§ 1. Introduction

As a continuation of the previous paper [7], we shall consider in this paper the problem of prediction given bounded intervals and obtain integral representations of predictors and prediction errors. For that purpose we shall introduce innovation processes well matched with bounded intervals. We follow the notation and terminology in [6].

Let \( X = (X(t); t \in \mathbb{R}) \) be a real separable and measurable stationary Gaussian process on a probability space \((\Omega, F, P)\) with expectation zero which is continuous in the mean and purely nondeterministic. Furthermore we suppose that \( X \) has the \( N \)-ple Markovian property in the broad sense ([7]). We then know that the spectral measure of \( X \) has a Hardy density \( \Delta \) whose outer part \( h \) is expressed in the form

\[
\begin{align*}
Q(\lambda) &= \sum_{n=0}^{N-1} c_n(-i\lambda)^n, \\
P(\lambda) &= \sum_{n=0}^{N-1} b_n(-i\lambda)^n, \\
\lambda &\in \mathbb{R}, \\
c_n, b_n &\in \mathbb{R}, \\
c_N &\neq 0 \\
V_P &\subset C^+, \\
V_Q &\subset C^+ \cup \mathbb{R}, \\
V_P \cap V_Q &= \phi,
\end{align*}
\]

where \( V_s \) denotes the set of zero points of a polynomial \( S \).

In [7], we have constructed an \( N \)-dimensional stationary Gaussian process \( \mathcal{X} = (\mathcal{X}(t); t \in \mathbb{R}) \) satisfying

\[
F^{+/-}_x(t) = F_{x^{+/-}}(t) = \sigma(\mathcal{X}(t)) \quad (t \in \mathbb{R}).
\]

Similarly we can obtain an \( N \)-dimensional stationary Gaussian process \( \mathcal{Y} = (\mathcal{Y}(t); t \in \mathbb{R}) \) satisfying
Using these processes $\mathcal{X}$ and $\mathcal{Y}$, we shall define in §2 for any $a \in \mathbb{R}$ innovation processes $\nu_a^+ = (\nu_a^+(t); t \geq 0)$ which are standard $(F_a^+(t); t \geq 0)$-Brownian motions, where $F_a^+(t) = \partial F_a(t) (t = 0)$, $F_a((a, a + t)) (t > 0)$ and $F_a^-(t) = \partial F_a(t) (t = 0)$, $F_a((a - t, a)) (t > 0)$.

In §3 we shall obtain integral representations of the predictors $E(\mathcal{X}(a + T)|F_X((a, a + T)))$ and $E(X(a + T + t)|F_X((a, a + T)))$ (resp. $E(\mathcal{Y}(a - T)|F_X((a - T, a)))$ and $E(X(a - T - t)|F_X((a - T, a)))$) in terms of innovation processes $\nu_a^+$ (resp. $\nu_a^-$) ($a \in \mathbb{R}$, $t > 0$, $T > 0$). As an application of these results, we shall find that Gaussian processes $Y_a = (Y_a(t); t \geq 0) = (X(\pm t) - E(X(\pm t)|\partial F_X(0)); t \geq 0)$ have canonical representations (33).

We shall obtain in §4 integral representations of the predictors $E(\mathcal{X}(a - t)|F_X((a, a + T)))$ and $E(X(a - t)|F_X((a, a + T)))$ (resp. $E(\mathcal{Y}(a + t)|F_X((a - T, a)))$ and $E(X(a + t)|F_X((a - T, a)))$) in terms of innovation processes $\nu_a^+$ (resp. $\nu_a^-$) ($a \in \mathbb{R}$, $t > 0$, $T > 0$). Representation kernels in representation theorems in §3 and §4 can be written using the solution of matrix Riccati equation.

In §5 we shall prove orthogonal decomposition theorems of integral representations of the predictors $E(\mathcal{X}(a + T)|F_X((a - a, a)))$ and $E(X(a + T)|F_X((a - a, a)))$ (resp. $E(\mathcal{Y}(a - t)|F_X((a - a, a)))$ and $E(X(a + t)|F_X((a - a, a)))$) in terms of innovation processes $\nu_a^+$ and $\nu_{-a}$ (resp. $\nu_a^-$ and $\nu_{-a}^-$) ($a > 0$, $t > 0$).

In §6 we shall give concrete computations in the space $Z_d$ of representation kernels in representation theorems in §3 and §4 and then obtain explicit representations of prediction errors of $E(X(a + T)|F_X((a - a, a)))$, $E(X(a - t)|F_X((a, a + T)))$ ($a \in \mathbb{R}$, $t > 0$, $T > 0$) and $E(X(\pm(a + T)|F_X((a - a, a)))$ ($a > 0$, $t > 0$).

