\mathscr{W} \subset \mathscr{Z}^*$. Obviously, $xTx^{-1} \in \mathscr{W}$ for every $T \in \mathscr{W}$ and $x \in \bar{\mathscr{G}}_+$, i.e., $\mathscr{W}$ is a normal subset of $\mathscr{Z}^*$ in the sense of [17I, 3.2]. Let $U = S(\mathfrak{m}, 1)$. Then every $S(\mathfrak{m}, c)$ is a normal subgroup of U. In view of [17I, Prop. 3.11], we only have to construct a weak canonical system

$$\{V_S, \varphi_S, J_{TS}(x), (S, T \in \mathscr{W}; x \in \bar{\mathscr{G}}_+)\}$$

relative to $\{\mathscr{W}, F, F'\}$ (see [17I, 3.2] for the definition). Actually, it suffices to construct a weak canonical system relative to $\{\mathscr{W}', F, F'\}$, where $\mathscr{W}'$ is the union of $\mathscr{W}_c$ with $c \geq c_0$ for some c_0.

4.2. We shall identify G with the subgroup $i(G)$ of $G^{\cdot}$, and drop the injection i from now on. Define

$$(S) = (S(\mathfrak{m}, c)) = F_c \cdot \{\alpha \in \bar{\mathscr{G}}^{\cdot}_+ | \alpha \equiv 1 \bmod_0 (\mathfrak{m}, c)\} .$$

Then $(S) \in \mathscr{Z}^{\cdot *}$. Let $S = S(\mathfrak{m}, c)$, $T = pSp^{-1}$ and $(T) = p(S)p^{-1}$, where $p \in G_A \subset G^{\cdot}_A$. Then $T \in \mathscr{Z}^*$, $(T) \in \mathscr{Z}^{\cdot *}$ and $T = (T) \cap \bar{\mathscr{G}}_+$. Note that $\nu(T) \subset \mu((T))$, hence $k_T \supseteq k_{(T)}$.

We have $\Gamma_T = \Gamma_{(T)} \cap G_{Q+}$. Therefore the holomorphic embedding $h: X \to \mathfrak{H}$ induces a rational map $h_T: \Gamma_T \backslash X \to \Gamma_{(T)} \backslash \mathfrak{H}$. For c sufficiently large (independent of p), say $c \geq c_0$, the quotient $\Gamma_T \backslash X$ and $\Gamma_{(T)} \backslash \mathfrak{H}$ are non-singular, and h_T is injective [2, Prop. 1.15]. Assume this is the case. Take the canonical model $(V_{(T)}, \varphi_{(T)})$ for $\Gamma_{(T)}\backslash\mathfrak{H}$, and let $V_T = \varphi_{(T)}(h(X))$, $\varphi_T = \varphi_{(T)} \circ h$. Then (V_T, φ_T) is a model for $\Gamma_T\backslash X$. Let $\mathscr{W}'$ be the union of $\mathscr{W}_c$ for all $c \geq c_0$.

4.3. Let $x \in \mathscr{G}_+ \subset \mathscr{G}^{\cdot}_+$, $U = x^{-1}Tx$ and $(U) = x^{-1}(T)x$. Then $J = J_{(T)(U)}(x)$ is a morphism of $V_{(U)}$ to $V_{(T)}^{\rho(x)}$ rational over $k_{(T)}$. Let k' be an arbitrary finite algebraic extension of k_T, and τ an isomorphism of k' into C so that $\tau = \rho(x)$ on k_T. Take an isolated fixed point $z \in X$ so that the field P' associated with it is linearly disjoint with k' over F' (Prop. 4). Then we can extend τ to an automorphism π of C over P'. We show that

(4.3.1) *there is* $\alpha \in G_{Q+}$ *so that* $\varphi_T(z)^\pi = J(\varphi_U(\alpha(z)))$.

We proceed as in [17I, 6.8].

By Prop. 6, there is $e \in F_c$, $\gamma \in G_{Q+}$, $x_1 \in D_+$ so that $x = ex_1\gamma$. Pick $d \in F_A'^{\times}$ so that $\lambda(d) = \nu(x_1)$. Then we have $[d^{-1}, F'] = \rho(x) = \pi$ on k_T. Take an element v of $P_A'^{\times}$ so that $\pi = [v, P']$ on P'_{ab}, and put $w = N_{P'/F'}(v) \in F_A'^{\times}$. Then from (1.8.2) we have $\nu(\eta(v)) = \lambda(w)$, where $\eta: P_A'^{\times} \to \mathscr{G}_+$ is defined as in 1.8. Note that $[w, F'] = \pi = [d^{-1}, F']$ on k_T, hence $\lambda(dw) = \nu(s)u$ with $s \in T$ and $u \in F_c$. Since $F_c = F^{\times}F_c^2$ [17II, 2.2], and T contains F_c, we can assume $u \in F^{\times}$. Then $u \in F_+^{\times}$, because $\lambda(d) = \nu(x_1)$, $\lambda(w) = \nu(\eta(v))$ and $\nu(s)$ are all positive at every infinite place. Therefore, there is $\varepsilon \in G_{Q+}$ so that $\nu(\varepsilon) = u$. Now

$$\nu(x_1^{-1}s\eta(v)^{-1}\varepsilon) = \lambda(d)^{-1}\nu(s)\lambda(w)^{-1}u = 1 .$$

