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HYPERBOLIC NONWANDERING SETS WITHOUT DENSE PERIODIC POINTS

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In this paper we give a negative answer to the problem which is suggested in [3]: if a nonwandering set Ω is hyperbolic, are the periodic points dense in Ω?

Newhouse and Palis proved that on two dimensional closed manifolds the answer is positive ([1], [2]).

Suppose that $f: M \to M$ is a diffeomorphism of a manifold M. A point $x \in M$ is a nonwandering point of f if for any neighbourhood $U \subset M$ of x there is a positive integer n such that $f^n(U) \cap U \neq \emptyset$. $\Omega = \{$nonwandering points of $f\}$ is called the nonwandering set of f. A point of $M - \Omega$ is a wandering point. A nonwandering set Ω of f is hyperbolic if Ω is compact and $TM|\Omega$ splits into a Whitney sum of Tf-invariant subbundles

$$TM|\Omega = E^s \oplus E^u ,$$

and there are $c > 0, 0 < \lambda < 1$ such that

$$\|Tf^n v\| \leqq c\lambda^n \|v\| \qquad \text{if } v \in E^s$$

and

$$\|Tf^{-n} v\| \leqq c\lambda^n \|v\| \qquad \text{if } v \in E^u$$

for $n > 0$.

We will prove the following.

THEOREM. *Suppose that M is a manifold with* $\dim M \geqq 4$. *Then there is a diffeomorphism $F: M \to M$ such that the nonwandering set Ω is hyperbolic but periodic points of F are not dense in Ω.*

Proof. **0.** An outline of Proof. To simplify the proof, we assume $\dim M = 4$. In 1 we construct an embedding of 2-dimensional disk $f: D$

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$\to D$, where f has a hyperbolic set consisting of finite fixed points and two non-periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ (13.2, 13.3). In 2 ~ 10 we will extend f to an embedding $F: N \to N$, where $M \supset N = D \times D^2 \cup$ 1-*handle*. The non-wandering set of F consists of a finite number of fixed points and the two orbits $\mathcal{O}_1$, $\mathcal{O}_2$, where points nearby $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) return near by $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) through the 1-handle (15). And other points are wandering (12, 14). Finally we extend F to a diffeomorphism of M.

1.

Let

$$D = [-2, 6] \times [-1, 3] \subset \boldsymbol{R}^2$$

and an embedding $f: D \to D$ satisfy the followings (Figure 1). Suppose that real numbers $a_{-1}, \cdots, a_6$ satisfy

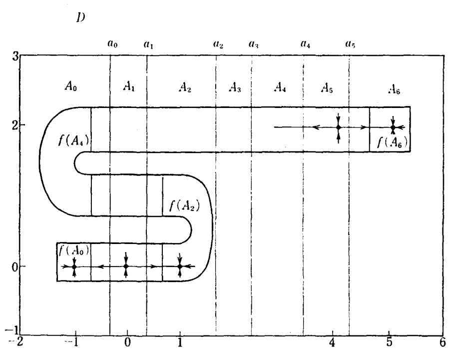

Figure 1

$$\text{(1.1)} \qquad \begin{aligned} a_{-1} = -2 < -1 < a_0 = -a_1 < 0 < a_1 < 1 < a_2 < a_3 \\ < a_4 < 4 < a_5 < 5 < a_6 = 6 \,, \end{aligned}$$

and the rectangle A_i $(i = 0, \cdots, 6)$ is given by

$$A_i = \{(x, y) \in D \mid a_{i-1} \leqq y \leqq a_i\} \,.$$

Then f satisfies (1.2) ~ (1.5).

(1.2) $f|A_0, f|A_2$ and $f|A_6$ are contractions with three sinks $(-1, 0)$, $(1, 0)$, $(5, 2)$,

(1.3) $f(A_4) \subset \operatorname{int} A_0$,

(1.4) $f|A_i: A_i \to f(A_i)$ $(i = 1, 3, 5)$ maps A_i linearly onto a rectangle $f(A_i)$, expanding the x-direction and contracting the y-direction. There are two hyperbolic fixed points $(0, 0)$ and $(4, 2)$.