Using the results of the previous section, we shall obtain in §7 integral representations of the predictors $E(\mathcal{X}(a + T)|F_X((a - T, a)))$ (resp. $E(\mathcal{Y}(a - t)|F_X((a, a + T)))$) in terms of innovation processes $\nu_a^+$ (resp. $\nu_a^-$) ($a \in \mathbb{R}$, $t > 0$, $T > 0$) and then the predictors $E(\mathcal{X}(a + t)|F_X((a - a, a)))$ (resp. $E(\mathcal{Y}(a - t)|F_X((a - a, a)))$) in terms of innovation processes $\nu_a^+$ and $\nu_{-a}$ (resp. $\nu_a^-$ and $\nu_{-a}^-$) ($a > 0$, $t > 0$).

§2. Innovation processes $\nu_a^+$ and $\nu_a^-$ ($a \in \mathbb{R}$)

We denote by $E$ the Fourier transform of $h$ in (1.1). Then we have the following canonical representation:
\[ X(t) = \sqrt{2\pi t} \int_{-\infty}^{t} E(t - s) dB(s), \]

where \((B(t); t \in \mathbb{R})\) is a standard Brownian motion satisfying
\[ F_T(t) = \sigma(B(s_1) - B(s_2); s_1, s_2 < t) \quad (t \in \mathbb{R}). \]

We define an \(N \times N\)-matrix \(A\) and an \(N \times 1\)-vector \(b\) by
\[
\begin{bmatrix}
0 & a_0 \\
-1 & 0 & a_1 \\
& -1 & 0 & a_2 \\
& & \ddots & \ddots & \ddots \\
& & & 0 & a_{N-2} \\
& & & & -1 & a_{N-1}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{N-1}
\end{bmatrix},
\]

where \(a_n = c_n c_{N-1}^{-1} \) \((0 \leq n \leq N - 1)\). Since the characteristic equation of \(A\) is equal to \((-1)^N c_N^{-1} P(i \omega)\) and so all eigenvalues of \(A\) have negative real parts by (1.1), we can define \(N\) real \(L^2\)-functions \(E_n (0 \leq n \leq N - 1)\) and a real \(L^2\)-function \(F\) by
\[
E_n(t) = \sqrt{2\pi^{-1}} \chi_{(0, \omega)}(t) (e^{itA} b)_n \quad (t \in \mathbb{R})
\]
and
\[
F(t) = -\sqrt{2\pi c_N^{-1}} \chi_{(0, \omega)}(t) \left( e^{itA} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \quad (t \in \mathbb{R}).
\]

Using these \(L^2\)-functions \(E_n\), we define an \(N\)-dimensional stationary Gaussian process \(\mathcal{X} = (\mathcal{X}(t); t \in \mathbb{R}) = ((X_0(t), \ldots, X_{N-1}(t))^*; t \in \mathbb{R})\) by
\[
X_n(t) = \sqrt{2\pi^{-1}} \int_{-\infty}^{t} E_n(t - s) dB(s) \quad (0 \leq n \leq N - 1, \ t \in \mathbb{R}).
\]

Then we see from the results in [7] that

THEOREM 2.1 ( i ) \(X_{N-1}(t) = (-2\pi)^{-1} c_N X(t) \) \((t \in \mathbb{R})\).
( ii ) \(X_n(t); 0 \leq n \leq N - 1\) is linearly independent in \(M_\mathcal{X}\) for any \(t \in \mathbb{R}\).
( iii ) \(M_\mathcal{X}(t) = M_\mathcal{X}^*(t)\) and \(F_\mathcal{X}(t) = F_\mathcal{X}^*(t) \) \((t \in \mathbb{R})\).
( iv ) \(M_\mathcal{X}^i\) \((t)\) is equal to the linear hull of \(\{X_n(t); 0 \leq n \leq N - 1\} \) \((t \in \mathbb{R})\).
( v ) \(F_\mathcal{X}^i\) \((t)\) = \(F_\mathcal{X}^i\) \((t)\) = \(\sigma(\mathcal{X}(t)) \) \((t \in \mathbb{R})\).
(vi) \[ \mathcal{X}(t) - \mathcal{X}(s) = \sqrt{2\pi}^{-1}(B(t) - B(s))b + \int_s^t A\mathcal{X}(u)du \quad (s < t). \]

(vii) \[ \mathcal{X}(t) = e^{(t-s)A}\mathcal{X}(s) + \sqrt{2\pi}^{-1}\int_s^t e^{(t-u)A}b dB(u) \quad (s < t). \]

(viii) \[ E(X(t)|F_x(s)) = \sum_{n=0}^{N-1} (-1)^n F^n(t-s)X_n(s) \quad (s < t). \]

(ix) \[ E(X(t)|F_x(s)) = e^{(t-s)A}\mathcal{X}(s) \quad (s < t). \]

[2.1] Now we fix any \( a \in \mathbb{R} \) and define the \( \sigma \)-fields \( F^+_a(t) \) \((t \geq 0)\) by

\[ F^+_a(t) = \begin{cases} \partial F_X(a) & (t = 0) \\ F_X((a, a + t)) & (t > 0) \end{cases} \]

