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ON AUGMENTED SCHOTTKY SPACES AND AUTOMORPHIC FORMS, I

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0. Introduction

With respect to Teichmüller spaces, many beautiful results are obtained by Teichmüller, Ahlfors, Bers, Maskit, Kra, Earle, Abikoff, and others. For example, the boundary consists of b-groups, and the augmented Teichmüller space is defined by attaching a part of the boundary to the Teichmüller space. By using the augmented Teichmüller space, a compactification of the moduli space of Riemann surfaces is accomplished (cf. Abikoff [1], Bers [2]).

On the other hand Schottky spaces and Schottky groups are studied by Akaza, Bers, Chuckrow, Marden, Maskit, Zarrow, and Sato, but results about this direction are inferior in comparison with those about Teichmüller spaces. For example, the augmented Schottky space corresponding to the augmented Teichmüller space has not yet been defined in the "natural" way. However it is reasonable that Schottky spaces have properties similar to Teichmüller spaces and have rather useful properties in some aspects. With respect to this, Bers [3] introduced the augmented Schottky space in his sense, and studied automorphic forms on the fiber spaces over the space. The augmented Schottky space in the sense of Bers means the space which consists of all Schottky groups of genus $g \geqq 2$ and all extended Schottky groups representing Riemann surfaces with only non-dividing nodes. Each point in the space are represented by the so called (λ, p, q)-method. However it seems that it is difficult to represent extended Schottky groups corresponding Riemann surfaces with dividing nodes by this method. If we do not attach such extended Schottky groups to the Schottky space, we can not define the augmented Schottky space corresponding to the augmented Teichmüller space. Then it gives rise to a problem whether or not the coordinates

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introduced by the (λ, p, q)-method are natural.

This is the first part of the paper entitled "On the augmented Schottky spaces and automorphic forms". The main objects of the first part are the following three: (1) To introduce new coordinates to the Schottky space (Theorem 1), (2) to define the augmented Schottky space corresponding to the augmented Teichmüller space, and (3) to consider relations between the augmented Schottky space and Riemann surfaces with or without nodes (Theorem 2). In the second part [8], we will treat automorphic forms on the fiber spaces over the augmented Schottky space.

In § 1 we will state definitions and give an example. In § 2 we will state some results about surface topology. In § 3 we will introduce multi-suffix which plays an important role to clarify and to simplify the later statements. In § 4 we will introduce the new coordinates to the Schottky space and in § 5 we will define the augmented Schottky space by using the new coordinates. In § 6 we will consider relations between the augmented Schottky space and Riemann surfaces with or without nodes. Last we have to note that in this paper we will only consider Riemann surfaces with at most g non-dividing nodes and at most $2g-3$ dividing nodes and extended Schottky groups representing the Riemann surfaces. For the general case, we will treat elsewhere.

The author wishes to express his deep gratitude Professor K. Oikawa for many advices and suggestions.

1. Definitions and an example

1–1. Definition of Schottky group. Let $C_1, C_1', \cdots, C_g, C_g'$ be a set of $2g, g \geqq 1$, mutually disjoint Jordan curves (we call them *defining curves*) on the Riemann sphere which complize the boundary of a $2g$-ply connected region ω. Suppose there are g Möbius transformations $A_1, \cdots, A_g$ which have the property that A_j maps C_j onto C_j' and $A_j(\omega) \cap \omega = \phi$, $1 \leqq j \leqq g$. Then g necessarily loxodromic transformations A_j generate a (marked) Schottky group $G = \langle A_1, \cdots, A_g \rangle$ of genus g with ω as a fundamental region. We call ω a *standard fundamental region for* G, and C_j and C_j' *defining curves of* A_j $(j = 1, 2, \cdots, g)$.

1–2. Definition of the Schottky space. We say two marked Schottky groups $G = \langle A_1, \cdots, A_g \rangle$ and $\hat{G} = \langle \hat{A}_1, \cdots, \hat{A}_g \rangle$ being *equivalent* if there exists a Möbius Transformation T such that $\hat{A}_j = TA_jT^{-1}$, $j = 1, 2, \cdots, g$.

The *Schottky space* of genus g, denoted by $\mathfrak{S}_g$, is the set of all equivalent classes of Schottky groups of genus $g \geqq 1$.

Let $G = \langle A_1, \cdots, A_g \rangle$ be a marked Schottky group. Let λ_j ($|\lambda_j| > 1$), p_j and q_j be the multiplier, the repelling and the attracting fixed points of A_j, respectively. We normalize G by setting $p_1 = 0$, $q_1 = \infty$ and $p_2 = 1$. Then a point in $\mathfrak{S}_g$ is identified with $\hat{\tau} = (\lambda_1, \cdots, \lambda_g; q_2, p_3, q_3, \cdots, p_g, q_g) \in C^{3g-3}$.

1–3. A remark on the coordinates of $\mathfrak{S}_g$. In the previous paper [7], we defined a Schottky space $\tilde{\mathfrak{S}}_g$ as the set of all points

$$\tilde{\tau} = (\lambda_1, p_1, q_1, \lambda_2, p_2, q_2, \cdots, \lambda_g, p_g, q_g) \in \hat{C}^{3g} = (C \cup \{\infty\})^{3g} ,$$

where, λ_j, p_j and q_j $(j = 1, 2, \cdots, g)$ are as defined in § 1–2. We define a boundary point $\tilde{\tau}_0$ of $\tilde{\mathfrak{S}}_g$ by setting $\tilde{\tau}_0 = \lim_{n\to\infty} \tilde{\tau}_n$ with $\tilde{\tau}_n \in \tilde{\mathfrak{S}}_g$ and $\tilde{\tau}_0 \notin \tilde{\mathfrak{S}}_g$. However we show by the following example that the definition of the boundary by using this coordinates is not complete.

EXAMPLE. Set

$$A_{r,n}(z) = ((1 + (r/n))z + ((2/n) + (r/n^2)))/(rz + (1 + (r/n)))$$

and $B_{r,n}(z) = (7z - 29)/(z - 4)$. Then $G_{r,n} = \langle A_{r,n}, B_{r,n} \rangle$ are Schottky groups. We have

$$\begin{aligned}\tilde{\tau}_{r,n} = (1 &+ (4r/n) + (2r^2/n^2) + \sqrt{(4r/n)(2 + (5r/n) + (4r^2/n^2) + (r^3/n^3))} ,\\ &- \sqrt{(2n + r)/rn^2}, \sqrt{(2n + r)/rn^2}, (7 + 3\sqrt{5})/2, (11 - \sqrt{5})/2, (11 + \sqrt{5})/2)) .\end{aligned}$$

We have

$$A_{r,\infty}(z) = \lim_{n\to\infty} A_{r,n}(z) = z/(rz + 1)$$

and

$$\tilde{\tau}_{r,\infty} = \lim_{n\to\infty} \tilde{\tau}_{r,n} = (1, 0, 0, *, *, *) ,$$

where $*$ denotes the same elements as in the above $\tilde{\tau}_{r,n}$.

Next let n fix and r tend to ∞. We have

$$A_{\infty,n}(z) = \lim_{r\to\infty} A_{r,n}(z) = 1/n$$

and

$$\tilde{\tau}_{\infty,n} = \lim_{n\to\infty} \tilde{\tau}_{r,n} = (\infty, -1/n, 1/n, *, *, *) .$$

Furthermore we have

$$\lim_{r\to\infty} \tilde{\tau}_{r,\infty} = (1, 0, 0, *, *, *)$$

and

$$\lim_{n\to\infty} \tilde{\tau}_{\infty,n} = (\infty, 0, 0, *, *, *) .$$

Hence the following diagram is not commutative:

$$\begin{array}{ccc} \tilde{\tau}_{r,n} & \xrightarrow{n\to\infty} & \tilde{\tau}_{r,\infty} \\ {\scriptstyle r\to\infty}\downarrow & & \downarrow{\scriptstyle r\to\infty} \\ \tilde{\tau}_{\infty,n} & \xrightarrow{n\to\infty} & \tilde{\tau}_{\infty,\infty} \end{array}$$

The second column means that the boundary points $(1, 0, 0, (7+3\sqrt{5})/2, (11-\sqrt{5})/2, (11+\sqrt{5})/2)$ represent infinite numbers of Riemann surfaces which are not conformally equivalent. Thus the definition of boundary by the (λ, p, q)-method is not complete.

In this paper, we will introduce new coordinates to the Schottky space $\mathfrak{S}_g$ and will consider the space which is the union of $\mathfrak{S}_g$ and a part of the boundary.

1–4. Riemann surfaces with nodes. A *closed Riemann surface with nodes* S, is a compact complex space each point P of which has a neighborhood isomorphic either to a disk $|z| < 1$ in C (with P corresponding to $z = 0$) or to the set $|z| < 1$, $|w| < 1$, $zw = 0$ in C^2 (with P corresponding to $z = w = 0$). In the later case, P is called a *node*. Every component of $S\backslash\{\text{nodes}\}$ is called a *part* of S.

We classify nodes into the following two kinds. Cut off a closed Riemann surface with nodes, S, at a node P and denote by $\tilde{S}$ the resulting set, that is, $\tilde{S} = S - \{P\}$.

(1) If $\tilde{S}$ is still connected, then P is called a *nondividing node*.

(2) If $\tilde{S}$ is not connected, then P is called a *dividing node*.

In this paper we mainly consider Riemann surfaces with at most g non-dividing nodes and at most $2g-3$ dividing nodes.

2. Topological preliminaries

2–1. Let S be a compact Riemann surface of genus $g \geqq 2$. If mutually disjoint, simple loops on S, $\delta_1, \delta_2, \cdots, \delta_n$, have the following property, then we call $\Sigma = \{\delta_1, \delta_2, \cdots, \delta_n\}$ a *basic system of loops*: Each component of $S - \bigcup_{j=1}^{n} \delta_j$ (we call it a cell) is a sphere with three disks removed, that is, a planar and triply connected domain.

PROPOSITION 1. (1) $n = 3g - 3$. (2) *The number of cells is* $2g - 2$. (3) *At least g numbers of* δ_j, *for example* $\delta_1, \cdots, \delta_g$, *are non-dividing, and* $S - \bigcup_{j=1}^{g} \delta_j$ *is a sphere with* $2g$ *disks removed.*

Proof. Let m be the number of cells. Each cell has three boundary loops and hence there are $3m$ disks altogether. It is trivial that $m > 1$. Each loop δ_j has two sides $\delta_j^{\pm}$. Since $m > 1$ and S is connected, for at least one j, δ_j^+ and δ_j^- are boundaries of distinct cells. If we join the two cells along the boundaries δ_j^+ and δ_j^-, we have a sphere with four disks removed. This and the remaining $m - 2$ cells constitute $m - 1$ blocks.

If $m - 1 > 1$, then for at least one k, δ_k^+ and δ_j^- are boundaries of distinct blocks. Again we join the two blocks along the boundaries δ_k^+ and δ_k^-. Whenever the above operation is performed, the total number of disks on the spheres decreases two. If we perform the above operation $m - 1$ times, all cells are connected together, and we get a planar surface S^*, since S is connected. The number of disks on S^* is $3m - 2(m - 1) = m + 2$, that is, S^* is a sphere with $m + 2$ disks removed. S^* is the part of S with a part of Σ, for example $\delta_1, \cdots, \delta_k$, removed: $S - \bigcup_{j=1}^{k} \delta_j = S^*$. Since S^* is connected and planar, we have $k = g$ from the definition of genus. We have the third assertion.