(1.5) There are numbers $\alpha > 1$ and $0 < \beta < 1$ such that

$$f(x, y) = \begin{cases} (\alpha x, \beta y) & \text{for } (x, y) \in A_1 \\ (\alpha(x-4)+4, \beta(y-2)+2) & \text{for } (x, y) \in A_5 . \end{cases}$$

2.

Let $D' \subset R^2$ satisfy the followings (Figure 2). D' is a neighbourhood of $(\{0\} \times [-1, 1]) \cup ([-2, 0] \times \{0\})$ which is diffeomorphic to a 2-dimensional disk, and there is a sufficiently small positive number ε such that

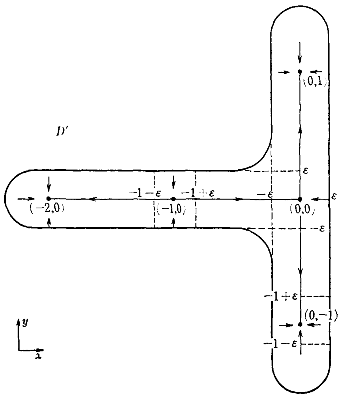

Figure 2

$$\{(x, y) \in D' \,|\, |y+1| \leqq \varepsilon\} = [-\varepsilon, \varepsilon] \times [-1-\varepsilon, -1+\varepsilon] .$$

and

$$\{(x, y) \in D' \,|\, |x+1| \leqq \varepsilon\} = [-1-\varepsilon, -1+\varepsilon] \times [-\varepsilon, \varepsilon] .$$

Let an embedding $g: D' \to D'$ satisfy (2.1) ~ (2.9).

(2.1) $g(D') \subset \operatorname{int} D'$,

(2.2) g is isotopic to the identity,

(2.3) $\bigcup_{n>0} g^n(D') = (\{0\} \times [-1, 1]) \cup ([-2, 0] \times \{0\})$,

(2.4) there are five fixed points, that is, three sinks $(-2, 0)$, $(0, 1)$, $(0, -1)$, and two saddle points $(0, 0)$, $(-1, 0)$,

(2.5) $W^u((0, 0)) = \{0\} \times (-1, 1)$,

(2.6) $W^u((-1, 0)) = (-2, 0) \times \{0\}$,

(2.7) $W^s((0, 0)) \cap D' = \{(x, 0) \in D' \,|\, -1 < x\}$,

where $W^s(p)$ (resp. $W^u(p)$) is the stable (resp. unstable) manifold through p. $(-1, 1)$ and $(-2, 0)$ denote open intervals.

(2.8) $g(x, y) = (\frac{1}{2}x, \frac{1}{2}(y+1) - 1)$ if $|y+1| \leqq \varepsilon$, (2.9) $g(x, y) = (2(x+1) - 1, \frac{1}{2}y)$ if $|x+1| \leqq \varepsilon$.

3.

Define

$$N = D \times D' \cup_{\psi} D^3(\delta) \times [0, 1]\,,$$

where

$$D^3(\delta) = \{(y_1, y_2, y_3) \in \boldsymbol{R}^3 \mid \sqrt{y_1^2 + y_2^2 + y_3^2} \leqq \delta\}$$

and

$$0 < \delta < \tfrac{1}{4}\varepsilon\,.$$

The attaching map

$$\psi\colon D^3(\delta) \times ([0, \varepsilon] \cup [1-\varepsilon, 1]) \to D \times D'$$

is given by

$$\psi(y_1, y_2, y_3, t) = \begin{cases} (y_1, y_2, t, y_3 - 1) & \text{if } 0 \leqq t \leqq \varepsilon \\ (y_1 + 5, y_2 + 2, y_3 - 1, 1 - t) & \text{if } 1 - \varepsilon \leqq t \leqq 1 \end{cases}$$

(Figure 3).