Then we shall show

**Theorem 2.2.** There exists a standard Brownian motion \( \nu^+_a = (\nu^+_a(t); t \geq 0) \) such that

(i) \( \nu^+_a(0) = 0 \),

(ii) \( F^+_a(t) = \partial F_X(a) \lor \sigma(\nu^+_a(s); 0 \leq s \leq t) \quad (t \geq 0) \),

(iii) \( \nu^+_a \) is independent of the \( \sigma \)-field \( \partial F_X(a) \),

(iv) \[ X_{n_0}(t+\alpha) - X_{n_0}(\alpha) = \sqrt{2\pi}^{-1}b_{n_0}^\nu \nu^+_a(t) + b_{n_0} e \int_{a}^{a+\alpha} E(\mathcal{X}(s)|F_X((a, s))) ds \]

\((t \geq 0)\),

where \( n_0 = \max \{ n \in \{0, 1, \ldots, N-1\}; b_n \neq 0 \} \) and \( b_{n_0} e \) is the \( n_0 + 1 \)-th row of the matrix \( A \).

**Proof.** By (iv) we define a stochastic process \( \nu^+_a = (\nu^+_a(t); t \geq 0) \) with continuous paths. It then follows from (2.2), Theorem 2.1 (iii) and (iv) that \( \nu^+_a \) is a square integrable \((F^+_a(t); t \geq 0)\)-martingale with expectation zero. We put \( M = N - 1 - n_0 \). It is easy to see from Lemma 2.4, Theorem 2.1 (i) and Lemma 4.1 (i) in [7] that \( E^{(n)}(0+) = 0 \) \((0 \leq n \leq M - 1)\). This implies by (2.1) that \( X(t) \) is \( M \)-times differentiable in the mean and a stationary Gaussian process \( X^{(M)} = (X^{(M)}(t); t \in \mathbb{R}) \) has the same property as \( X \). Applying Theorem 2.1 in this paper to the process \( X^{(M)} \), we have an \( N \)-dimensional stationary Gaussian process \( \mathcal{X}_M = (\mathcal{X}_M(t); t \in \mathbb{R}) \) such that \( \mathcal{X}_{M,N-1}(t) = (-2\pi)^{-1}c_n X^{(M)}(t) \)

\[ \mathcal{X}_{M,N-1}(t + \alpha) - \mathcal{X}_{M,N-1}(\alpha) = (-1)^\nu b_{n_0} \sqrt{2\pi}^{-1}(B(t + \alpha) - B(\alpha)) + \int_{a}^{a+\alpha} (A\mathcal{X}_M(u))_{N-1} du \quad (t \geq 0). \]

Using this process \( \mathcal{X}_M \), we define an \( \mathbb{R}^N \)-valued stochastic process
Then it follows from the results of [4] and [5] that \( (\eta^*_t(t); t \geq 0) \) is a standard \( \mathcal{F}_+(t); t \geq 0 \)-Brownian motion for which \( \sigma(\eta^*_t(s); 0 \leq s \leq t) \) is equal to \( \sigma(Y(s); 0 \leq s \leq t) \) and \( \sigma(\eta^*_u(u) - \eta^*_v(v); u, v > t) \) is independent of \( \mathcal{F}_+(t) \). By the definition of the process \( (Y(t); t \geq 0) \), \( Y(t) = (-1)^m \sqrt{2\pi \varepsilon} \int_0^t \mathcal{X}(s)ds + B(t + a) - B(a) \), and then a real valued stochastic process \( (\eta^*_t(t); t \geq 0) \) by

\[
\eta^*_t(t) = Y(t) - (-1)^m \sqrt{2\pi \varepsilon} \int_0^t E(\mathcal{X}(s)|\mathcal{F}_+(a, a + s))ds.
\]

Next we define the \( \sigma \)-fields \( \mathcal{F}_+(t) (t \geq 0) \) by

\[
\mathcal{F}_+(t) = \begin{cases} \{ \partial \mathcal{F}_+(a) \} & (t = 0), \\ \mathcal{F}_+(a - t, a) & (t > 0). \end{cases}
\]

Noting that \( \vec{h} = \vec{E} \), we see that there exists a standard Brownian motion \( (B_-(t); t \in \mathbb{R}) \) for which the followings hold:

\[
X(t) = \sqrt{2\pi^{-1}} \int_0^t E(s - t)dB_-(s),
\]

\[
\mathcal{F}_+(t) = \sigma(B_-(s_1) - B_-(s_2); s_1, s_2 > t) \quad (t \in \mathbb{R}).
\]

Using this Brownian motion \( (B_-(t); t \in \mathbb{R}) \) and \( N \) real \( L^2 \)-functions \( E_n \) \((0 \leq n \leq N - 1)\) in (2.4), we define an \( N \)-dimensional stationary Gaussian process \( \mathcal{V} = (\mathcal{V}(t); t \in \mathbb{R}) = ((Y_0(t), \cdots, Y_{N-1}(t))^*; t \in \mathbb{R}) \) by

\[
Y_n(t) = \sqrt{2\pi^{-1}} \int_0^t E_n(s - t)B_-(s) \quad (0 \leq n \leq N - 1, t \in \mathbb{R}).
\]

Similarly as in Theorem 2.1, we have
THEOREM 2.3. \( (i) \) \( Y_{n-1}(t) = X_{n-1}(t) = (-2\pi)^{-1}e^N X(t) \) \( (t \in \mathbb{R}) \).

