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Nagoya Math. J.
Vol. 75 (1979), 145-150

A GEOMETRIC CHARACTERIZATION OF C^n AND OPEN BALLS

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Introduction

The purpose of this paper is to give a result concerning the problem of geometric characterizations of the Euclidean n-space C^n and bounded domains. It is well known that a simply connected Riemann surface is biholomorphic to one of the Riemann sphere, the complex plane and the unit disc. And there are several results concerning the geometric characterization of these spaces. To show that some simply connected open Riemann surface is biholomorphic to the complex plane or the unit disc, it is sufficient to see that there exist non constant bounded subharmonic functions or not. But in the higher dimensional case, there is no uniformization theorem. By this reason to show that some complex manifold is biholomorphic to C^n or an open ball, we must construct a biholomorphic mapping directly.

The following problems are given by R. E. Greene and H. Wu in [1].

PROBLEM 1. If a complete Kähler manifold M is diffeomorphic to R^{2n} and its sectional curvature is non positive and larger than $-A/r^{2+\varepsilon}$ outside a compact set, then is M biholomorphic to C^n? Here r is the distance function from a fixed point of M and ε, A are some positive constants.

PROBLEM 2. If a complete Kähler manifold M is diffeomorphic to R^{2n} and its sectional curvature is non positive and smaller than $-A/r^2$ outside a compact set, then is M biholomorphic to a bounded domain?

In this paper we consider the above problems under very restrictive conditions, using a theorem of J. Milnor [4].

Received August 21, 1978.

1. A theorem of J. Milnor

In this section we recall a theorem of J. Milnor and his construction of a holomorphic mapping for the later use. Let M be a simply connected open Riemann surface with a complete hermitian metric g and o be a point of M. Let $\exp: M_o \to M$ be the exponential mapping with respect to g, where M_o is the tangent space at o. We assume exp is a diffeomorphism and g is rotationally symmetric at o, *i.e.*, every rotation of M_o is induced by an isometry of M. In the geodesic polar coordinate system $\{r, \theta\}, g$ takes the form $g = dr^2 + f(r)^2 d\theta^2$. The Gauss curvature depends only on r and we denote it by $K(r)$. The function f satisfies the following Jacobi equation,

$f''(r) = -K(r)f(r)$, with initial conditions $f(0) = 0$ and $f'(0) = 1$.

Introduce a new coordinate $\rho(r)$ in place of r by setting $\rho(r) = \int_1^r \frac{ds}{f(s)}$. Then the metric takes the form

$$g = dr^2 + f(r)^2 d\theta^2 = f(r)^2(d\rho^2 + d\theta^2) .$$

Hence $\rho + i\theta$ is a holomorphic coordinate on $M - \{o\}$. We define a holomorphic mapping $\Phi: M - \{o\} \to M_o$ by $\Phi(\rho + i\theta) = e^{\rho + i\theta}$. We can easily show that o is a removable singularity of Φ and Φ is a biholomorphic mapping from M into M_o. If $\int_1^\infty \frac{ds}{f(s)} = \infty$, M is biholomorphic to C and if $\int_1^\infty \frac{ds}{f(s)} < \infty$, M is biholomorphic to an open disc.

THEOREM (J. Milnor [4]). *Let M be a simply connected open Riemann surface with a complete hermitian metric. We assume the metric is rotationally symmetric at a point o of M. We denote by r the distance from o and by $K(r)$ the Gauss curvature.*

1. *If $K(r) \geqq -1/r^2 \log r$, for large r, then M is biholomorphic to C.*
2. *If $K(r) \leqq -(1 + \varepsilon)/r^2 \log r$ for large r and if $f(r)$ is not bounded, then M is biholomorphic to an open disc. Here ε is some positive constant.*

Greene and Wu [2] generalized this theorem to the case when the metric is not necessarily rotationally symmetric.

2. Theorem and its proof

Let M be an n-dimensional complete Kähler manifold and o be a point of M. We assume the exponential mapping $\exp: M_o \to M$ is a diffeomorphism. We call these manifolds *Kähler manifolds with a pole* (Greene and Wu [2]). We consider the hermitian inner product on M_o induced from the Kähler metric on M. We denote by $U(M_o)$ the unitary transformation group of M_o with respect to this inner product.

DEFINITION. A Kähler manifold with a pole (M, o) is a *Kählerian model* iff every $\phi \in U(M_o)$ is realized as the differential of an isometry Φ of M, *i.e.*, $\Phi(o) = o$ and $\Phi_{*o} = \phi$.