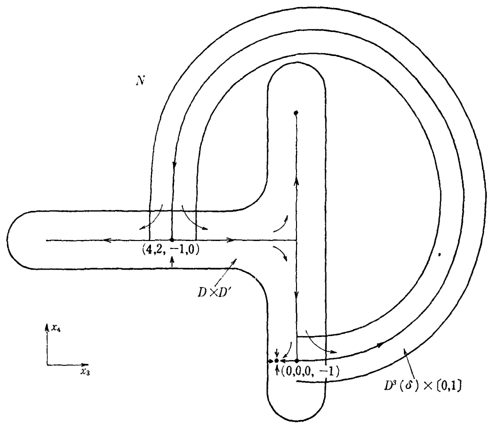

Figure 3

In $4 \sim 10$, we will construct an embedding $F\colon N \to N$. After this, (x_1, x_2, x_3, x_4) (resp. (y_1, y_2, y_3, t)) denotes a point of $D \times D' \subset N$ (resp. $D^3(\delta) \times [0, 1] \subset N$).

4.

For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $|x_3 + 1| \geqq \varepsilon$ and $|x_4 + 1| \geqq \varepsilon$, define

(4.1) $F(x_1, x_2, x_3, x_4) = (f(x_1, x_2), g(x_3, x_4))$.

5.

For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $\frac{1}{4}\varepsilon \leqq |x_4 + 1| \leqq \varepsilon$, define

(5.1) $F(x_1, x_2, x_3, x_4) = (f_{|x_4+1|}(x_1, x_2), g(x_3, x_4))$, where $f_t: D \to D (0 \leq t \leq \varepsilon)$ is an isotopy satisfying (5.2) $\sim$ (5.6). Suppose that b_i $(i = 1, \cdots, 4)$ is a positive number with

$$0 < b_1 < b_2 < \delta < b_3 < b_4 < a_1, \qquad \alpha b_1 < b_2 , \tag{5.2}$$

and

$$b_4 < \min\{4 - a_4, a_5 - 4\} .$$

Then

$$f_t(x_1, x_2) = f(x_1, x_2) \qquad \text{if } |x_1| < b_1 \text{ or } |x_1| > b_4 , \tag{5.3}$$

$$f_t = f \qquad \text{for } \tfrac{1}{2}\varepsilon \leqq t \leqq \varepsilon , \tag{5.4}$$

$$f_t = f_0 \qquad \text{for } 0 \leqq t \leqq \tfrac{1}{4}\varepsilon , \tag{5.5}$$

and

$$f_t(x_1, x_2) = (\bar{f}_t(x_1), \beta x_2) \qquad \text{for } |x_1| \leqq b_4 , \tag{5.6}$$

where $\bar{f}_t$ is an isotopy of a neighbourhood of 0 in $\boldsymbol{R}^1$ and $\bar{f}_0$ has five fixed points: three sources $0, \pm b_3$, and two sinks $\pm b_2$ (Figure 4).

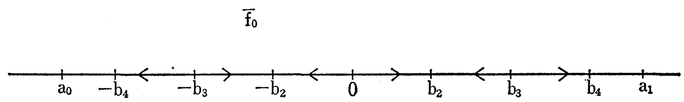

Figure 4

6.

For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $|x_4 + 1| < \frac{1}{4}\varepsilon$, F is defined as follows. Let

$$U = \{(x_1, x_2, x_3, x_4) \in D \times D' \mid \sqrt{x_1^2 + x_2^2 + (x_4 + 1)^2} \leqq \delta\} , \tag{6.1}$$

and

$$U_1 = \{(x_1, x_2, x_3, x_4) \in D \times D' \mid \sqrt{x_1^2 + x_2^2 + (x_4 + 1)^2} \leqq \delta_1\} , \tag{6.2}$$

where $b_2 < \delta_1 < \delta$. Then F is defined as follows.

$$F(x_1, x_2, x_3, x_4) = (f_0(x_1, x_2), g(x_3, x_4)) \quad \text{if } (x_1, x_2, x_3, x_4) \in D \times D' - U \text{ and } |x_4 + 1| \leqq \tfrac{1}{4}\varepsilon , \tag{6.3}$$

F is written in the form such that

$$F(x_1, x_2, x_3, x_4) = (f_0(x_1, x_2), \bar{g}(x_1, x_2, x_3, x_4), \tfrac{1}{2}(x_4 + 1) - 1) \quad \text{if } (x_1, x_2, x_3, x_4) \in U \text{ and } x_3 > -\tfrac{1}{2}\varepsilon , \tag{6.4}$$

where $\bar{g}$ satisfies the followings.