\( (ii) \) \( \{ Y_n(t); 0 \leq n \leq N - 1 \} \) is linearly independent in \( M_X \) for any \( t \in \mathbb{R} \).

\( (iii) \) \( M_t^X(t) = M_t^X(t) \) and \( F_t^X(t) = F_t^X(t) \) \( (t \in \mathbb{R}) \).

\( (iv) \) \( M_t^{X^*}(t) \) is equal to the linear hull of \( \{ Y_n(t); 0 \leq n \leq N - 1 \} \) \( (t \in \mathbb{R}) \).

\( (v) \) \( F_t^X(t) = F_t^{X^*}(t) = \sigma(\mathcal{F}(t)) \) \( (t \in \mathbb{R}) \).

\( (vi) \) \( \mathcal{B}(s) - \mathcal{B}(t) = \sqrt{2\pi}^{-1}(B_+(t) - B_-(s))b + \int_s^t A\mathcal{B}(u)du \) \( (s < t) \).

\( (vii) \) \( \mathcal{B}(s) = \int_0^t e^{(t-s)A}\mathcal{B}(t) + \sqrt{2\pi}^{-1}\int_s^t e^{(t-s)A}d\mathcal{B}(u) \) \( (s < t) \).

\( (viii) \) \( E(X(s) | F_t^X(t)) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t - s)Y_n(t) \) \( (s < t) \).

\( (ix) \) \( E(\mathcal{B}(s) | F_t^X(t)) = e^{(t-s)A}\mathcal{B}(t) \) \( (s < t) \).

By virtue of Theorem 2.3, in the same way as Theorem 2.2, we obtain

THEOREM 2.4. There exists a standard Brownian motion \( \nu_a^- = (\nu_a^-(t); t \geq 0) \) such that

\( (i) \) \( \nu_a^-(0) = 0 \),

\( (ii) \) \( F_a^-(t) = \partial F_a^X(a) \vee \sigma(\mathcal{F}(s); 0 \leq s \leq t) \) \( (t \geq 0) \),

\( (iii) \) \( \nu_a^- \) is independent of the \( \sigma \)-field \( \partial F_a^X(a) \),

\( (iv) \) \( Y_{n_0}(a - t) - Y_{n_0}(a) = \sqrt{2\pi}^{-1}b_{n_0}\nu_a^-(t) + b_{n_0}e^{-1} \mathcal{B}(s) | F_X((s,a))ds \) \( (t \geq 0) \).

DEFINITION 2.1. We call the standard Brownian motions \( \nu_a^+ \) (resp. \( \nu_a^- \)) \( (F_t^+(t); t \geq 0) \) (resp. \( (F_t^-(t); t \geq 0) \)) innovation processes associated with the stationary Gaussian process \( X \).

[2.3] Finally in this subsection we shall give a relation between the family of innovation processes \( \nu_a^\pm \) \( (a \in \mathbb{R}) \). We have the unitary transformation group \( (U(t); \ t \in \mathbb{R}) \) acting on the space \( M_X \) defined by

\[ U(t)X(s) = X(t + s) \quad (t, s \in \mathbb{R}) \].

Then we shall show

THEOREM 2.5. \( U(t)\nu_a^\pm = \nu_{a+t}^\pm \) for any \( a \in \mathbb{R} \) and \( t \in \mathbb{R} \).

Proof. It is easy to see that \( U(t)M_a^X((a, b)) = M_a^X((a + t, b + t)) \), \( U(t)\partial M_a^X(a) = \partial M_a^X(a + t) \) and \( U(t)M_a^{X^*}(a) = M_a^{X^*}(a + t) \). We define an \( N \)-dimensional stationary Gaussian process \( \mathcal{F}(t) = (U(t)X_0(0), \ldots, U(t)X_{N-1}(0)) \).
Then it follows from Theorem 2.1 (ii) and (iv) that $\mathcal{X}(t)$ is continuous in the mean, each component of $\mathcal{X}(t)$ belongs to the space $M_{\mathcal{X}}^+(t)$ and $\{U(t)X_n(0) ; 0 \leq n \leq N-1\}$ is linearly independent in $M_{\mathcal{X}}$ for any $t \in \mathbb{R}$. Therefore, we see from Theorem 5.1 in [7] that there uniquely exists a constant $N \times N$-matrix $\tilde{T}$ for which $\mathcal{X}(t) = \tilde{T}\mathcal{X}(t)$ $(t \in \mathbb{R})$. Since $\mathcal{X}(0) = \mathcal{X}(0)$, we find that $\tilde{T}$ is the unit matrix and so $\mathcal{X}(t) = \mathcal{X}(t)$. By Theorem 2.2 (iv) this implies that $U(t)\nu^+_s(s) = \nu^+_s(s)$. Similarly, we have $U(t)\nu^-_s(s) = \nu^-_s(s)$. (Q.E.D.)