LEMMA 1. *Let (M, o) be a Kählerian model and Π_o be a complex linear subspace of M_o. Then $\Pi = \exp \Pi_o$ is a totally geodesic complex submanifold of M.*

Proof. There exists $\phi \in U(M_o)$ such that $\Pi_o = \{u \in M_o | \phi(u) = u\}$. Let Φ be the isometry of M such that $\Phi(o) = o$ and $\Phi_{*o} = \phi$. Then Π is just the fixed point set of Φ. Since Π is the fixed point set of an isometry, Π is a totally geodesic submanifold (c.f. S. Kobayashi [3]). Let p be a point of Π and γ be the geodesic joining o to p. We denote by Π_p the tangent space of Π at p. Let $X \in \Pi_p$ and $\tilde{X}$ be the parallel translation of X along γ. Since Π is totally geodesic, $\tilde{X}_0 \in \Pi_0$. Since the complex structure J is parallel, $J\tilde{X}$ is parallel along γ. On the other hand $J\tilde{X}_0 \in \Pi_0$ and Π is totally geodesic, and so $\tilde{X}$ and $J\tilde{X}$ are tangent to Π. This means $JX \in \Pi_p$. Thus Π_p is invariant under J for every point of $p \in \Pi$. Consequently Π is a complex submanifold of M.

Let (M, o) be a Kählerian model. For a point $p \in M - \{o\}$, there exists a unique geodesic $\gamma: [0, b] \to M$ with arc length parameter such that $\gamma(0) = o$ and $\gamma(b) = p$. We define a vector field ∂ on $M - \{o\}$ by $\partial(p) = \dot{\gamma}(b)$, and we call this vector field the radial vector field. And we consider the holomorphic sectional curvature in the direction ∂, *i.e.*, the sectional curvature of the tangent plane spanned by ∂ and $J\partial$, and we call it *the holomorphic radial curvature*. Since (M, o) is a Kählerian model, the holomorphic radial curvature depends only on r where r is the distance from the point o (Lemma 1), and we denote it by $K(r)$. Now, we shall prove

THEOREM. *Let (M, o) be a Kählerian model and $K(r)$ be the holomorphic radial curvature. Then*

(1). *If* $K(r) \geqq -1/r^2 \log r$ *for large* r, *then* M *is biholomorphic to the Euclidean n-space* C^n.

(2). *If* $K(r)$ *is non positive and* $K(r) \leqq -(1+\varepsilon)/r^2 \log r$ *for large* r, *where* ε *is some positive constant, then* M *is biholomorphic to an open ball in* C^n.

Proof. Let Π_o be a 1-dimensional complex subspace of M_o. Then $\Pi = \exp \Pi_o$ is a totally geodesic complex submanifold of M (Lemma 1). The curvature with respect to the induced metric is equal to the curvature with respect to the metric of M. In the geodesic polar coordinate $\{r, \theta\}$ on Π, the induced metric takes the form $dr^2 + f(r)^2 d\theta^2$, since the induced metric is rotationally symmetric at o. Put $\rho(r) = \int_1^r \frac{ds}{f(s)}$ and $\mu(r) = e^{\rho(r)}$.

Let $\{\tilde{r}, \tilde{\Theta}\}$ be the polar coordinate system of M_o and $\{r, \Theta\}$ be the corresponding geodesic polar coordinate system of M, where $\tilde{\Theta}$ and Θ are spherical coordinates. They are related by $\tilde{r} = r \circ \exp.$ and $\tilde{\Theta} = \Theta \circ \exp.$

Now we define a mapping $\Phi : M - \{o\} \to M_o$ by

$$\tilde{r} = \mu(r) \quad \text{and} \quad \tilde{\Theta} = \Theta .$$

LEMMA 2. *The mapping* Φ *is holomorphic.*

Proof. First we consider the case of $n = 2$. Let J and J_o be the complex structure on M and M_o respectively. We shall prove $\Phi_*(JX) = J_o\Phi_*(X)$ for all $X \in M_p$ and $p \in M - \{o\}$. Take an orthonormal linear coordinate system $\{z_1, z_2\}$ on M_o. Put $z_i = x_i + iy_i$, then $\{x_1, y_1, x_2, y_2\}$ is a real linear orthonormal coordinate system on M_0. We consider $\{x_1, y_1, x_2, y_2\}$ also as a coordinate system of M via the exponential mapping, the normal coordinate system of M at o. Let p be a point of $M - \{o\}$ and Π_o be the 1-dimensional subspace of M such that $\Pi = \exp \Pi_o$ contains p. We assume Π_o is the subspace $\{z_2 = 0\}$ of M_o without loss of generality. Π is a totally geodesic complex submanifold of M and $\Phi|_\Pi : \Pi - \{o\} \to \Pi_o$, the restriction of Φ to $\Pi - \{o\}$ is the same with the mapping constructed by Milnor. Then $\Phi|_\Pi$ is holomorphic and $\Phi_*(JX) = J_0\Phi_*(X)$ for any vector $X \in \Pi_p$. Next we consider the case $X \in \Pi_p^\perp$ where $\Pi_p^\perp$ is the orthogonal complement of Π_p with respect to the Kähler metric on M. It is clear that Π_p is spanned by $(\partial/\partial x_1)_p, (\partial/\partial y_1)_p$. First we see that $\Pi_p^\perp$ is spanned by $(\partial/\partial x_2)_p, (\partial/\partial y_2)_p$. Obviously $\Pi_o^\perp$ is spanned by $(\partial/\partial x_2)_o, (\partial/\partial y_2)_o$. Let γ be the geodesic joining o to p, then γ is contained in Π. If we consider a unitary transformation ϕ_θ of the form