Since $m + 2 = 2g$, we have $m = 2g - 2$, which is the second assertion. Furthermore the number of the loops is $g + (m - 1) = 3g - 3$, which is the first assertion. Our proof is now complete.

Remark. The choice of loops $\delta_1, \cdots, \delta_g$ in (8) is not necessarily unique, that is, there may be non-dividing loops among $\delta_{g+1}, \cdots, \delta_n$ (see the Fig. 1 below). We note that they divide S together with $\delta_1, \cdots, \delta_g$. There may or may not be dividing loops among $\delta_1, \cdots, \delta_n$.

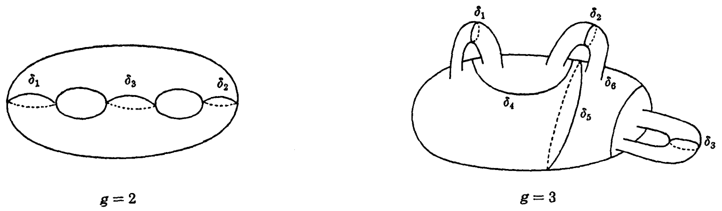

Fig. 1

2–2. If the number of non-dividing loops in a basic system of loops Σ is equal to g, then we call Σ a *standard system of loops*. From now on we let Σ denote a standard system of loops. We denote by $\alpha_1, \cdots, \alpha_g$ the non-dividing loops in Σ and by $\gamma_1, \cdots, \gamma_{2g-3}$ the dividing loops. The $\delta_1, \cdots, \delta_g$ in (3) of Proposition 1 are uniquely determined in this case.

PROPOSITION 2. *The two sides* $\alpha_j^{\pm}$ *of each loop* α_j $(j = 1, \cdots, g)$ *are boundary of the same cell.*

Proof. $S^* = S - \bigcup_{j=1}^{g} \alpha_j$ is a sphere with $2g$ disks removed. If α_j^+ and α_j^- are not boundary curves of the same cell, then there is a loop γ_k which separates α_j^+ from α_j^-. Then γ_k does not divide S, which contradicts γ_k being a dividing loop. Our proof is now complete.

Next we consider the third boundary curve of the cell containing α_j. In the case of $g = 2$, the boundary curves of the two cells are two sides of the same loop (we denote it by γ_1). In the case of $g \geqq 3$, the boundary curves are all distinct for $j = 1, 2, \cdots, g$. Thus the number of the boundary curves is equal to g. Let γ_j $(j = 1, 2, \cdots, g)$ be the boundary of the cell containing α_j. In the case of $g = 3$, $g = 2g - 3$. Hence there are no "γ-loops" other than $\gamma_1, \cdots, \gamma_g$. However in the case of $g \geqq 4$, there are "γ-loops" other than $\gamma_1, \cdots, \gamma_g$ and we denote them by $\gamma_{g+1}, \cdots, \gamma_{2g-3}$.

PROPOSITION 3. *In the case of genus* $g \geqq 4$, *either of two components of* $S - \gamma_j$ *contains more than one (two or more) "α-loops" for each* $j = g + 1, \cdots, 2g - 3$.

Proof. We remove the cells bounded by $\alpha_j^{\pm}$ and γ_j $(j = 1, \cdots, g)$ from $S^* = S - \bigcup_{j=1}^{g} \alpha_j$. Let S^{**} be the resulting surface. Then S^{**} is a sphere with g disks removed. Each component of $S^{**} - \gamma_j$ is neither a disk nor a ring domain. Thus either component contains more than one γ_k loops. Our proof is now complete.

Remark. Let "α-loops" on two components of $S-\gamma_j$ be $\alpha_{j(1)}, \alpha_{j(2)}, \cdots, \alpha_{j(m)}; \alpha_{\hat{j}(1)}, \alpha_{\hat{j}(2)}, \cdots, \alpha_{\hat{j}(n)}$ $(j(1)<j(2)<\cdots<j(m); \hat{j}(1)<\hat{j}(2)<\cdots<\hat{j}(n))$, respectively. Then we say that the loop γ_j gives a *partition* $\{j(1), j(2), \cdots, j(m)\} \cup \{\hat{j}(1), \hat{j}(2), \cdots, \hat{j}(n)\}$ of $\{1, 2, \cdots, g\}$.

There are g numbers of cells bounded by $\alpha_j^{\mp}$ and γ_j, and we denote them by σ_j for $j=2, 3, \cdots, g$ and σ_0 for $j=1$. We call them *terminal cells*. We call the $g-3$ remaining cells *not terminal cells*. We call γ_j $(j=1, 2, \cdots, g)$ *terminal loops* and γ_j $(j=g+1, \cdots, 2g-3)$ *not terminal loops*.

3. Multi-suffix

3–1. Let G be a fixed marked Schottky group: $G=\langle A_1, \cdots, A_g\rangle$. Let $\Sigma=\{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ be a standard system of loops on $S=\Omega(G)/G$ as in § 2, that is, $\alpha_1, \cdots, \alpha_g$ are terminal loops and $\gamma_{g+1}, \cdots, \gamma_{2g-3}$ are not terminal loops. We will represent γ_j as $\gamma(1, i_1, \cdots, i_\mu)$ by using multi-suffix $(1, i_1, \cdots, i_\mu)$ introduced in the following paragraphs.

The dividing boundary loop of the cell σ_0 containing the loop α_1 is γ_1. Let σ_1 be the cell which is the opposite side of σ_0 with respect to γ_1 and we write it as $\sigma_1=(\sigma_0; \gamma_1)$. In the case of $g=2$, $\sigma_1=\sigma_2$. In the case of $g\geqq 3$, σ_1 is not terminal. We denote by $\gamma(1,0)$ and $\gamma(1,1)$ the boundary loops of σ_1 other than γ_1 according with the following rule: We denote by $[\sigma_1; \gamma(1,0)]$ (resp. $[\sigma_1; \gamma(1,1)]$ be the union of all cells which lie in the opposite part of σ_1 with respect to $\gamma(1,0)$ (resp. $\gamma(1,1)$). Let $i(1,0)$ (resp. $i(1,1)$) be the minimum of i with $\alpha_i \subset [\sigma_1; \gamma(1,0)]$ (resp. $\alpha_i \subset [\sigma_1; \gamma(1,1)]$. Then we should have $i(1,0)<i(1,1)$.

We denote by $\sigma(1,0)$ (resp. $\sigma(1,1)$) the cells $(\sigma_1; \gamma(1,0))$ (resp. $(\sigma_1; \gamma(1,1))$) which lies on the opposite side of σ_1 with respect to $\gamma(1,0)$ (resp. $\gamma(1,1)$). If $\sigma(1,0)$ (resp. $\sigma(1,1)$) is not terminal, then we denote by $\gamma(1,0,0)$ and $\gamma(1,0,1)$ (resp. $\gamma(1,1,0)$ and $\gamma(1,1,1)$) the boundary loops of the cell $\sigma(1,0)$ (resp. $\sigma(1,1)$) other than $\gamma(1,0)$ (resp. $\gamma(1,1)$) according with the following rule; Parts $[\sigma(1,0); \gamma(1,0,0)]$, $[\sigma(1,0); \gamma(1,0,1)]$, $[\sigma(1,1); \gamma(1,1,0)]$, and $[\sigma(1,1); \gamma(1,1,1)]$ are defined similarly to the above. Let $i(1,0,0)$ (resp. $i(1,0,1)$, $i(1,1,0)$ and $i(1,1,1)$) be the minimum of i with $\alpha_i \subset [\sigma(1,0); \gamma(1,0,0)]$ (resp. $[\sigma(1,0); \gamma(1,0,1)]$, $[\sigma(1,1); \gamma(1,1,0)]$ and $[\sigma(1,1); \gamma(1,1,1)]$). Then we should have $i(1, 0, 0) < i(1, 0, 1)$ and $i(1, 1, 0) < i(1, 1, 1)$.

If $\sigma(1,0,0)$, $\sigma(1,0,1)$, $\sigma(1,1,0)$ or $\sigma(1,1,1)$ are not terminal, the same process is repeated. That is, in general, suppose $\gamma_1, \gamma(1, i_1), \cdots, \gamma(1, i_1, \cdots, i_\mu)$ have been determined and $(\sigma(1, i_1, \cdots, i_{\mu-1}); \gamma(1, i_1, \cdots, i_\mu)) = \sigma(1, i_1, \cdots, i_\mu)$ is not terminal, where $i_\nu = 0$ or 1 for $\nu = 1, 2, \cdots, \mu$. The boundary loops of the cell $\sigma(1, i_1, \cdots, i_\mu)$ are $\gamma(1, i_1, \cdots, i_\mu)$, $\gamma(1, i_1, \cdots, i_\mu, 0)$, and $\gamma(1, i_1, \cdots, i_\mu, 1)$, where $\gamma(1, i_1, \cdots, i_\mu, 0)$ and $\gamma(1, i_1, \cdots, i_\mu, 1)$ are determined as follows. The parts $[\sigma(1, i_1, \cdots, i_\mu); \gamma(1, i_1, \cdots, i_\mu, 0)]$ and $[\sigma(1, i_1, \cdots, i_\mu); \gamma(1, i_1, \cdots, i_\mu, 1)]$ are defined similarly to the above. Let $i(1, i_1, \cdots, i_\mu, 0)$ and $i(1, i_1, \cdots, i_\mu, 1)$ be the minimum of i with $\alpha_i \subset [\sigma(1, i_1, \cdots, i_\mu); \gamma(1, i_1, \cdots, i_\mu, 0)]$ and $\alpha_i \subset [\sigma(1, i_1, \cdots, i_\mu); \gamma(1, i_1, \cdots, i_\mu, 1)]$, respectively. Then we should have $i(1, i_1, \cdots, i_\mu, 0) < i(1, i_1, \cdots, i_\mu, 1)$.

Thus $2g-3$ loops γ_j $(j = 1, 2, \cdots, 2g-3)$ are expressed as $\gamma(1, i_1, \cdots, i_\mu)$ by using multi-suffix.

3–2. Here we present two examples.

EXAMPLE 1. Let Riemann surface S, "α-loops" $\alpha_1, \cdots, \alpha_g$ and "γ-loops" $\gamma_1, \cdots, \gamma_{2g-3}$ be as in the following Fig. 2.

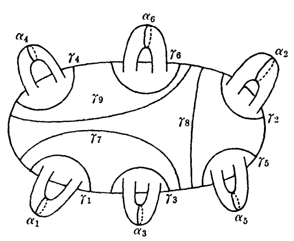

Fig. 2

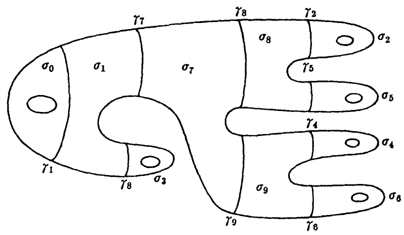

Fig. 3

We denote cells by $\sigma_1, \cdots, \sigma_9$ as in Fig. 3, where the picture of S is destorted for the sake of convenience. Next we express it as a tree in Fig. 4 below. Here every dot $\bullet$ denotes a terminal cell, every white circle $\circ$ denotes a not terminal cell and every segment—denotes a "γ-loop". If we represent "γ-loops" in Fig. 4 by multi-suffix, we have Fig. 5 below. We have the following: $\gamma_7 = \gamma(1, 0)$, $\gamma_3 = \gamma(1, 1)$, $\gamma_8 = \gamma(1, 0, 0)$, $\gamma_9 = \gamma(1, 0, 1)$, $\gamma_2 = \gamma(1, 0, 0, 0)$, $\gamma_5 = \gamma(1, 0, 0, 1)$, $\gamma_4 = \gamma(1, 0, 1, 0)$, and $\gamma_6 = \gamma(1, 0, 1, 1)$.