§ 3. Integral representations of the predictors (I)

[3.1] In this subsection we shall obtain integral representations of the predictors $E(\mathcal{X}(a + T) | F_a(T))$ ($a \in \mathbb{R}$, $T \geq 0$). For any $a \in \mathbb{R}$ we define $N \times N$-matrices $P_a(t)$ ($t \geq 0$) by

\begin{equation}
P_a(t) = E\left[(\mathcal{X}(a + t) - E(\mathcal{X}(a + t) | F_a(t))(\mathcal{X}(a + t) - E(\mathcal{X}(a + t) | F_a(t)))^*\right]
\end{equation}

and then $N \times 1$-vectors $f_a(t, s)$ ($0 \leq s \leq t < \infty$) by

\begin{equation}
f_a(t, s) = e^{(t-s)A} \cdot (P_a(s)e^* + \sqrt{2\pi}^{-1}b).
\end{equation}

At first we shall show

**Lemma 3.1.** $f_a(t, s) = (\partial/\partial s)E(\nu^+_a(s) \cdot \mathcal{X}(a + t))$ ($0 \leq s \leq t < \infty$).

**Proof.** We put $\tilde{\mathcal{X}}(s) = \mathcal{X}(s) - E(\mathcal{X}(a + s) | F_a(s))$. It then follows from Theorems 2.1 (vi) and 2.2 (iv) that

\begin{equation}
\nu^+_a(t) = e \cdot \int_0^t \tilde{\mathcal{X}}(s)ds + B(a + t) - B(a) \quad (t \geq 0).
\end{equation}

Therefore we see from (2.2) and Theorem 2.1 (vii) that

\begin{align*}
E(\nu^+_a(s) \cdot \mathcal{X}(a + t)) &= \int_0^t E(\mathcal{X}(a + t) \cdot \tilde{\mathcal{X}}(\tau))e^* \cdot d\tau \\
&\quad + E((B(a + s) - B(a)) \cdot \mathcal{X}(a + t)) \\
&= \int_0^t e^{(t-\tau)A}E(\mathcal{X}(a + \tau) \cdot \tilde{\mathcal{X}}(\tau))e^*d\tau \\
&\quad + \sqrt{2\pi}^{-1} \int_a^t e^{(a+t-\tau)A}b d\tau \\
&= \int_0^t e^{(t-\tau)A}(P_a(\tau)e^* + \sqrt{2\pi}^{-1}b)d\tau.
\end{align*}
On the other hand, by Corollaries 2.1 and 2.2 in [6] and the results in [7], we find that \( P_a(\cdot) \) is continuous in \( \cdot \) and so this implies Lemma 3.1.

**LEMMA 3.2.** For any \( a \in \mathbb{R} \) and any \( T \in (0, \infty) \),

\[
E(\mathcal{X}(a + T)|\mathcal{F}_X((a, a + T))) = E(\mathcal{X}(a + T)|\partial \mathcal{F}_X(a)) + \int_0^T f_\alpha(T, s) d\nu_\alpha(s).
\]

**Proof.** We put \( Y = \mathcal{X}(a + T) - E(\mathcal{X}(a + T)|\partial \mathcal{F}_X(a)) - \int_0^T f_\alpha(T, s) d\nu_\alpha(s) \).

It then follows from Theorem 2.2 (i), (iii) and Lemma 3.1 that \( (d/ds)E(Y \cdot \nu_\alpha(s)) = 0 \) and so \( E(Y \cdot \nu_\alpha(s)) = E(Y \cdot \nu_\alpha(0)) = 0 \) for any \( s \in [0, T] \). Since \( Y \) is orthogonal to the space \( \partial \mathcal{M}_X(a) \), we see that \( Y \) is orthogonal to the closed linear hull of \( \{ \nu_\alpha(s); 0 \leq s \leq T \} \cup \partial \mathcal{M}_X(a) \) and so \( Y \) is independent of the \( \sigma \)-field generated by \( \{ \nu_\alpha(s); 0 \leq s \leq T \} \cup \partial \mathcal{M}_X(a) \). Therefore, by virtue of Theorem 2.2 (ii), we find that \( E(Y|\mathcal{F}_X((a, a + T))) = 0 \) and this implies Lemma 3.2. (Q.E.D.)

Next we shall derive a differential equation which \( P_a(t) \) satisfies.

**LEMMA 3.3.** For any \( a \in \mathbb{R} \), \( P_a(t) \) is the unique solution of the following matrix Riccati equation:

\[
\begin{cases}
\frac{dP_a(t)}{dt} = (A - \sqrt{2\pi^{-1}} b \cdot e) P_a(t) + P_a(t)(A - \sqrt{2\pi^{-1}} b \cdot e)^* - P_a(t) e^* e P_a(t) \\
P_a(0) = K_\alpha(0) - \Sigma_\alpha(0),
\end{cases}
\]

where \( \Sigma_\alpha(0) = E[E(\mathcal{X}(a)|\partial \mathcal{F}_X(a)) \cdot E(\mathcal{X}(a)|\partial \mathcal{F}_X(a))^*] \).