$$\phi_\theta = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{bmatrix},$$

$\Pi_o^\perp$ is invariant under ϕ_θ. There is an isometry of Φ_θ of M such that $\Phi_\theta(o) = o, \Phi_{\theta*o} = \phi_\theta$ by the assumption of the theorem. ϕ_θ and Φ_θ operate in the same manner on M_o and M through the exponential mapping. $\Pi_p^\perp$ is obtained by the parallel translation of $\Pi_o^\perp$ along γ. If $\tilde{X}$ is a parallel vector field along γ then $\Phi_{\theta*}\tilde{X}$ is also parallel along γ. Hence $\Pi_p^\perp$ is invariant under $\Phi_{\theta*}$. On the other hand the invariant subspace under $\Phi_{\theta*}$ complemental to Π_p is spanned by $(\partial/\partial x_2)_p, (\partial/\partial y_2)_p$. This means that $\Pi_p^\perp$ is spanned by $(\partial/\partial x_2)_p, (\partial/\partial y_2)_p$.

The restriction of the complex structure J to $\Pi_o^\perp$ is $\phi_{\pi/2}$. Let $X \in \Pi_o^\perp$ and $\tilde{X}$ be the parallel vector field along γ such that $\tilde{X}_o = X$. Since $J\tilde{X}$ and $\Phi_{\pi/2*}\tilde{X}$ are parallel along γ with the same value at o, we have $J\tilde{X} = \Phi_{\pi/2*}\tilde{X}$. More explicitly $J(\partial/\partial x_2)_p = (\partial/\partial y_2)_p$ and $J(\partial/\partial y_2)_p = -(\partial/\partial x_2)_p$. Let S_b be the geodesic sphere of radius b around o and $\tilde{S}_b$ be the sphere in M_o. Since Φ maps S_b into $\tilde{S}_{\mu(b)}$ and preserves the coordinate of spherical part, we have

$$\Phi_*(\partial/\partial x_2)_p = \mu(b)/b \cdot \partial/\partial x_2 , \qquad \Phi_*(\partial/\partial y_2)_p = \mu(b)/b \cdot \partial/\partial y_2 ,$$

Since $J_o \cdot \partial/\partial x_2 = \partial/\partial y_2$, $J_o \cdot \partial/\partial y_2 = -\partial/\partial x_2$ and $J(\partial/\partial x_2)_p = (\partial/\partial y_2)_p$, $J(\partial/\partial y_2)_p = -(\partial/\partial x_2)_p$, we have

$$\Phi_*(J(\partial/\partial x_2)_p) = J_o\Phi_*(\partial/\partial x_2)_p , \qquad \Phi_*(J(\partial/\partial y_2)_p) = J_o\Phi_*(\partial/\partial y_2)_p .$$

This means $J_o\Phi_*(X) = \Phi_*(JX)$ for all $X \in \Pi_p^\perp$. This completes the proof of the lemma in the case of $n = 2$.

Next we consider the case $n \geqq 3$. Let p be a point of $M - \{o\}$ and X be a tangent vector at p. There is a 2-dimensional complex subspace Π_o of M_o such that $\Pi = \exp \Pi_o$ contains p and X is tangent to Π. Π is a complex submanifold and (Π, o) is a 2-dimensional Kählerian model with respect to the induced metric. Then $\Phi|_\Pi$ is holomorphic. Hence $\Phi_*(JX) = J_o\Phi_*(X)$, and this completes the proof of Lemma 2.

We continue the proof of the theorem. Since Φ is holomorphic and o is a removable singularity, we can extend Φ across o. Obviously Φ is a one to one mapping. So M is biholomorphic to $\Phi(M)$. In the case of

(1), $\lim_{r\to\infty}\mu(r)=\infty$, then $\Phi(M)=M_o$. In the case of (2), $f(r)$ is larger than r, since $f(r)$ satisfies the Jacobi equation. Hence $f(r)$ is unbounded. By the theorem of Milnor $\mu(r)$ is bounded and $\Phi(M)$ is an open ball of radius $\lim_{r\to\infty}\mu(r)$. This completes the proof of the theorem.

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