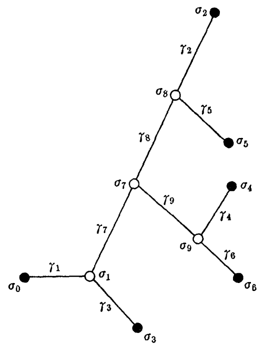

Fig. 4

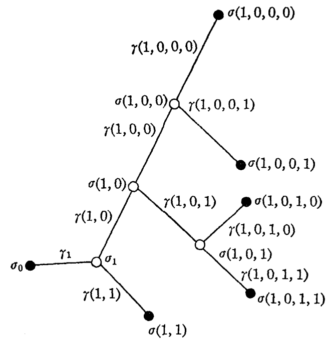

Fig. 5

EXAMPLE 2. Let S be the same as in Example 1. Let a standard system of loops Σ is as in Fig. 2′. By similar ways to Example 1, we have the following Fig. 3′, Fig. 4′, and Fig. 5′. Thus we have the following: $\gamma_7 = \gamma(1, 0)$, $\gamma_3 = \gamma(1, 1)$, $\gamma_8 = \gamma(1, 0, 0)$, $\gamma_5 = \gamma(1, 0, 1)$, $\gamma_9 = \gamma(1, 0, 0, 0)$, $\gamma_4 = \gamma(1, 0, 0, 1)$, $\gamma_2 = \gamma(1, 0, 0, 0, 0)$, and $\gamma_6 = \gamma(1, 0, 0, 0, 1)$.

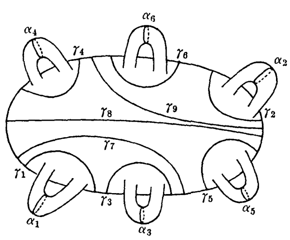

Fig. 2′

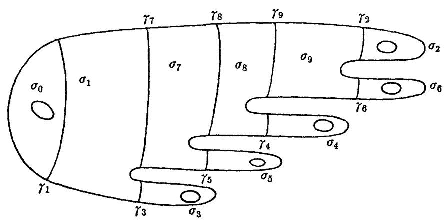

Fig. 3′

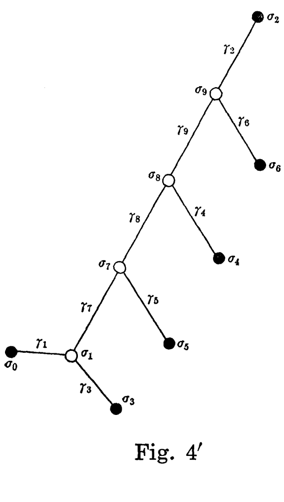

Fig. 4′

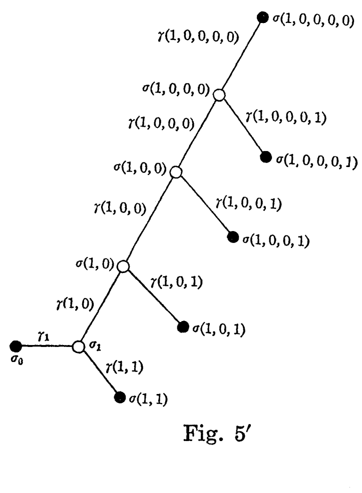

Fig. 5′

4. New coordinates

4–1. Here we will introduce new coordinates to the Schottky space. We fix a marked Schottky group $G_0 = \langle A_{10}, \cdots, A_{g0} \rangle$. Let $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ be a fixed standard system of loops on the Riemann surface $S_0 = \Omega(G_0)/G_0$ as in the previous section, that is, $\alpha_1, \cdots, \alpha_g$ are "α-loops" associated with the above marking and $\gamma_1, \cdots, \gamma_g$ are terminal loops and $\gamma_{g+1}, \cdots, \gamma_{2g-3}$ are not terminal loops. We remark that each γ_j $(j = 1, 2, \cdots, 2g-3)$ gives a partition of $\{1, 2, \cdots, g\}$. From now on we use these partitions associated with Σ. If necessary, we use the representation of "γ-loops" based on multi-suffix.

4–2. Let $G = \langle A_1, \cdots, A_g \rangle$ be a marked Schottky group. Let λ_j $(|\lambda_j| > 1)$, p_j and g_j be the multiplier, the repelling and the attracting fixed points of A_j $(j = 1, 2, \cdots, g)$, respectively. We normalize G by setting $p_1 = 0$, $q_1 = \infty$ and $p_2 = 1$. Then a point in $\mathfrak{S}_g$ is identified with

$$\tilde{\tau} = (\lambda_1, \cdots, \lambda_g; q_2, p_3, q_3, \cdots, p_g, q_g) \in C^{3g-3} .$$

Now we will introduce new coordinates

$$\tau = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in C^{3g-3} ,$$

where t_i and ρ_j $(i = 1, 2, \cdots, g; j = 1, 2, \cdots, 2g-3)$ are defined as follows.

We define t_i by setting $t_i = 1/\lambda_i$ $(i = 1, 2, \cdots, g)$. Thus $t_i \in D^* = \{z | 0 < |z| < 1\}$.

Next in order to define ρ_j $(j = 1, 2, \cdots, 2g-3)$, we determine the numbers $k(j)$, $\ell(j)$, $m(j)$, and $n(j)$ which are $\geqq 1$ and $\leqq g$, by the following Table 1; here we write $j = (1, i_1, \cdots, i_\mu)$ if $\gamma_j = \gamma(1, i_1, \cdots, i_\mu)$.

Table 1

j	1	$j = (1, i_1, \cdots, i_\mu)$ $(2 \leqq j \leqq g)$	$j = (1, i_1, \cdots, i_{\mu-1}, 0)$ $(\mu \geqq 1, g+1 \leqq j \leqq 2g-3)$	$j = (1, i_1, \cdots, i_{\mu-1}, 1)$ $(\mu \geqq 1, g+1 \leqq j \leqq 2g-3)$
$k(j)$	1	j	1	1
$\ell(j)$	1	j	$(1, i_1, \cdots, i_{\mu-1}, 1, 0, \cdots, 0)$	$(1, i_1, \cdots, i_{\mu-1}, 0, 0, \cdots, 0)$
$m(j)$	2	1	$(1, i_1, \cdots, i_{\mu-1}, 0, 0, \cdots, 0)$	$(1, i_1, \cdots, i_{\mu-1}, 1, 0, \cdots, 0)$

$n(j)$	$(1, 1, 0, \cdots, 0)$	$(1, i_1, \cdots, i_{\mu-1}, 1, 0, \cdots, 0)$ if $i_\mu = 0$ $(1, i_1, \cdots, i_{\mu-1}, 0, 0, \cdots, 0)$ if $i_\mu = 1$	$(1, i_1, \cdots, i_{\mu-1}, 0, 1, 0, \cdots, 0)$	$(1, i_1, \cdots, i_{\mu-1}, 1, 1, 0, \cdots, 0)$

In Table 1, $\gamma(1, i_1, \cdots, i_{\mu-1}, 0, 0, \cdots, 0)$, $\gamma(1, i_1, \cdots, i_{\mu-1}, 1, 0, \cdots, 0)$, $\gamma(1, i_1, \cdots, i_{\mu-1}, 1, 1, 0, \cdots, 0)$ and so on denote terminal loops. If $\mu = 1$, then we regard $\gamma(1, i_1, \cdots, i_{\mu-1}, 0)$ and $\gamma(1, i_1, \cdots, i_{\mu-1}, 1)$ as $\gamma(1, 0)$ and $\gamma(1, 1)$, respectively.

4–3. As examples, we consider the two cases stated in § 3.

Case 1. (Example 1 in § 3)

j	1	2	3	4	5	6	7	8	9
$k(j)$	1	2	3	4	5	6	1	1	1
$\ell(j)$	1	2	3	4	5	6	3	4	2
$m(j)$	2	1	1	1	1	1	2	2	4
$n(j)$	3	5	2	6	2	4	4	5	6

Case 2. (Example 2 in § 3).

j	1	2	3	4	5	6	7	8	9
$k(j)$	1	2	3	4	5	6	1	1	1
$\ell(j)$	1	2	3	4	5	6	3	5	4
$m(j)$	2	1	1	1	1	1	2	2	2
$n(j)$	3	6	2	2	2	2	5	4	6

4–4. The coordinate ρ_j is now defined as follows:

(1) For $j = 1, 2, \cdots, g$ we determine $T_j \in$ Möb (the set of all Möbius transformations) by $T_j(p_{k(j)}) = 0$, $T_j(q_{k(j)}) = \infty$, and $T_j(p_{m(j)}) = 1$ and define ρ_j by setting $\rho_j = T_j(p_{n(j)})$.

(2) For $j = g + 1, \cdots, 2g - 3$, we determine $T_j \in$ Möb by $T_j(p_{k(j)})$

$= 0$, $T_j(p_{\ell(j)}) = \infty$, and $T_j(p_{m(j)}) = 1$ and define ρ_j as $\rho_j = T_j(p_{n(j)})$.

We note that $\rho_j \in \boldsymbol{C} - \{0, 1\}$ $(j = 1, 2, \cdots, 2g-3)$. By the above things, we obtain a mapping $\tilde{\varphi}$ by setting $\tilde{\varphi}(G) = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in D^{*g} \times (\boldsymbol{C} - \{0, 1\})^{2g-3}$.

PROPOSITION 4. *Two equivalent marked Schottky groups* $G = \langle A_1, \cdots, A_g \rangle$ *and* $\hat{G} = \langle \hat{A}_1, \cdots, \hat{A}_g \rangle$, *that is,* $\hat{A}_k = UA_kU^{-1}$, $U \in$ Möb, *have the same coordinates* t_i *and* ρ_j.

Proof. Set $\tilde{\varphi}(G) = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3})$ and $\tilde{\varphi}(\hat{G}) = (\hat{t}_1, \cdots, \hat{t}_g; \hat{\rho}_1, \cdots, \hat{\rho}_{2g-3})$. It is trivial that $t_i = \hat{t}_i$ $(i = 1, 2, \cdots, g)$. For ρ, first we consider the case (1). We set $\hat{p}_k = U(p_k)$ and $\hat{q}_k = U(q_k)$ $(k = 1, 2, \cdots, g)$, where p_k and q_k are the repelling and the attracting fixed points of A_k, respectively. Then we easily see that $\hat{p}_k$ and $\hat{q}_k$ are the repelling and the attracting fixed points of $\hat{A}_k$, respectively. We determine $\hat{T}_j \in$ Möb by setting $\hat{T}_j(\hat{p}_{k(j)}) = 0$, $\hat{T}_j(\hat{q}_{k(j)}) = \infty$, and $\hat{T}_j(\hat{p}_{m(j)}) = 1$ and define $\hat{\rho}_j$ by setting $\hat{\rho}_j = \hat{T}_j(\hat{p}_{n(j)})$. Then we have $\hat{T}_j = T_jU^{-1}$. Thus we have $\rho_j = \hat{\rho}_j$, since $\hat{\rho}_j = \hat{T}_j(\hat{p}_{n(j)}) = T_jU^{-1}(U(p_{n(j)})) = T_j(p_{n(j)}) = \rho_j$. We can similarly prove the case (2). Our proof is now complete.