**Proof.** We put \( \Sigma_\alpha(t) = E[E(\mathcal{X}(a + t)|\mathcal{F}_X(t)) \cdot E(\mathcal{X}(a + t)|\mathcal{F}_X(t))^*] \). Then it follows from Theorems 2.1 (xi), 2.2 (iii) and Lemma 3.2 that

\[
\Sigma_\alpha(t) = e^{tA} \Sigma_\alpha(0) e^{tA^*} + \int_0^t f_\alpha(t, s) f_\alpha^*(t, s) d\lambda(s) .
\]

Noting that \( P_a(t) = K_\alpha(0) - \Sigma_\alpha(t) \), we see from Lemma 5.2 in [7] that

\[
\frac{dP_a(t)}{dt} = -A \cdot \Sigma_\alpha(t) - \Sigma_\alpha(t) A^* - P_a(t) e^* e P_a(t) \\
- \sqrt{2\pi^{-1}} P_a(t) e^* b^* - \sqrt{2\pi^{-1}} b \cdot e \cdot P_a(t) - (2\pi)^{-1} b \cdot b^* \\
= A \cdot (K_\alpha(t) - \Sigma_\alpha(t)) + (K_\alpha(0) - \Sigma_\alpha(t)) A^* \\
- P_a(t) e^* e P_a(t) - \sqrt{2\pi^{-1}} P_a(t) e^* b^* - \sqrt{2\pi^{-1}} b \cdot e \cdot P_a(t)
\]
\begin{align*}
&= A \cdot P_a(t) + P_a(t)A^* - P_a(t)e^* \cdot e \cdot P_a(t) \\
&\quad - \sqrt{2\pi}^{-1} P_a(t)e^* \cdot b^* - \sqrt{2\pi}^{-1} b \cdot e \cdot P_a(t) \\
&= (A - \sqrt{2\pi}^{-1} b \cdot e) P_a(t) + P_a(t)(A - \sqrt{2\pi}^{-1} b \cdot e)^* - P_a(t)e^* \cdot e \cdot P_a(t) . \\
\end{align*}

We define an $N \times (N - n_0)$-matrix $J$ by
\begin{equation}
J = (K_x(0)_{m_n})_{n_0 \leq m \leq N-1} \cdot (K_x(0)_{m_n})_{m \leq n_0, n \leq N-1} .
\end{equation}

Then we shall show

**Lemma 3.4.** $\Sigma_a(0) = J \cdot (K_x(0)_{m_n})_{n_0 \leq m \leq N-1} = J \cdot (K_x(0)_{m_n})_{m \leq n_0, n \leq N-1} \cdot J^* .

In particular, $\Sigma_a(0)$ is independent of $a$.

**Proof.** Since the dimension of the space $\partial M_1(a)$ is $N - n_0$ ([1]), it follows from Theorem 2.1 (i) (ii) (vi) that $\partial M_1(a)$ equals the linear hull of $\{X_n(a) ; n_0 \leq n \leq N-1\}$. Therefore there uniquely exists an $N \times (N - n_0)$-matrix $J(a)$ such that $E(\mathbb{F}(a) | \partial F_1(a)) = J(a)(X_{n_0}(a) \cdots X_{N-1}(a))^*$. This implies that
\begin{align*}
J(a) &= E(\mathbb{F}(a) \cdot (X_{n_0}(a) \cdots X_{N-1}(a)) | E((X_{n_0}(a) \cdots X_{N-1}(a))^* \cdot (X_{n_0}(a) \cdots X_{N-1}(a))^*)^{-1} \\
&= E(\mathbb{F}(0) \cdot (X_{n_0}(0) \cdots X_{N-1}(0)) | E((X_{n_0}(0) \cdots X_{N-1}(0))^* \cdot (X_{n_0}(0) \cdots X_{N-1}(0))^*)^{-1}
\end{align*}
and so we have Lemma 3.4. (Q.E.D.)

By the uniqueness of local solutions, we see from Lemmas 3.3 and 3.4 that

**Lemma 3.5.** For any $a, b \in R$ $P_a(t) = P_b(t)$ ($t \geq 0$).

Consequently, combining above lemmas, we obtain

**Theorem 3.1.** For any $a \in R$ and any $T \in (0, \infty)$,
\begin{align*}
E(\mathbb{F}(a + T) | F_1((a, a + T))) &= E(\mathbb{F}(a + T) | \partial F_1(a)) + \int_0^T e^{(T-s)A}(P(s)e^* + \sqrt{2\pi}^{-1} b)v_a^*(s) \\
&= e^{TA} J \cdot (X_{n_0}(a) \cdots X_{N-1}(a))^* + \int_0^T e^{(T-s)A}(P(s)e^* + \sqrt{2\pi}^{-1} b)v_a^*(s) ,
\end{align*}

where $J$ is the $N \times (N - n_0)$-matrix given by (3.4) and $P(t)$ is the unique solution of the following matrix Riccati equation:
In this subsection we shall obtain integral representations of the predictors $E(\mathcal{Y}(a - T) | F_\alpha^-(T))$ $(a \in \mathbb{R}, T \geq 0)$. Similarly as in Lemmas 3.1 and 3.2, we see from Theorems 2.3 and 2.4 that