$$\bar{g}(x_1, x_2, x_3, x_4) = \tfrac{1}{2}x_3 \qquad \text{near the frontier of } U, \tag{6.5}$$

$$\bar{g}(x_1, x_2, x_3, x_4) = 2x_3 \quad \text{if } (x_1, x_2, x_3, x_4) \in U_1 \quad \text{and} \quad -\tfrac{1}{4}\varepsilon \leqq x_3 < \tfrac{1}{2}\varepsilon , \tag{6.6}$$

and

(6.7) $\bar{g}(x_1, x_2, x_3, x_4)$ does not depend on x_1 if $|x_1| < b_1$. $\bar{g}$ as above can be induced from a vector field (b_1 is assumed to be sufficiently small).

$$F(\{(x_1, x_2, x_3, x_4) \in U | x_3 < 0\}) \subset \{(x_1, x_2, x_3, x_4) \in U | x_3 < 0\} . \tag{6.8}$$

In $\{(x_1, x_2, x_3, x_4) \in U | x_3 < 0\}$ there are only a finite number of nonwandering points, which are hyperbolic fixed points. Furthermore F satisfies the conditions in 10.

7.

On $D^3(\delta) \times [0, 1 - \varepsilon]$, F is given as follows.

$$F(y_1, y_2, y_3, t) = (f_0(y_1, y_2), \tfrac{1}{2}y_3, \phi(y_1, y_2, y_3, t)) , \tag{7.1}$$

where ϕ satisfies the followings.

If $\sqrt{y_1^2 + y_2^2 + y_3^2} < \delta_1$ or $\frac{1}{2} < t$

(7.2) $\phi(y_1, y_2, y_3, t)$ depends only on t,

and

$$\frac{d\phi}{dt} > 0 . \tag{7.3}$$

$$\phi(y_1, y_2, y_3, t) = 1 - \tfrac{1}{2}(1 - t) \qquad \text{for } 1 - 2\varepsilon \leqq t \leqq 1 - \varepsilon . \tag{7.4}$$

$$\phi(y_1, y_2, y_3, t) = \bar{g}(y_1, y_2, t, y_3 - 1) \qquad \text{if } 0 \leqq t \leqq \varepsilon . \tag{7.5}$$

Moreover F satisfies 10.

8.

For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $|x_3 + 1| < \frac{1}{4}\varepsilon$, F is given as follows. Let $h_t\colon D \to D$ $(0 \leqq t \leqq \varepsilon)$ be an isotopy such that

$$h_t = f \qquad \text{if } \tfrac{1}{2}\varepsilon \leqq t \leqq \varepsilon\,, \tag{8.1}$$

$$h_t(x_1, x_2) = f(x_1, x_2) \qquad \text{if } -2 \leqq x_1 \leqq 4 - b_4 \text{ or } 4 + b_4 \leqq x_1 \leqq 6\,, \tag{8.2}$$

and

$$h_t(x_1, x_2) = f_t(x_1 - 4, x_2 - 2) + (4, 2) \qquad \text{if } |x_1 - 4| < b_4\,. \tag{8.3}$$

Then F is written in the form such that

$$F(x_1, x_2, x_3, x_4) = (h_0(x_1, x_2), \bar{h}(x_1, x_2, x_3, x_4), \tfrac{1}{2}x_4)\,, \tag{8.4}$$

where $\bar{h}$ satisfies the followings.

$$\bar{h}(x_1, x_2, x_3, x_4) = \tfrac{1}{2}(x_3 + 1) - 1 \quad \text{if } \sqrt{(x_1 - 4)^2 + (x_2 - 2)^2 + (x_3 + 1)^2} \leqq \delta \text{ and } x_4 > \tfrac{2}{3}\varepsilon\,, \tag{8.5}$$

$$\bar{h}(x_1, x_2, x_3, x_4) = 2(x_3 + 1) - 1 \quad \text{if } \sqrt{(x_1 - 4)^2 + (x_2 - 2)^2 + (x_3 + 1)^2} \geqq \delta_2 \text{ or } x_4 < \tfrac{1}{3}\varepsilon\,, \tag{8.6}$$

where $\delta < \delta_2 < \frac{1}{4}\varepsilon$.

(8.7) $\bar{h}(x_1, x_2, x_3, x_4)$ does not depend on x_1 if $|x_1 - 4| < b_1$. Furthermore F satisfies 10.