Remark. We will see later that reverse of this proposition holds.

Thus we can define a mapping φ of $\mathfrak{S}_g$ into $D^{*g} \times (\boldsymbol{C} - \{0, 1\})^{2g-3}$ by setting $\varphi([G]) = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3})$, where $[G]$ denotes the equivalence class of G, that is, a point in $\mathfrak{S}_g$. We denote by $\mathfrak{S}_g(\Sigma)$ the image of $\mathfrak{S}_g$ under the mapping.

4–5. Next we consider the converse. Let G_0, S_0 and Σ be as in 4–1. We will show that λ_j, p_j and q_j, $j = 1, 2, \cdots, g$, are uniquely determined from a given point

$$\tau = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in D^{*g} \times (\boldsymbol{C} - \{0, 1\})^{2g-3}$$

by the process opposite to the above, where we set $p_1 = 0$, $q_1 = \infty$, and $p_2 = 1$.

LEMMA 1. (1) *Let* α, β, γ *are mutually distinct points in* $\boldsymbol{C}$ *and let* $T \in$ Möb *be determined by setting* $T(0) = \alpha$, $T(\infty) = \beta$, *and* $T(1) = \gamma$. *If* $\rho \in \boldsymbol{C} - \{0, 1\}$, *then* $T(\rho) \neq \alpha, \beta, \gamma$.

(2) *Let* α, γ, δ *are mutually distinct points in* $\boldsymbol{C}$. *Suppose* $\rho \in \boldsymbol{C} - \{0, 1\}$. *Let* $T \in$ Möb *be determined by setting* $T(0) = \alpha$, $T(1) = \gamma$, *and* $T(\rho) = \delta$. *Then* $T(\infty) \neq \alpha, \gamma, \delta$.

The proof is easy and so we omit it here.

(1) We determine $p_{n(j)} = p(1, 1, 0, \cdots, 0)$ by setting $p_{n(j)} = \rho_1$.

(2) Next we determine $p_{n(j)} = p(1, i_1, \cdots, i_{\mu-1}, 0, 1, 0, \cdots, 0)$ and $p_{n(j)} = p(1, i_1, \cdots, i_{\mu-1}, 1, 1, 0, \cdots, 0)$ $(\mu \geqq 1)$. For $\mu = 1$, $p(1, 0, 1, 0, \cdots, 0)$ and $p(1, 1, 1, 0, \cdots, 0)$ are dertermined from p_1, p_2 and $p(1, 1, 0, \cdots, 0)$ by the process opposite to one in 4–2. Suppose $p(1, i_1, \cdots, i_{\mu-1}, 1, 0, \cdots, 0)$ and $p(1, i_1, \cdots, i_{\mu-1}, 0, 0, \cdots, 0)$ are dertermined. Then by the process opposite to one in 4–2, we can determine $p(1, i_1, \cdots, i_{\mu-1}, 0, 1, 0, \cdots, 0)$ and $p(1, i_1, \cdots, i_{\mu-1}, 1, 1, 0, \cdots, 0)$. From (1) and (2) by the induction, we determine $p_3, p_4, \cdots, p_g$. That is, we have the following.

LEMMA 2. *Set* $r(1) = (1, 1, 0, \cdots, 0)$, $r(j) = (1, i_1, \cdots, i_{\mu-1}, 0, 1, 0, \cdots, 0)$ *for* $j = (1, i_1, \cdots, i_{\mu-1}, 0)$ *and* $r(j) = (1, i_1, \cdots, i_{\mu-1} 1, 1, 0, \cdots, 0)$ *for* $j = (1, i_1, \cdots, i_{\mu-1}, 1)$. *Then the mapping* $j \mapsto r(j)$ *of the set* $\{1, g+1, g+2, \cdots, 2g-3\}$ *onto* $\{3, 4, \cdots, g\}$ *is bijective.*

(3) For $j = 2, 3, \cdots, g$, we determine q_j by setting $q_j = \hat{T}_j(\infty)$, where $\hat{T}_j \in$ Möb $(j = (1, i_1, \cdots, i_\mu))$ is determined by setting $\hat{T}_j(0) = p_j$, $\hat{T}_j(1) = p_1$, and $\hat{T}_j(\rho_j) = p(1, i_1, \cdots, i_{\mu-1}, 1, 0, \cdots, 0)$ if $i_\mu = 0$ and $\hat{T}_j(0) = p_j$, $\hat{T}_j(1) = p_1$, and $\hat{T}_j(\rho_j) = p(1, i_1, \cdots, i_{\mu-1}, 0, \cdots, 0)$ if $i_\mu = 1$.

(4) We define λ_i $(i = 1, 2, \cdots, g)$ by setting $\lambda_i = 1/t_i$.

Thus we obtain $\tilde{\tau} = (\lambda_1, \cdots, \lambda_g; q_2, p_3, q_3, \cdots, p_g, q_g)$ from $\tau = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3})$. Furthermore we determine $A_j \in$ Möb from $\tilde{\tau}$ as follows: The multiplier, the repelling and the attracting fixed points of A_j are λ_j, p_j and q_j, respectively. Thus we obtain a mapping ψ of $D^{*g} \times (C - \{0, 1\})^{2g-3}$ into Möb^g by setting $\psi(\tau) = \langle A_1, \cdots, A_g \rangle$ (we denote it by $G(\tau)$).

Remark. $G(\tau)$ may or may not be a Schottky group.

4–6. Now we have the following theorem.

THEOREM 1. *Let the mappings* $\varphi: \mathfrak{S}_g \to D^{*g} \times (C - \{0, 1\})^{2g-3}$ *and* $\psi: D^{*g} \times (C - \{0, 1\})^{2g-3} \to$ Möb^g *be as above. Then* $\psi\varphi =$ id. *and* $\varphi\psi | \mathfrak{S}_g(\Sigma) =$ id, *where* id. *and* $\psi | \mathfrak{S}_g(\Sigma)$ *denote the identity mapping and the restriction of the mapping* ψ *to the set* $\mathfrak{S}_g(\Sigma)$, *respectively.*

5. Augmented Schottky spaces

5–1. Let G_0 and Σ be a fixed marked Schottky group and a standard system of loops on $S_0 = \Omega(G_0)/G_0$ as in § 3. Let $I \subset \{1, 2, \cdots, g\}$, $J \subset \{1, 2,$

$\cdots, 2g-3\}$, $|I|$ = number of elements in I, and $|J|$ = number of elements in J. We define subsets $\delta^{I,J}\mathfrak{S}_g(\Sigma)$ of $\overline{\mathfrak{S}}_g(\Sigma) = \mathfrak{S}_g(\Sigma) \cup \partial\mathfrak{S}_g(\Sigma) \subset \bar{D}^g \times \hat{C}^{2g-3}$, where $D = \{z: |z| < 1\}$.

(1) For $I = J = \phi$, we define $\delta^{\phi,\phi}\mathfrak{S}_g(\Sigma)$ as $\delta^{\phi,\phi}\mathfrak{S}_g(\Sigma) = \mathfrak{S}_g(\Sigma)$.

(2) For $I \neq \phi$ and $J = \phi$, $\delta^{I,\phi}\mathfrak{S}_g(\Sigma)$ (we denote it by $\delta^I\mathfrak{S}_g(\Sigma)$) is the set of all points

$$\tau = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in D^g \times (C - \{0, 1\})^{2g-3}$$

having the following properties (i)–(iv).

(i) $t_i = 0$ for $i \in I$ and $t_i \neq 0$ for $i \notin I$.

By the same way as in the previous section 4–4, we can uniquely determine $p_1 = 0$, $q_1 = \infty$, $p_2 = 1$, q_2, $p_3, \cdots, p_g$, q_g from τ, since $\rho_j \neq 1$ for $j = 1, 2, \cdots, 2g-3$. Thus for each $i \notin I$, we have the Möbius transformation $A_i(\tau, z)$ whose multiplier, the repelling and the attracting fixed points are $\lambda_i = 1/t_i, p_i$ and q_i, respectively.

(ii) $A_i(\tau, z)$, $i \notin I$, generate a Schottky group (we denote it by $G(\tau)$).

(iii) The $2|I|$ points p_i, q_j $(i, j \in I)$ are distinct.

(iv) The $2|I|$ points p_i and q_j $(i, j \in I)$ lie in some standard fundamental region for $G(\tau)$.

We call p_i and q_i with $i \in I$ the extended repelling and the extended attracting fixed points, respectively for the later convenience.

5–2. (3) For $I = \phi$ and $J \neq \phi$, $\delta^{\phi,J}\mathfrak{S}_g(\Sigma)$ (we denote it by $\hat{\delta}^J\mathfrak{S}_g(\Sigma)$) is the set of all points

$$\tau = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in D^{*g} \times (C - \{0\})^{2g-3}$$

having the following properties (i)–(iv).

(i) $\rho_j = 1$ for $j \in J$ and $\rho_j \neq 1$ for $j \notin J$.

5–3. For the purpose of stating the remaining properties, we need a preparation. Let $J = \{k_1, k_2, \cdots, k_m\} \subset \{1, 2, \cdots, 2g-3\}$ with $k_1 < k_2 < \cdots < k_m$. Then $\rho_{k_1} = \rho_{k_2} = \cdots = \rho_{k_m} = 1$. First suppose $k_1 = 1$, that is, $\rho_1 = 1$.

1) We set $p_1 = 0$, $q_1 = \infty$, and $\lambda_1 = 1/t_1$. Let A_1 be the Möbius transformation whose multiplier, the repelling, and the attracting fixed points are λ_1, p_1 and q_1, respectively. Set $G_0(\tau) = \langle A_1 \rangle$. Then the point 1 lies in some standard fundamental region for $G_0(\tau)$. We call the point 1 the *right distinguished point associated with* k_1 and denote it by $\tilde{p}_1^+ = 1$.

2) We consider all of the following sequences:

Case 1. $\rho(1, i_1) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_1 - 1}) \neq 1$, $\rho(1, i_1, \cdots, i_{\mu_1}) \neq 1$ and $\gamma(1, i_1, \cdots, i_{\mu_1})$ is terminal.

Case 2. $\rho(1, i_1) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_1 - 1}) \neq 1$, $\rho(1, i_1, \cdots, i_{\mu_1}) = 1$.

We perform the same process as in the previous section by using $0, \infty$, and 1 instead of $p_1, p_2 = p(1, 0, \cdots, 0)$ and $p(1, 1, 0, \cdots, 0)$, respectively. Then we can determine a number for each $\rho(1, i_1, \cdots, i_{\nu_1})$ ($1 \leqq \nu_1 \leqq \mu_1$ for Case 1, $1 \leqq \nu_1 \leqq \mu_1 - 1$ for Case 2). That is, we get the following:

1. Suppose $\rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1) \neq 1$, $\rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1, 0) \neq 1$, $\rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1, 0, 0) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1}) \neq 1$ ($\nu_1 \leqq \mu_1$) and $\gamma(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1})$ is terminal. That is, $\rho(1, i_1), \cdots, \rho(1, i_1, \cdots, i_{\nu_1 - 1})$, $\rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1)$, $\rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1, 0), \cdots, \rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1})$ is a sequence in Case 1. Then we determine $p(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1})$ from $\rho(1, i_1, \cdots, i_{\nu_1 - 1})$ by the same way as in the previous section.