By the strong approximation theorem for G^u, we can write $x_1^{-1}s\eta(v)^{-1}\varepsilon$ as $m\psi$, where $\psi \in G_Q^u$ and $m \in G_A^u \cap (x_1^{-1}Tx_1)$. Put $\alpha = \gamma^{-1}\psi\varepsilon^{-1} \in G_{Q+}$ and $t = s^{-1}x_1mx_1^{-1} \in T$. Then we have $\eta(v)^{-1} = te^{-1}x\alpha$. In view of (3.5.1) and the properties of canonical models for $G^{\cdot}$ at isolated fixed points, we have

$$\varphi_T(z)^\pi = \varphi_{(T)}(h(z))^\pi = J_{(T)(R)}(\eta(v)^{-1})(\varphi_{(R)}(h(z))) ,$$

where $(R) = \eta(v)(T)\eta(v)^{-1} = \alpha^{-1}x^{-1}(T)x\alpha = \alpha^{-1}(U)\alpha$. Now

$$\begin{aligned} J_{(T)(R)}(\eta(v)^{-1}) &= J_{(T)(R)}(te^{-1}x\alpha) \\ &= J_{(T)(U)}(x) \circ J_{(U)(R)}(\alpha) = J \circ J_{(U)(R)}(\alpha) . \end{aligned}$$

Hence

$$\begin{aligned}\varphi_T(z)^\pi &= J \circ J_{(U)(R)}(\alpha)(\varphi_{(R)}(h(z)))\\ &= J(\varphi_{(U)}(\alpha(h(z)))) = J(\varphi_{(U)}(h(\alpha(z))))\\ &= J(\varphi_U(\alpha(z))) .\end{aligned}$$

4.4. We show that V_T *is defined over* k_T. First note that if $z \in X$ is an isolated fixed point, and P' the reflex field associated with it, then $\varphi_T(z) \in V_T$ is rational over P'_{ab}. For $\beta \in G_{Q+}$, $\beta(z)$ is an isolated fixed point with the same P' as its reflex field. Hence for any $\beta \in G_{Q+}, \varphi_T(\beta(z))$ is also defined over P'_{ab}. Since $\{\varphi_T(\beta(z)) \mid \beta \in G_{Q+}\}$ is dense in V_T, this shows V_T is defined over a finite algebraic extension k_1 of k_T. Take k_1 as k' in 4.3. Let x, τ, z and π be as what they stand for in 4.3. Then we have (4.3.1). This still holds if we replace z by $\beta(z)$ for any $\beta \in G_{Q+}$. Since the points $\varphi_T(\beta(z))$ are dense in V_T, and V_T is defined over k_1, we see that

(4.4.1) $$J^{-1} \textit{ sends } V_T \textit{ into } V_U .$$

Now take x to be the identity element. Then $U = T$ and $J = \mathrm{id}$. Hence from (4.4.1) it follows that $V_T^\tau = V_T$. This being true for any isomorphism τ of k_1 into C over k_T, we conclude that V_T is defined over k_T.

4.5. We have constructed, for any $T \in \mathscr{W}'$, a model (V_T, φ_T) of $\Gamma_T \backslash X$ with V_T rational over k_T. Let $T = p^{-1}Sp \in \mathscr{W}'$, $x \in \overline{\mathscr{G}}_+$ and $U = x^{-1}Tx$. Consider the members $(T) = p^{-1}(S)p$ and $(U) = x^{-1}(T)x$ of $\mathscr{X}^*$. Then $J = J_{(T)(U)}(x)$ is a morphism of $V_{(U)}$ onto $V_{(T)}^{\rho(x)}$ rational over $k_{(T)}$. Since V_T is rational over k_T, it follows from (4.4.1) that J sends V_U onto $V_T^{\rho(x)}$. Denote the restriction of J to V_U by $J_{TU}(x)$. Then $J_{TU}(x)$ is a morphism of V_U onto $V_T^{\rho(x)}$. It is rational over k_T, because J is rational over $k_{(T)}$, a subfield of k_T.

Now it is clear that

$$\{V_U, \varphi_U, J_{TU}(x), (T, U \in \mathscr{W}'; x \in \overline{\mathscr{G}}_+)\}$$

is a weak canonical system relative to $\{\mathscr{W}', F, F'\}$. From this, as pointed out in 4.1, we can produce a system of canonical models for G using a standard procedure.

5. Remarks

Once the canonical models V_S are constructed, we can talk about some typical problems concerning them. For example, there is the problem of determining the zeta-functions of these varieties [6], [7]. Another one deals with the number of connected components of the real points on V_S [19]. We mention here a related fact about the actions of "negative elements" of G_Q [13].

Let $\alpha \in G_Q$ be such that $\nu(\alpha)$ is negative at $\tau_1, \cdots, \tau_r$. Then the element $\alpha_0 \in G_{A+}$, whose component is α at a finite place, and 1 at an infinite place, belongs to $\mathscr{G}_+$. The action of α_0 is given as follows: for $S \in \mathscr{X}^*$ and $T = \alpha S \alpha^{-1}$, we have

$$J_{ST}(\alpha_0)[\varphi_T(z)] = \bar{\varphi}_S(\alpha(\bar{z})) \qquad (z \in X) .$$

In view of our construction, this follows directly from the main theorem of [13].

Postscript. This work was completed in the spring of 1977. A different approach to the problem is given in Deligne [20]. When I learned of the work of Deligne, I decided to write up a short note [21] constructing canonical models in the sense of Deligne [2]. However, it has been suggested to me that it would be useful to have available a more explicit, down-to-earth construction of the canonical models in the sense of Shimura [17]. I hope this paper serves that end for the cases considered herein.

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