9.

For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $\frac{1}{4}\varepsilon \leqq |x_3 + 1| < \varepsilon$, define

$$F(x_1, x_2, x_3, x_4) = (h_{|x_3+1|}(x_1, x_2), 2(x_3 + 1) - 1, \tfrac{1}{2}x_4)\,. \tag{9.1}$$

10.

F is an embedding of N such that

$$F(N) \subset \operatorname{int} N\,, \tag{10.1}$$

and

(10.2) F is isotopic to the identity.

11.

Straightening the corner (and modifying F near the corner), we can regard N as a submanifold of M which is diffeomorphic to $D^3 \times S^1$. Extend F to a diffeomorphism of M such that the nonwandering set of

$F|M - N$ consists of a finite number of hyperbolic fixed points.

12.

In $12 \sim 15$, we will show that the number of nonwandering points of $F|N$ is countably infinite. This implies that the diffeomorphism of M as above is the required one, because its periodic points are finite. (12.1) and (12.2) follow from the construction of F.

(12.1) If $(x_3, x_4) \notin (\{0\} \times [1, -1]) \cup ([-2, 0] \times \{0\})$, then $(x_1, x_2, x_3, x_4) \in N$ is a fixed point or a wandering point.

(12.2) If $(x_1, x_2) \neq (0, 0)$, then $(x_1, x_2, x_3, x_4) \in N$ is a fixed point or a wandering point.

13.

The maximal invariant set of $F|(D \times \{0\} \times \{0\})$ consists of points satisfying one of the conditions (13.1) $\sim$ (13.3).

(13.1) $(x_1, x_2, 0, 0) \in D \times D$ such that there is an integer n_0 with

$$f^n(x_1, x_2) \in A_i \ (0 \leqq i \leqq 6) \qquad \text{for } n \in Z$$

and

$$f^n(x_1, x_2) \in A_i \ (i = 0, 2, 5, 6) \qquad \text{for } n > n_0\,,$$

where Z denotes the integers.

(13.2) $(x_1, x_2, 0, 0) \in D \times D'$ such that there is $n_0 \in Z$ with

$$\begin{aligned} f^n(x_1, x_2) &\in A_5 \qquad \text{if } n < n_0 \\ f^n(x_1, x_2) &\in A_3 \qquad \text{if } n = n_0 \end{aligned}$$

and

$$f^n(x_1, x_2) \in A_1 \qquad \text{if } n > n_0\,.$$

(13.3) $(x_1, x_2, 0, 0) \in D \times D'$ such that there is $n_0 \in Z$ with

$$f^n(x_1, x_2) \in A_5 \qquad \text{for } n < n_0$$

and

$$f^n(x_1, x_2) \in A_1 \qquad \text{for } n \geqq n_0\,.$$

Denote

$$\mathcal{O}_1 = \{(x_1, x_2, 0, 0) \in D \times D' | (x_1, x_2, 0, 0) \text{ satisfies (13.2)}\}$$

and

$$\mathcal{O}_2 = \{(x_1, x_2, 0, 0) \in D \times D' | (x_1, x_2, 0, 0) \text{ satisfies (13.3)}\}$$

Then $\mathcal{O}_i$ $(i = 1, 2)$ is an orbit of one point, and $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \{(4, 2, 0, 0), (0, 0, 0, 0)\}$ is a hyperbolic set.

14.

We will show that any point satisfying (13.1) is a fixed point or a wandering point. Suppose $(x_1, x_2, 0, 0)$ is not a fixed point and satisfies (13.1). Let $W_1 \subset D$ be a neighbourhood of (x_1, x_2) with

$$f^n(W_1) \subset A_i \ (i = 0, 2\ 6) \qquad \text{for } n > n_0 , \tag{14.1}$$

where n_0 is given in (13.1). (It does not occur that $f^n(x_1, x_2) \in A_5$ for $n > n_0$, because $(x_1, x_2) \neq (4, 2)$.) Choose a neighbourhood $W_2 \subset D'$ of $(0, 0)$ such that

$$g^n(W_2) \cap \{(x_3, x_4) \in D' | |x_4 + 1| \leqq \varepsilon\} = \emptyset \qquad \text{for } n \leqq n_0 . \tag{14.2}$$

If $n \leqq n_0$ (resp. $n > n_0$), it follows from (14.2) (resp. (14.1) and (5.3)) that

$$F^n(w) \cap \{(z_1, z_2, z_3, z_4) \in D \times D' | \sqrt{z_1^2 + z_2^2 + (z_3 + 1)^2} < \delta\} = \emptyset$$
$$\text{for } w \in W_1 \times W_2 .$$

Therefore $(x_1, x_2, 0, 0)$ is a wandering point, because a non-periodic point is nonwandering only if its nearby points return near by the point through $D^3(\delta) \times [0, 1]$.