2. Suppose $\rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1) \neq 1$, $\rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1, 0) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1 - 1}) \neq 1$ and $\sigma(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1}) = 1$, that is, $\rho(1, i_1), \cdots, \rho(1, i_1, \cdots, i_{\nu_1 - 1}), \cdots, \rho(1, i_1, \cdots, i_{\nu_1 - 1}, \cdots, 1, 0, \cdots, 0)$ is a sequence in Case 2. Then we determine a number from $\rho(1, i_1, \cdots, i_{\nu_1 - 1})$ by the same way as in the previous section. We denote it by $\tilde{p}^+(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1})$ and call it the *right distinguished point associated with* $(1, i_1, \cdots, i_{\nu_1 - 1}, 1, \underbrace{0, \cdots, 0}_{\ell_1})$.

3. We determine $q(1, i_1, \cdots, i_{\mu_1})$ from $\rho(1, i_1, \cdots, i_{\mu_1})$ in Case 1 by the same way as in the previous section.

Let there be g_1 numbers of sequences of Case 1 and $n_1 - 1$ numbers of sequences of Case 2. We denote by $\gamma_{1(1)}, \gamma_{1(2)}, \cdots, \gamma_{1(g_1)}$ the terminal loops of sequences belonging to Case 1. We regard the point 0 as the left distinguished point associated with k_1 and denote it by $\tilde{p}_1^-$. We denote by $\tilde{p}_{1(1)}^+, \tilde{p}_{1(2)}^+, \cdots, \tilde{p}_{1(n_1 - 1)}^+$ the right distinguished points which occur in sequences of Case 2.

Remark. If $\rho(1, 0) \neq 1, \cdots, \rho(1, \underbrace{0, \cdots, 0}_{\ell_1 - 1}) \neq 1$, $\rho(1, \underbrace{0, \cdots, 0}_{\ell_1}) = 1$, then we set $\tilde{p}^+(1, \underbrace{0, \cdots, 0}_{\ell_1}) = \infty$. Similarly if $\rho(1, 1) \neq 1, \cdots, \rho(1, 1, \underbrace{0, \cdots, 0}_{\ell_1 - 1}) \neq 1$,

$\rho(1, 1, \underbrace{0, \cdots, 0}_{\ell_1}) = 1$, then we set $\tilde{p}^+(1, 1, \underbrace{0, \cdots, 0}_{\ell_1}) = 1$.

We set $\lambda_{1(i)} = 1/t_{1(i)}$ $(i = 1, 2, \cdots, g_1)$. Let $A_{1(i)}$ be the Möbius transformation whose multiplier, the repelling, and the attracting fixed points are, $\lambda_{1(i)}, p_{1(i)}$ and $q_{1(i)}$, respectively. We set $G_1(\tau) = \langle A_{1(1)}, \cdots, A_{1(g_1)}\rangle$.

3) Next we consider the general case. We treat the following two cases:

Case 1. $\rho(1, i_1, \cdots, i_{\mu_\ell}) = 1, \rho(1, i_1, \cdots, i_{\mu_\ell+1}) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_{\ell+1}}) \neq 1$ and $\gamma(1, i_1, \cdots, i_{\mu_{\ell+1}})$ is terminal.

Case 2. $\rho(1, i_1, \cdots, i_{\mu_\ell}) = 1, \rho(1, i_1, \cdots, i_{\mu_\ell+1}) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_{\ell+1}-1}) \neq 1$ and $\rho(1, i_1, \cdots, i_{\mu_{\ell+1}}) = 1$.

Let $\gamma(1, i_1, \cdots, i_{\mu_\ell}) = \gamma_{k_\ell}$. We use $0, \infty$ and 1 instead of $p_1, p(1, i_1, \cdots, i_{\mu_\ell}, 0, \cdots, 0)$ and $p(1, i_1, \cdots, i_{\mu_\ell}, 1, 0, \cdots, 0)$, respectively. By the same way as in the case 2), we determine p and q from each $\rho(1, i_1, \cdots, i_{\nu_\ell})$ $(1 \leqq \nu_\ell \leqq \mu_{\ell+1}$ for Case 1, and $1 \leqq \nu_\ell \leqq \mu_{\ell+1} - 1$ for Case 2).

Let there be g_ℓ numbers of sequences of Case 1 and $n_\ell - 1$ numbers of sequences of Case 2. Then we get $n_\ell - 1$ numbers of right distinguished points $\tilde{p}^+_{\tilde{\ell}(1)}, \cdots, \tilde{p}^+_{\tilde{\ell}(n-1)}$ associated with $\tilde{\ell}(1), \cdots, \tilde{\ell}(n_\ell - 1)$, respectively. We denote by $\gamma_{\ell(1)}, \gamma_{\ell(2)}, \cdots, \gamma_{\ell(g_\ell)}$ terminal loops of sequences belonging to Case 1. We regard the point 0 as the left distinguished points associated with k_ℓ, where $\gamma_{k_\ell} = \gamma(1, i_1, \cdots, i_{\mu_\ell})$, and denote it by $\tilde{p}^-_{k_\ell}$. We set $\lambda_{\ell(i)} = 1/t_{\ell(i)}$ $(i = 1, 2, \cdots, g)$. Let $A_{\ell(i)}$ be the Möbius transformation whose multiplier, the repelling, and the attracting fixed points are $\lambda_{\ell(i)}, p_{\ell(i)}$ and $q_{\ell(i)}$, respectively. We set $G_\ell(\tau) = \langle A_{\ell(1)}, \cdots, A_{\ell(g_\ell)}\rangle$.

4) Last we consider the following case: $\rho(1, i_1, \cdots, i_{\mu_{\ell-1}}) = 1$, $\rho(1, i_1, \cdots, i_{\mu_{\ell-1}+1}) = 1$ and $\gamma(1, i_1, \cdots, i_{\mu_\ell})$ is terminal $(\mu_\ell = \mu_{\ell-1} + 1)$.

Let $(1, i_1, \cdots, i_{\mu_\ell}) = k_\ell$, that is, $\gamma(1, i_1, \cdots, i_{\mu_\ell}) = \gamma_{k_\ell}$. We set $p_{k_\ell} = 0$, $q_{k_\ell} = \infty$ and $\lambda_{k_\ell} = 1/t_{k_\ell}$. Let A_{k_ℓ} be the Möbius transformation whose multiplier, the repelling, and the attracting fixed points are $\lambda_{k_\ell}, p_{k_\ell}$ and q_{k_ℓ}, respectively. Set $G_\ell(\tau) = \langle A_{k_\ell}\rangle$. Then the point 1 lies in some standard fundamental region for $G_\ell(\tau)$. We call the point the left distinguished point associated with k_ℓ and denote it by $\tilde{p}^-_{k_\ell} = 1$.

Remark. For $g = 2$, we have the following. We set $p_2 = 0$, $q_2 = \infty$ and $\lambda_2 = 1/t_2$. Let A_2 be the Möbius transformation whose multiplier, the repelling, and the attracting fixed points are λ_2, p_2, and q_2, respectively. Set $G_2(\tau) = \langle A_2\rangle$. Then the point 1 lies in some standard fundamental region for $G_2(\tau)$. We call the point 1 the left distinguished point associated with k_1 and denote it by $\tilde{p}_1^- = 1$.

We treat the case of $k_1 \neq 1$, that is, the case of $\rho_1 \neq 1$, similarly to the above. We consider the following sequences:

Case 1. $\rho_1 \neq 1$, $\rho(1, i_1) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_0}) \neq 1$ and $\gamma(1, i_1, \cdots, i_{\mu_0})$ is terminal.

Case 2. $\rho_1 \neq 1, \rho(1, i_1) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_0-1}) \neq 1$ and $\rho(1, i_1, \cdots, i_{\mu_0}) = 1$.

Let there be g_0 numbers of the sequences of Case 1 and n_0 numbers of the sequences of Case 2. Then by the same way as in the above, we determine n_0 right distinguished points $\tilde{p}^+_{\bar{0}(1)}, \cdots, \tilde{p}^+_{\bar{0}(n_0)}$ for $\rho(1, i_1, \cdots, i_{\nu_0})$ ($1 \leqq \nu_0 \leqq \mu_0$ for Case 1 and $1 \leqq \nu_0 \leqq \mu_0 - 1$ for Case 2) and a Möbius transformation group $G_0(\tau) = \langle A_{0(1)}, \cdots, A_{0(g_0)} \rangle$. We omit the detail here.

By the above things, we get $|J| + 1$ $(=m + 1)$ numbers of groups (some of which may be the trivial group): $G_0(\tau)$, $G_1(\tau), \cdots, G_m(\tau)$. Furthermore we obtain distinguished points $\tilde{p}^+_{\bar{\ell}(1)}, \cdots, \tilde{p}^+_{\bar{\ell}(n-1)}, \tilde{p}^-_{\bar{k}_\ell}$ for each $\ell = 1, 2, \cdots, m$ and $\tilde{p}^+_{\bar{0}(1)}, \cdots, \tilde{p}^+_{\bar{0}(n_0)}$.

5–4. Now we can write the remaining requirements as follows.

(ii) For each $i = 0, 1, \cdots, m$, $G_i(\tau)$ is a Schottky group or the trivial group.

(iii) For each $i = 1, 2, \cdots, m$, n_i distinguished points $\tilde{p}^+_{\bar{i}(1)}, \cdots, \tilde{p}^+_{\bar{i}(n_i-1)}$, $\tilde{p}^-_{\bar{k}_i}$ are distinct and n_0 distinguished points $p^+_{\bar{0}(1)}, \cdots, \tilde{p}^+_{\bar{0}(n_0)}, \tilde{p}^-_{\bar{k}_i}$ are distinct.

(iv) For each $i = 0, 1, \cdots, m$, the above n_i distinguished points lie in some standard fundamental region $\omega_i(\tau)$ for $G_i(\tau)$.

5–5. (4) For $I \neq \phi$ and $J \neq \phi$, the set $\delta^{I,J}\mathfrak{S}_g(\Sigma)$ is defined by combining the definitions of the case (2) and case (3).

Remark. In the case (3) and the case (4), we have $m + 1$ $(=|J| + 1)$ compact Riemann surfaces

$$S_0(\tau) = \Omega(G_0(\tau))/G_0(\tau),\ S_1(\tau) = \Omega(G_1(\tau))/G_1(\tau), \cdots, S_m(\tau) \\ = \Omega(G_m(\tau))/G_m(\tau).$$

We say that τ represents $m + 1$ Riemann surfaces $S_0(\tau), S_1(\tau), \cdots, S_m(\tau)$ and write $S(\tau) = \bigcup_{j=0}^{m} S_j(\tau)$. Let $\prod_\ell : \Omega(G_\ell(\tau)) \to S_\ell(\tau)$ be the natural projection. Set $\hat{p}^+_{\bar{\ell}(\nu)} = \prod_\ell(\tilde{p}^+_{\bar{\ell}(\nu)})$ and $\tilde{p}^+_{\bar{\ell}(\nu)} = \prod_\ell(\hat{p}^-_{\bar{\ell}(\nu)})$. We call them right and left distinguished points on $S_\ell(\tau)$ associated with $\tilde{p}^+_{\bar{\ell}(\nu)}$ and $\tilde{p}^-_{\bar{\ell}(\nu)}$, respectively.