**Lemma 3.6.** For any $a \in \mathbb{R}$ and any $T \in (0, \infty)$,

$$E(\mathcal{Y}(a - T) | F_X((a - T, a))) = E(\mathcal{Y}(a - T) | \partial F_X(a)) + \int_0^T e^{(T-t)A}(Q_a(s)e^*) + \sqrt{2\pi^{-1}b} \, d\omega^-(s),$$

where $Q_a(t) = E\{[(\mathcal{Y}(a - t) - E(\mathcal{Y}(a - t) | F_\alpha^-(t))) \cdot (\mathcal{Y}(a - t) - E(\mathcal{Y}(a - t) | F_\alpha^-(t)))^*]\}.$

In the same way as Lemma 3.3, by Theorems 2.3, 2.4 and Lemma 3.6, we have

**Lemma 3.7.** For any $a \in \mathbb{R}$, $Q_a(t)$ satisfies the following matrix Riccati equation

$$\begin{align*}
    \frac{dQ_a(t)}{dt} &= (A - \sqrt{2\pi^{-1}b} \cdot e)Q_a(t) + Q_a(t)(A - \sqrt{2\pi^{-1}b} \cdot e)^* - Q_a(t) \cdot e^* \cdot e \cdot Q_a(t), \\
    Q_a(0) &= K_\alpha(0) - II_a(0),
\end{align*}$$

where $II_a(0) = E[E(\mathcal{Y}(a) | \partial F_X(a)) \cdot E(\mathcal{Y}(a) | \partial F_X(a))^*].$

Now we shall show

**Lemma 3.8.** For any $a \in \mathbb{R}$, $Q_a(0) = P(0)$.

**Proof.** Similarly as in Lemma 3.4, we see that

$$II_a(0) = \bar{J} \cdot (K_\alpha(0)_{mn})_{n \leq m} \cdot \bar{J}^*,$$

where $\bar{J} = (K_\alpha(0)_{mn})_{n \leq m \leq N - 1} \cdot (K_\alpha(0)_{mn})_{n \leq m \leq N - 1}^{-1}$. On the other hand, it follows from (2.6) and (2.11) that $K_\alpha(0) = K_\alpha(0)$. This implies that $Q_a(0) = P(0)$. (Q.E.D.)

Therefore, we find from Lemmas 3.6, 3.7 and 3.8 that

**Theorem 3.2.** For any $a \in \mathbb{R}$ and any $T \in (0, \infty)$,
\[ E(\mathcal{B}(a - T) | F_X((a - T, a))) \]
\[ = E(\mathcal{B}(a - T) | \partial F_X(a)) \] + \( \int_0^T e^{(T - s)A}(P(s)e^* \alpha + \sqrt{2\pi^{-1}}b)d\nu^*_\alpha(s) \)
\[ = e^{T}\mathcal{B}_0 \cdot (Y_{n_0}(a) \ldots Y_{N-1}(a)) + e^{(T-1)A}(P(s)e^* \alpha + \sqrt{2\pi^{-1}}b)d\nu^*_\alpha(s), \]

where \( J \) is the \( N \times (N - n_0) \)-matrix given by (3.4) and \( P(t) \) is the unique solution of the matrix Riccati equation (3.5).

[3.3] As an application of Theorems 3.1 and 3.2, we shall show

**Theorem 3.3.** For any \( a \in R, t \in (0, \infty) \) and \( T \in (0, \infty), \)
\( (i) \) \( E(X(a + T + t) | F_X((a, a + T))) \)
\[ = E(X(a + T + t) | \partial F_X(a)) \] + \( (0 \ldots 0(-c_N)^{-1}2\pi) \int_0^T f(T + t, s)d\nu^*_\alpha(s) \)
\( (ii) \) \( E(X(a - T - t) | F_X((a - T, a))) \)
\[ = E(X(a - T - t) | \partial F_X(a)) \] + \( (0 \ldots 0(-c_N)^{-1}2\pi) \int_0^T f(T + t, s)d\nu^*_\alpha(s), \]

where \( f(t, s) \) is the \( N \times 1 \)-vector function for \( a = 0 \) in (3.2).

**Proof.** By Theorems 2.1 (xi) and 3.1, we have
\[ E(\tilde{\mathcal{B}}(a + T + t) | F_X((a, a + T))) \]
\[ = E(e^{T}\tilde{\mathcal{B}}(a + T) | F_X((a, a + T))) \] + \( \int_0^T f(T + t, s)d\nu^*_\alpha(s) \)
\[ = e^{T}(E(\tilde{\mathcal{B}}(a + T) | \partial F_X(a)) + \int_0^T f(T + t, s)d\nu^*_\alpha(s)). \]

Therefore we obtain (i) noting Theorem 2.1 (i). By Theorem 2.3 (i) (xi) and Theorem 3.2, (ii) is similarly proved. (Q.E.D.)