15.

Here we will prove that a point of $\mathcal{O}_1 \cup \mathcal{O}_2$ is nonwandering. Suppose that $(x_1, x_2, 0, 0) \in D \times D'$ satisfies (13.2) or (13.3). Let W be a neighbourhood of $(x_1, x_2, 0, 0)$ such that

$$W = \{(z_1, z_2, z_3, z_4) \in D \times D' | |z_i - x_i| \leqq \sigma, |z_j| \leqq \sigma \quad \text{for } i = 1, 2,\ j = 3, 4\} \tag{15.1}$$

for $0 < \sigma < b_1$. Choose a sequence $(x_1, x_2, y_3^{(i)}, y_4^{(i)}) \in W$ $(i = 1, 2, \cdots)$ with

$$y_3^{(i)} > 0 , \qquad y_4^{(i)} < 0 , \tag{15.2}$$

and

$$y_3^{(i)} \to 0 , \quad y_4^{(i)} \to 0 \qquad (\text{as } i \to \infty) .$$

Then there is a sequence of integers $\{n_i\}_{i=1,2,\dots}$ which satisfies $n_i \to \infty$ (as $i \to \infty$) and

$$F^{n_i}(x_1, x_2, y_3^{(i)}, y_4^{(i)}) \to (0, 0, 0, -1) \qquad \text{as } i \to \infty . \tag{15.3}$$

This implies that there is a sequence $\{m_i\}_{i=1,2,\dots}$ with $m_i > n_i$ and

$$F^{m_i}(x_1, x_2, y_3^{(i)}, y_4^{(i)}) \to (4, 2, -1, 0) \qquad \text{as } i \to \infty . \tag{15.4}$$

This implies that there is a sequence $\{\ell_i\}_{i=1,2,\dots}$ with $\ell_i > m_i$ and

$$F^{\ell_i}(x_1, x_2, y_3^{(i)}, y_4^{(i)}) \to (4, 2, 0, 0) \qquad \text{as } i \to \infty . \tag{15.5}$$

It follows from (1.5), (4.1), (5.1), (5.3), (6.3), (6.4), (6.7), (7.1), (7.2), (8.3), (8.4), (8.7) and (9.1) that on a neighbourhood of (4, 2, 0, 0) F satisfies

$$\begin{aligned} &F^{\ell_i}([x_1 - \sigma, x_1 + \sigma] \times \{x_2\} \times \{y_3^{(i)}\} \times \{y_4^{(i)}\}) \\ &\qquad \supset [4 - \sigma, 4 + \sigma] \times \{v_2^{(i)}\} \times \{v_3^{(i)}\} \times \{v_4^{(i)}\} , \end{aligned} \tag{15.6}$$

where

$$v_j^{(i)} = \mathrm{pr}_j\, F^{\ell_i}(x_1, x_2, y_3^{(i)}, y_4^{(i)}) ,$$

and pr_j is the projection to the x_j-factor. It follows from (15.5) and (15.6) that for sufficiently large k

$$\bigcup_{i>0} F^{\ell_i+k}([x_1 - \sigma, x_1 + \sigma] \times \{x_2\} \times \{y_3^{(i)}\} \times \{y_4^{(i)}\}) \cap W \neq \varnothing . \tag{15.7}$$

Thus $(x_2, x_2, 0, 0)$ is a nonwandering point. We have constructed F such that $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \{\textit{fixed points}\}$ is hyperbolic. This completes the proof.

After this paper was written the author was informed that A. Danker also constructed a counter-example to this problem. (c.f. On Smale's Axiom A dynamical systems, Ann. of Math. **107** (1978) 517–553.)

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