We give definitions of the following sets by using $\delta^{I,J}\mathfrak{S}_g(\Sigma)$: $\mathfrak{S}_g^I(\Sigma) = \bigcup_{L\subset I} \delta^L\mathfrak{S}_g(\Sigma)$, $\hat{\mathfrak{S}}_g^J(\Sigma) = \bigcup_{M\subset J} \hat{\delta}^M\mathfrak{S}_g(\Sigma)$, $\mathfrak{S}_g^{I,J}(\Sigma) = \bigcup_{\substack{L\subset I\\ M\subset J}} \delta^{L,M}\mathfrak{S}_g(\Sigma)$, $\mathfrak{S}_g^*(\Sigma) = \mathfrak{S}_g^I(\Sigma)$ with $I = \{1, 2, \cdots, g\}$, $\hat{\mathfrak{S}}_g(\Sigma) = \hat{\mathfrak{S}}_g^J(\Sigma)$ with $J = \{1, 2, \cdots, 2g-3\}$ and $\hat{\mathfrak{S}}_g^*(\Sigma) = \mathfrak{S}_g^{I,J}(\Sigma)$ with $I = \{1, 2, \cdots, g\}$ and $J = \{1, 2, \cdots, 2g-3\}$.

DEFINITION. We call the set $\hat{\mathfrak{S}}_g^*(\Sigma)$ the augmented Schottky space associated with Σ.

Remark 1. The mappings φ and ψ defined in § 4 are naturally extended to $\mathfrak{S}_g^*$ and $\mathfrak{S}_g^*(\Sigma)$, respectively, where $\mathfrak{S}_g^*$ is the augmented Schottky space in the sense of Bers (cf. Bers [3]).

Remark 2. From Theorem 1, we see, for the above mappings $\varphi\colon \mathfrak{S}_g^* \to D^g \times (C - \{0, 1\})^{2g-3}$ and $\psi\colon D^g \times (C - \{0, 1\})^{2g-3} \to \text{Möb}^g$, $\psi\varphi = \text{id.}$ and $\varphi\psi | \mathfrak{S}_g^*(\Sigma) = \text{id.}$

From Remarks 1 and 2, we may identified $\mathfrak{S}_g^*$ with $\mathfrak{S}_g^*(\Sigma)$.

Remark 3. Let $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ and $\hat{\Sigma} = \{\hat{\alpha}_1, \cdots, \hat{\alpha}_g; \hat{\gamma}_1, \cdots, \hat{\gamma}_{2g-3}\}$ be standard systems of loops on a compact Riemann surface of genus g. Then there is a canonical bijection of $\mathfrak{S}_g^*(\Sigma)$ onto $\mathfrak{S}_g^*(\hat{\Sigma})$. Let ψ (resp. $\tilde{\varphi}$) be the mapping of $\mathfrak{S}_g^*(\Sigma)$ (resp. $\mathfrak{S}_g^*$) onto $\mathfrak{S}_g^*$ (resp. $\mathfrak{S}_g^*(\hat{\Sigma})$) stated in the above Remark 1. Then $\tilde{\varphi}\psi$ is the desired mapping.

Remark 4. Let Σ and $\hat{\Sigma}$ be distinct standard systems of loops on a compact Riemann surface of genus g. We can not necessarily identify $\hat{\mathfrak{S}}_g(\Sigma)$ with $\hat{\mathfrak{S}}_g(\hat{\Sigma})$ for the following reason. For $\tau \in \hat{\delta}^J\mathfrak{S}_g(\Sigma)$, we have $|J| + 1$ Schottky groups $G_0(\tau), G_1(\tau), \cdots, G_m(\tau)$ ($m = |J|$) and $m + 1$ Riemann surfaces $S_0(\tau), S_1(\tau), \cdots, S_m(\tau)$. But any $\hat{\tau} \in \hat{\mathfrak{S}}_g(\hat{\Sigma})$ may not represent Riemann surfaces $S_0(\tau), S_1(\tau), \cdots, S_m(\tau)$. For example, let Σ and $\hat{\Sigma}$ be the standard systems of loops associated with Case 1 and Case 2 in § 3, respectively. Let

$$\tau = (t_1, \cdots, t_6; \rho_1, \rho_2, \cdots, \rho_6, 1, 1, 1) \in \hat{\delta}^{\{7,8,9\}}\mathfrak{S}_6(\Sigma) .$$

Then we have four Riemann surfaces $S_0(\tau)$, $S_1(\tau)$, $S_3(\tau)$, and $S_4(\tau)$. But there are no $\hat{\tau} \in \hat{\mathfrak{S}}_g(\hat{\Sigma})$ representing the Riemann surfaces.

Remark 5. $\hat{\mathfrak{S}}_g^*(\Sigma) \subset D^g \times (C - \{0\})^{2g-3}$.

Remark 6. Each point $\tau = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \subset \hat{\mathfrak{S}}_g^*(\Sigma) \backslash \mathfrak{S}_g(\Sigma)$ has the following property: At least one of ρ_j, $j = 1, 2, \cdots, 2g-3$, is equal to one or at least one of t_i, $i = 1, 2, \cdots, g$, is equal to zero. Thus $\hat{\mathfrak{S}}_g^*(\Sigma) \backslash \mathfrak{S}_g(\Sigma)$ is the intersection of $\partial\mathfrak{S}_g(\Sigma)$ $(\subset \bar{D}^g \times \hat{C}^{2g-3})$ with finitely many complex hyperplanes.

5–6. PROPOSITION 5. *The augmented Schottky space $\hat{\mathfrak{S}}_g^*(\Sigma)$ is a domain in C^{3g-3}, and a subset of $\mathfrak{S}_g(\Sigma) \cup \partial\mathfrak{S}_g(\Sigma)$. For each $I \subset \{1, 2, \cdots, g\}$ and each $J \subset \{1, 2, \cdots, 2g-3\}$, $\delta^{I,J}\mathfrak{S}_g(\Sigma)$ is a domain in $D^{*g-|I|} \times (C - \{0, 1\})^{2g-3-|J|}$ and $\mathfrak{S}_g^{I,J}(\Sigma)$ is a subdomain of $\hat{\mathfrak{S}}_g^*(\Sigma)$.*

Proof. That $\mathfrak{S}_g^*(\Sigma)$ is a domain is proved by a method similar to Bers [3]. Here we will prove that $\hat{\mathfrak{S}}_g(\Sigma)$ is a domain. By using the techniques in the above two cases, we may prove that $\hat{\mathfrak{S}}_g^*(\Sigma)$ is a domain in C^{3g-3}.

Let $\tau \in \hat{\mathfrak{S}}_g(\Sigma)$. Suppose $\tau \in \hat{\delta}^J\mathfrak{S}_g(\Sigma)$. Let the Riemann surface $S(\tau)$ associated with τ have $m+1$ $(m = |J|)$ parts, that is, $S(\tau) = \bigcup_{j=0}^{m} S_j(\tau)$. For τ, we have $m+1$ Schottky groups (including the trivial group) $G_0(\tau)$, $G_1(\tau), \cdots, G_m(\tau)$. We write

$$G_\ell(\tau) = \langle A_{\ell(1)}(\tau), \cdots, A_{\ell(g_\ell)}(\tau)\rangle \qquad (\ell = 0, 1, \cdots, m) .$$

Then let the distinguished points be $\tilde{p}_{\ell(1)}(\tau), \cdots, \tilde{p}_{\ell(n_\ell)}(\tau)$, where $\tilde{p}_{\ell(n_\ell)}(\tau) = \tilde{p}_{k_\ell}^-(\tau)$ $(\ell = 1, 2 \cdots, m)$. Let $p_{\ell(j)}(\tau)$ and $q_{\ell(j)}(\tau)$ be the repelling and the attracting fixed points of $A_{\ell(j)}(\tau)$, $j = 1, 2, \cdots, g_\ell$, respectively. $\tilde{p}_{\ell(1)}(\tau)$, $\cdots, \tilde{p}_{\ell(n_\ell)}(\tau)$ lie in a suitable fundamental region $\omega_\ell(\tau)$ for $G_\ell(\tau)$ by the definition of the augmented Schottky space.

We normalize $G_\ell(\tau)$ by setting $\tilde{p}_{\ell(n_\ell)}(\tau) = 1$, $p_{\ell(1)}(\tau) = 0$, and $q_{\ell(1)}(\tau) = \infty$ $(\ell = 1, 2, \cdots, m)$ (for $\ell = 0$, we set $\tilde{p}_{0(1)}^+(\tau) = 1$ instead of $\tilde{p}_{\ell(n_\ell)}(\tau) = 1$). We denote it by $G_\ell(\tau)$ again. Let defining curves of the groups $G_\ell(\tau)$ be $C_{\ell(1)}(\tau), C'_{\ell(1)}(\tau), \cdots, C_{\ell(g_\ell)}(\tau), C'_{\ell(g_\ell)}(\tau)$. Let $\tau' \in D^g \times (C - \{0\})^{2g-3}$ satisfy $|\tau' - \tau| < \varepsilon$ for sufficiently small $\varepsilon > 0$. Let $L = \{k | \rho_k(\tau') = 1\}$. We may assume that $L \subset J$. We will show that $\tau' \in \hat{\delta}^L\mathfrak{S}_g(\Sigma) \subset \hat{\mathfrak{S}}_g(\Sigma)$. Let $L = J\backslash\{k_\ell\}$. Suppose that the components of τ_1 are equal to those of τ except ρ_{k_ℓ}, which satisfies the condition $0 < |\rho_{k_\ell}(\tau_1) - \rho_{k_\ell}(\tau)| < \varepsilon$. First we will show $\tau_1 \in \hat{\delta}^L\mathfrak{S}_g(\Sigma)$ and then $\tau' \in \hat{\delta}^L\mathfrak{S}_g(\Sigma)$.

Let $\rho(1, i_1, \cdots, i_{\mu_{\ell'}})(\tau) = 1$, $\rho(1, i_1, \cdots, i_{\mu_{\ell'}+1})(\tau) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_\ell - 1})(\tau) \neq 1$, $\rho(1, i_1, \cdots, i_{\mu_\ell})(\tau) = 1$, $\rho(1, i_1, \cdots, i_{\mu_\ell + 1})(\tau) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_\ell^* - 1})(\tau) \neq 1$, $\rho(1, i_1, \cdots, i_{\mu_\ell^*})(\tau) = 1$.

Remember that $\gamma_{k_\ell} = \gamma(1, i_1, \cdots, i_{\mu_\ell})$ and $\gamma_{k_{\ell'}} = \gamma(1, i_1, \cdots, i_{\mu_{\ell'}})$. We assume that $g_\ell \geqq 1$ and $g_{\ell'} \geqq 1$. For τ_1, we define

$$\tilde{G}_{k_\ell}(\tau_1) = \langle A_{\ell(1)}(\tau_1, z), \cdots, A_{\ell(g_\ell)}(\tau_1, z)\rangle$$

and

$$G_{k_{\ell'}}(\tau_1) = \langle A_{\ell'(1)}(\tau_1, z), \cdots, A_{\ell'(g_{\ell'})}(\tau_1, z)\rangle .$$

It is easily seen that both $\tilde{G}_{k_\ell}(\tau_1)$ and $G_{k_{\ell'}}(\tau_1)$ $(=G_{k_{\ell'}}(\tau))$ are Schottky groups, since the difference between τ_1 and τ is small. We normalize $\tilde{G}_{k_\ell}(\tau_1)$ by setting $p_{\ell(1)}(\tau_1) = 0$, $q_{\ell(1)}(\tau_1) = \infty$ and $\tilde{p}_{\bar{\ell}(n_\ell)}(\tau) = 1$, where we note that $\tilde{p}_{\bar{\ell}(n_\ell)}(\tau)$ lies in a standard fundamental region $\omega_\ell(\tau_1)$ for $\tilde{G}_{k_\ell}(\tau_1)$ $(=G_{k_\ell}(\tau_1))$. Similarly we normalize $G_{k_{\ell'}}(\tau_1)$ by setting $p_{\ell'(1)}(\tau_1) = 0$, $q_{\ell'(1)}(\tau_1) = \infty$, and $\tilde{p}^+_{k_\ell}(\tau) = 1$ (we set $\tilde{p}^+_{k_\ell}(\tau) = \tilde{p}_{\bar{\ell}'(1)}(\tau)$). We represent the normalized groups by the same notations. We choose their standard fundamental regions as follows and denote them by $\omega_\ell(\tau_1)$ and $\omega_{\ell'}(\tau_1)$: $C_{\ell(1)}(\tau)$, $C'_{\ell(1)}(\tau_1) = A_{\ell(1)}(\tau_1, C_{\ell(1)}(\tau))$, $\cdots$, $C_{\ell(g_\ell)}(\tau)$, $C'_{\ell(g_\ell)}(\tau_1) = A_{\ell(g_\ell)}(\tau_1, C_{\ell(g_\ell)}(\tau))$; $C_{\ell'(1)}(\tau)$, $C'_{\ell'(1)}(\tau_1) = A_{\ell'(1)}(\tau_1, C_{\ell'(1)}(\tau))$, $\cdots$, $C_{\ell'(g_{\ell'})}(\tau)$, $C'_{\ell'(g_{\ell'})}(\tau_1) = A_{\ell'(g_{\ell'})}(\tau_1, C_{\ell'(g_{\ell'})}(\tau))$ are defining curves of $\tilde{G}_\ell(\tau_1)$ and $G_{\ell'}(\tau_1)$, respectively. Then $\tilde{p}_{\bar{\ell}(1)}(\tau_1), \cdots, \tilde{p}_{\bar{\ell}(n_\ell-1)}(\tau_1)$ lie in $\omega_\ell(\tau_1)$ and $\tilde{p}_{\bar{\ell}'(2)}(\tau_1), \cdots, \tilde{p}_{\bar{\ell}'(n_{\ell'})}(\tau_1)$ lie in $\omega_{\ell'}(\tau_1)$.

We determine $T \in$ Möb by setting $T(0) = 1$, $T(\infty) = \rho_{k_\ell}(\tau_1)$, and $T(1) = 0$. For simplicity we write ρ instead of $\rho_{k_\ell}(\tau_1)$. Then we have

$$T(z) = \rho(z-1)/(z-\rho)$$

and $T(\rho) = \infty$. Since $\omega_{\ell'}(\tau) = \omega_{\ell'}(\tau_1)$ and the distinguished points of $G_{\ell'}(\tau)$ and $G_{\ell'}(\tau_1)$ are the same, the point 1 may be regarded as an interior point to $\omega'_{\ell'}(\tau_1)$, where $\omega'_{\ell'}(\tau_1) = \omega_{\ell'}(\tau_1) - \{\text{distinguished points}\}$. Since τ_1 is sufficiently close to $\tau, \sigma_{k_\ell}(\tau_1)$ is sufficiently close to 1. Thus we may regard

$$\{z \,|\, |z-1| < |1-\rho_{k_\ell}(\tau_1)|^{1/2}\} \subset \omega'_{\ell'}(\tau_1) .$$

Set $\tilde{c}: |z-1| = |1-\rho_{k_\ell}(\tau_1)|^{1/2}$. Under the mapping T, the circle $\tilde{c}$ is mapped to the small circle c:

$$|w-1| = |z||\rho-1|/|z-\rho| < 4|1-\rho|^{1/2} ,$$

where $w = T(z)$. $\rho - 1$ is sufficiently small so we have the following properties:

(i) ρ is contained in the interior to c.

(ii) The defining curves of the group $\tilde{G}_\ell(\tau_1) = \langle A_{\ell(1)}(\tau_1, z), \cdots, A_{\ell(g_\ell)}(\tau_1, z)\rangle = \langle A_{\ell(1)}(\tau, z), \cdots, A_{\ell(g_\ell)}(\tau, z)\rangle$ lie in the exterior to c.

We set $\tilde{G}_{\ell'}(\tau_1) = T(G_{\ell'}(\tau_1))T^{-1}$. Then the multiplier $\lambda_{\ell'(j)}(\tau_1)$ of

$TA_{\ell'(j)}(\tau_1, z)T^{-1}$ $(j = 1, 2, \cdots, g)$ satisfies $|\lambda_{\ell'(j)}(\tau_1)| > 1$. Furthermore $T(C_{\ell'(1)})$, $T(C'_{\ell'(1)}, \cdots, T(C'_{\ell'(g_{\ell'})})$ are defining curves of $\tilde{G}_{\ell'}(\tau_1)$ and are contained in the interior to c. Thus the free product of $\tilde{G}_\ell(\tau_1)$ and $\tilde{G}_{\ell'}(\tau_1)$ is a Schottky group by Maskit's Combination Theorem (we denote it by $G_\ell(\tau_1)$). Hence $\tau_1 \in \hat{\delta}^L \mathfrak{S}_g(\Sigma)$.

Next we consider the case $g_\ell = 0$ or $g_{\ell'} = 0$. For example, let $g_\ell = 0$ and $g_{\ell'} = 0$. Then $G_\ell(\tau)$, $G_\ell(\tau_1)$, $G_{\ell'}(\tau)$, and $G_{\ell'}(\tau_1)$ are all the trivial groups. We normalize $G_\ell(\tau_1)$, $G_{\ell'}(\tau_1)$, $\tilde{p}_{\ell(1)}(\tau_1) = 0$, $\tilde{p}_{\ell(2)}(\tau_1) = \infty$, and $\tilde{p}^-_{k_\ell}(\tau) = 1$; $\tilde{p}_{\ell'(1)}(\tau_1) = 0$, $\tilde{p}_{\ell'(2)}(\tau_1) = \infty$ and $\tilde{p}^+_{k_\ell}(\tau) = 1$. Since τ_1 is sufficiently close to τ, the distinguished points $\tilde{p}_{\ell'(2)}(\tau_1), \cdots, \tilde{p}_{\ell'(n_{\ell'})}(\tau_1)$ lie in the exterior to a small circle c around the point 1. We determine $T \in$ Möb by setting $T(0) = 1$, $T(\infty) = \rho_{k_\ell}(\tau_1)$, and $T(1) = 0$. By a similar way to the above, we can show that $\tau_1 \in \hat{\delta}^L \mathfrak{S}_g(\Sigma)$.

Finally if we note that $\hat{\delta}^L \mathfrak{S}_g(\Sigma)$ is open, which is easily seen, we see $\tau' \in \hat{\delta} \mathfrak{S}_g(\Sigma)$ from the fact $\tau_1 \in \hat{\delta}^L \mathfrak{S}_g(\Sigma)$ and the following: the difference between τ' and τ_1 is small except ρ_k for $k \in J$ and $\rho_k(\tau') = \rho_k(\tau_1)$ for $k \in J$.

In general, if $L \subset J$, we perform the above operation $|J| - |L|$ times. Then we see that $\tau' \in \hat{\delta}^L \mathfrak{S}_g(\Sigma)$. Thus $\hat{\mathfrak{S}}_g(\Sigma)$ is open.

That $\hat{\mathfrak{S}}_g(\Sigma)$ is connected is seen from the following well-known fact: If A is connected and $A \subset B \subset \bar{A}$, then B is connected.

Since the other parts of the proposition are easily proved, we omit the proof in detail here. Our proof is now complete.

6. Augmented Schottky spaces and Riemann surfaces

6–1. Throughout this section, let a marked Schottky group $G_0 = \langle A_{10}, \cdots, A_{g0} \rangle$ be given and let a standard system of loops Σ on $S_0 = \Omega(G_0)/G_0$ be given. Let S be a marked Riemann surface with nodes. We call the set $\Sigma' = \{\alpha'_1, \cdots, \alpha'_g; \gamma'_1, \cdots, \gamma'_{2g-3}\}$ of loops and nodes on S satisfying the following condition a standard system of loops and nodes: Each component of $S - \bigcup_{j=1}^{g} \alpha'_j - \bigcup_{j=1}^{2g-3} \gamma'_j$ is a planar and triply connected region of type [3, 0], [2, 1], [1, 2], and [0, 3], where a surface of type $[m, n]$ means a sphere with m disks removed and n points delated. From now on we only consider standard systems of loops and nodes satisfying the following condition: Each γ'_j $(j = 1, 2, \cdots, 2g-3)$ gives the same partition of $\{1, 2, \cdots, g\}$ as γ_j.

6–2. (1) Let $\tau \in \mathfrak{S}_g(\Sigma)$. In this case, it is well-known that $S(\tau) =$

$\Omega(G(\tau))/G(\tau)$ is a compact Riemann surface without nodes of genus g.

(2) Let $\tau \in \delta^I \mathfrak{S}_g(\Sigma)$ $(I \neq \phi)$. By the same way as in § 5, we can determine

$$p_1(\tau) = 0,\ q_1(\tau) = \infty,\ p_2(\tau) = 1,\ q_2(\tau), \cdots, p_g(\tau),\ q_g(\tau)$$

from τ. Let $G(\tau)$ be the Schottky group determined from τ (see § 5). Then $S(\tau) = \Omega(G(\tau))/G(\tau)$ is a compact Riemann surface of genus $g - |I|$ on which there are $|I|$ distinguished pairs of points: $\hat{p}_i(\tau)$, $\hat{q}_i(\tau)$, $i \in I$, where $\hat{p}_i(\tau)$ is the image of $p_i(\tau)$ under the mapping $\Omega(G(\tau)) \to S(\tau)$ and $\hat{q}_i(\tau)$ the image of $q_i(\tau)$. We denote by $\hat{S}(\tau)$ the Riemann surface $S(\tau)$ with the points $\hat{p}_i(\tau)$ and $\hat{q}_i(\tau)$ identified for every $i \in I$. Then $\hat{S}(\tau)$ is a compact Riemann surface of genus g with $|I|$ non-dividing nodes.

(3) Let $\tau \in \hat{\delta}^J \mathfrak{S}_g(\Sigma)$ $(J \neq \phi)$. Let $J = \{k_1, \cdots, k_m\}$. By the same way as in the previous section, we have $m + 1$ Schottky groups (including the trivial group) $G_0(\tau), G_1(\tau), \cdots, G_m(\tau)$. Then each $\Omega(G_\ell(\tau))/G_\ell(\tau) = S_\ell(\tau)$ $(\ell = 0, 1, \cdots, m)$ is a compact Riemann surface of genus g_ℓ on which there are n_ℓ distinguished points $\hat{p}_{\ell(1)}, \cdots, \hat{p}_{\ell(n_\ell)}$. Exactly in the same way as in § 4, we consider the following sequence: $\cdots, \rho(1, i_1, \cdots, i_{\mu_{\ell-1}}) = 1$; $\rho(1, i_1, \cdots, i_{\mu_{\ell-1}+1}) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_\ell - 1}) \neq 1, \rho(1, i_1, \cdots, i_{\mu_\ell}) = 1$; $\rho(1, i_1, \cdots, i_{\mu_\ell+1}) \neq 1, \cdots, \rho(1, i_1, \cdots, i_{\mu_{\ell+1}-1}) \neq 1, \rho(1, i_1, \cdots, i_{\mu_{\ell+1}}) = 1$ (or $\rho(1, i_1, \cdots, i_{\mu_{\ell+1}}) \neq 1$ if $\gamma(1, i_1, \cdots, i_{\mu_{\ell+1}})$ is terminal).

Let $\gamma_{k_\ell} = \gamma(1, i_1, \cdots, i_{\mu_\ell})$ and $\gamma_{k_{\ell'}} = \gamma(1, i_1, \cdots, i_{\mu_{\ell-1}})$. Then we denote by $\hat{p}^+_{k_\ell}$ (resp. $\hat{p}^-_{k_\ell}$) the image of right (resp. left) distinguished point $\tilde{p}^+_{k_\ell}$ (resp. $\tilde{p}^-_{k_\ell}$) under the natural projection $\Omega(G_{k_{\ell'}}(\tau)) \to S_{k_{\ell'}}(\tau)$ (resp. $\Omega(G_{k_\ell}(\tau)) \to S_{k_\ell}(\tau)$). By identifying $\hat{p}^+_{k_\ell}$ with $\hat{p}^-_{k_\ell}$ we connect $S_{k_\ell}(\tau)$ and $S_{k_{\ell'}}(\tau)$. The resulting surface is expressed as $S_{k_{\ell'}}(\tau) + S_{k_\ell}(\tau)$. We perform this operation for each $S_0(\tau), \cdots, S_{k_m}(\tau)$ and each distinguished pairs of points (for $S_0(\tau)$, we need a trivial modification). At last we have a compact Riemann surface

$$S(\tau) = S_0(\tau) + S_{k_1}(\tau) + \cdots + S_{k_m}(\tau)$$

of genus $g_0 + g_1 + \cdots + g_m$ $(=g)$ with m dividing nodes.

(4) Let $\tau \in \delta^{I,J} \mathfrak{S}_g(\tau)$ $(I \neq \phi, J \neq \phi)$. By using the above methods (2) and (3), we obtain a compact Riemann surface of genus g with $|J|$ dividing nodes and $|I|$ nondividing nodes:

$$S(\tau) = \hat{S}_0(\tau) + \hat{S}_{k_1}(\tau) + \cdots + \hat{S}_{k_m}(\tau)\,.$$

We call $S(\tau)$ the Riemann surface with nodes associated with τ.

6–3. Next we consider the converse. Let S be a compact Riemann surface of genus g without (with) nodes. Suppose there is a standard system of loops (and nodes) $\Sigma' = \{\alpha'_1, \cdots, \alpha'_g; \gamma'_1, \cdots, \gamma'_{2g-3}\}$ on S such that each γ'_j gives the same partition of $\{1, 2, \cdots, g\}$ as $\gamma_j \in \Sigma$. We call Σ' is compatible with Σ and denote by $\Sigma' \sim \Sigma$. We denote by (S, Σ') a pair of the above S and Σ'.

(1) Let S be a compact Riemann surface of genus g without nodes and let $\Sigma' \sim \Sigma$. Then it is well-known that there exists a $\tau \in \mathfrak{S}_g(\Sigma)$ with $\Omega(G(\tau))/G(\tau) = S$.

(2) Let S be a compact Riemann surface of genus g with only non-dividing nodes and $\Sigma' \sim \Sigma$. Let $I = \{i_1, \cdots, i_\ell\} \subset \{1, 2, \cdots, g\}$ be the set of i such that α'_i are non-dividing nodes on S. Along each α'_{i_k} $(k = 1, 2, \cdots, \ell)$, we cut off S and then we have a Riemann surface $\tilde{S}$ with $|I|$ pairs of punctures. We attach a point to each puncture. Then we have a compact Riemann surface $\hat{S}$ of genus $g - |I|$. The 2ℓ attached points are denoted by $\hat{p}_{i_1}, \hat{q}_{i_1}, \cdots, \hat{p}_{i_\ell}, \hat{q}_{i_\ell}$. We regard that each α'_i $(i \notin I)$ is a loop on $\hat{S}$. Let G be the marked Schottky group $G = \langle A_i | i \notin I \rangle$ having a standard fundamental region ω whose defining curves are projected to α'_i $(i \notin I)$. Let λ_i, p_i, and q_i $(i \notin I)$ be the multiplier, the repelling, and the attracting fixed points of $A_i(z)$, respectively. We denote by $\tilde{p}_{i_1}, \tilde{q}_{i_1}, \cdots, \tilde{p}_{i_\ell}, \tilde{q}_{i_\ell}$ the lifts of $\hat{p}_{i_1}, \hat{q}_{i_1}, \cdots, \hat{p}_{i_\ell}, \hat{q}_{i_\ell}$ in the interior to ω. Then by the same way as in § 4, we can determine $\rho_1, \cdots, \rho_{2g-3} \in \boldsymbol{C} - \{0, 1\}$ from p_i, q_i $(i \notin I)$ and $\tilde{p}_i, \tilde{q}_i$ $(i \in I)$ corresponding to the given partitions of $\{1, 2, \cdots, g\}$ associated with Σ. We set $t_i = 1/\lambda_i$ $(i \notin I)$ and $t_i = 0$ $(i \in I)$. Then, from the given S we obtain

$$\tau_S = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in D^g \times (\boldsymbol{C} - \{0, 1\})^{2g-3}$$

such that $\Omega(G(\tau_S))/G(\tau_S) = S$. Obviously $\tau_S \in \delta^I \mathfrak{S}_g(\Sigma)$.

Then we have the following. For the given (S, Σ') we determine a Schottky group G representing $\hat{S}$ and distinguished points $\hat{p}_i, \hat{q}_i$ $(i \in I)$ on $\hat{S}$.

Remark. The determination of $\tilde{p}_i$ and $\tilde{q}_i$ $(i \in I)$ from $\hat{p}_i$ and $\hat{q}_i$ is not unique. Thus τ_S is not uniquely determined from S. It depends on a choice of a standard fundamental region ω for G. We show it by the following example. Let $g = 2$ and $I = \{2\}$. Then $t_2 = 0$. Let $p_1 = 0$, $q_1 = \infty$ and $t_1 = 1/4$. We may $\tilde{p}_2 = 1$. Let ω_1 and ω_2 be standard fundamental regions for G as in Fig. 6 and Fig. 7, respectively. We take q_2

$= 2$ in Case 1. Then $q_2 = 1/2$ in Case 2. Thus

$$\tau_S = \begin{cases} (4, 0; 2) & \text{in Case 1} \\ (4, 0; 1/2) & \text{in Case 2.} \end{cases}$$

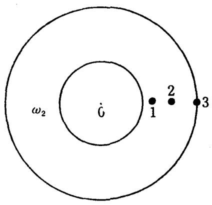

Fig. 6 (Case 1)

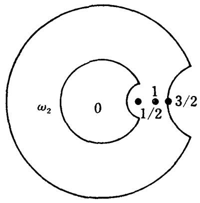

Fig. 7 (Case 2)

(3) Let S be a compact Riemann surface of genus g with only m dividing nodes and $\Sigma' \sim \Sigma$. Let $J = \{k_1, \cdots, k_m\} \subset \{1, 2, \cdots, 2g-3\}$ be the set of $\{k_\ell\}$ such that γ'_{k_ℓ} are dividing nodes on S. Along each γ_{k_ℓ} ($\ell = 1, 2, \cdots, m$) we cut off S and obtain $m+1$ Riemann surfaces, $S_0, S_1, \cdots, S_m$, with punctures. We attached a point to each puncture. Then we have compact Riemann surfaces S_ℓ ($\ell = 0, 1, \cdots, m$). We call the attached points distinguished points. Let the "α-loops" on each S_ℓ be $\alpha_{\ell(1)}, \cdots, \alpha_{\ell(g_\ell)}$. Let G_ℓ be the marked Schottky group $G_\ell = \langle A_{\ell(1)}, \cdots, A_{\ell(g_\ell)} \rangle$ having a standard fundamental region ω_ℓ whose defining curves are projected to $\alpha_{\ell(i)}$. We denote by $\tilde{p}_{\ell(1)}, \cdots, \tilde{p}_{\ell(n_\ell)}$ the lifts of $\hat{p}_{\ell(1)}, \cdots, \hat{p}_{\ell(n_\ell)}$, respectively to the interior of a standard fundamental region ω_ℓ. Let $\lambda_{\ell(j)}, p_{\ell(j)}$, and $q_{\ell(j)}$ be the multiplier, the repelling, and the attracting fixed points of $A_{\ell(j)}(z)$. By the process opposite to one in § 5, we determine h_ℓ numbers of ρ from $p_{\ell(j)}, q_{\ell(j)}$ ($j = 1, 2, \cdots, g_\ell$) and $\hat{p}_{\ell(j)}$ ($j = 1, 2, \cdots, n_\ell$). We easily see that $h_\ell = 2g_\ell + n_\ell - 3$. We set $t_{\ell(j)} = 1/\lambda_{\ell(j)}$ ($j = 1, 2, \cdots, g_\ell$; $\ell = 0, 1, \cdots, m$). Thus we determine $h_0 + h_1 + \cdots + h_m$ numbers of ρ. We set $\rho_j = 1$ for $j \in J$. By the above things, we determine $h_0 + h_1 + \cdots + h_m + m$ numbers of ρ. We have

$$\sum_{\ell=0}^{m} h_\ell + m = \sum_{\ell=0}^{m} (2g_\ell + n_\ell - 3) + m = 2g - 3 .$$

Thus from the given S and Σ', we determine

$$\tau_S = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in D^{*g} \times (C - \{0\})^{2g-3}$$

such that S coincides with the Riemann surface associated with τ_S. It is easily seen that $\tau_S \in \hat{\partial}^J \mathfrak{S}_g(\Sigma)$. In this case, we note that the same remark as in the previous case (2) holds.

(4) Let S be a compact Riemann surface of genus g with ℓ non-dividing nodes and m dividing nodes. Then by a method similar to the above two cases (2) and (3), we can determine

$$\tau_S = (t_1, \cdots, t_g; \rho_1, \cdots, \rho_{2g-3}) \in D^g \times (C - \{0\})^{2g-3}$$

from the given S such that $\tau_S \in \delta^{I,J}\mathfrak{S}_g(\Sigma)$ and S coincides with the Riemann surface associated with τ_S.

6–4. By collecting the above results, we have the following theorem.

THEOREM 2. (1) *For $\tau \in \delta^{I,J}\mathfrak{S}_g(\Sigma)$, there exists a Riemann surface S of genus g with $|I|$ non-dividing nodes and $|J|$ dividing nodes associated with τ in the sense of* 6–2.

(2) *Conversely, give any (S, Σ'), where S is a compact Riemann surface of genus g with $|I|$ non-dividing nodes and $|J|$ dividing nodes and Σ' is a standard system of loops and nodes on S compatible with Σ. Then there exists a $\tau \in \delta^{I,J}\mathfrak{S}_g(\Sigma)$ such that S coincides with the Riemann surface associated with τ in the sense of* 6–2.

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