Immediately from Theorem 3.3, we have

**Theorem 3.4.** For any \( a \in R \) and \( t \in (0, \infty), \)
\( (i) \) \( X(a + t) = E(X(a + t) | \partial F_X(a)) \)
\[ + (0 \ldots 0(-c_N)^{-1}2\pi) \int_0^T f(t, s)d\nu^*_\alpha(s), \]
\( (ii) \) \( X(a - t) = E(X(a - t) | \partial F_X(a)) \)
\[ + (0 \ldots 0(-c_N)^{-1}2\pi) \int_0^T f(t, s)d\nu^-_\alpha(s). \]
We define two Gaussian processes \( Y_\pm = (Y_\pm(t); \ t \geq 0) \) by
\[
Y_\pm(t) = X(\pm t) - E(X(\pm t)|\partial F_X(0)) .
\]
From Theorem 3.4, we have the following representations
\[
Y_\pm(t) = (0 \cdots 0(-c_X)^{-1}2\pi) \cdot \int_0^t f(t, s) du_\pm(s) .
\]
It is easy to see from Theorems 2.2 (ii) and 2.4 (ii) that \( \sigma(Y_\pm(s); \ 0 \leq s \leq t) \vee \partial F_X(0) = \sigma(u_\pm(s); \ 0 \leq s \leq t) \vee \partial F_X(0) \). Moreover \( Y_\pm \) and \( u_\pm \) are independent of \( \partial F_X(0) \). Therefore we obtain
\[
\sigma(Y_\pm(s); \ 0 \leq s \leq t) = \sigma(u_\pm(s); \ 0 \leq s \leq t) \quad (t \geq 0) .
\]
This implies that representations (3.7) are canonical ([3]).

§4. Integral representations of the predictors (II)

In the previous section we have obtained integral representations of
\[
E(X(b + t)|F_X((a, b))) - E(X(b + t)|F_X(a)) \quad (a < b, \ t > 0).
\]
The aim of this section is to obtain integral representations of
\[
E(X(b + t)|F_X((a, b))) - E(X(b + t)|F_X(b)) \quad (a < b, \ t > 0).
\]

For any \( a \in \mathbb{R} \) we define \( N \times 1 \)-vectors \( g_a(t, s) \ (0 \leq s, \ t < \infty) \) by
\[
g_a(t, s) = E[\theta(a + t) \cdot (\varphi(a - s) - E(\varphi(a - s)|F_a(s))^*)\cdot e^* .
\]
Then we shall prove

**Lemma 4.1.** \( g_a(t, s) = (\partial/\partial s)E(\nu_a(-s)\varphi(a + t)) \).

**Proof.** We put \( \varphi(s) = \varphi(a - s) - E(\varphi(a - s)|F_a(s)) \). It then follows from Theorems 2.3 (vi) and 2.4 (iv) that
\[
\nu_a(t) = e \cdot \int_0^t \varphi(s) ds + B_-(a) - B_-(a + t) \quad (t \geq 0) .
\]
Therefore, by (2.10) and Theorem 2.3 (vii), we have
\[
E(\nu_a(-s)\varphi(a + t))
\]
\[
= e^{-tA}E\{e^{(a-u)A} B_-\varphi(u)\}
\]
\[
= e^{-tA}E\{e \cdot \int_0^t \varphi(u) du \cdot (\varphi(a) - E(\varphi(a)|F_a(s))^*)\cdot e^* .
\]
This implies Lemma 4.1. (Q.E.D.)
Similarly as in Lemma 3.2, we can see from Theorem 2.4 and Lemma 4.1 that

**Lemma 4.2.** For any \( a \in \mathbb{R} \), \( t \in (0, \infty) \) and \( T \in (0, \infty) \),

\[
E(\mathcal{Y}(a + t) \mid F_X((a - T, a))) = E(\mathcal{Y}(a + t) \mid \partial F_X(a)) + \int_0^T g_a(t, s) d\gamma(s).
\]

Next we shall obtain an explicit form of \( g_a(t, s) \). We define for any \( t \in [0, \infty) \) an \( N \times N \)-matrix \( R(t) \) by

\[
R(t) = E[\mathcal{Y}(t) \cdot (\mathcal{Y}(0) - E(\mathcal{Y}(0) \mid \partial F_X(0)))^*].
\]

**Lemma 4.3.** For any \( a \in \mathbb{R} \)

\[
g_a(t, s) = R(t) \Phi^*(s) \mathbf{e}^* \quad (0 \leq s, t < \infty),
\]

where \( \Phi(s) \) is the unique solution of the following linear differential equation

\[
\begin{cases}
\frac{d\Phi(s)}{ds} = (A - P(s)\mathbf{e}^*\mathbf{e} - \sqrt{2\pi^{-1}b \cdot \mathbf{e}})\Phi(s) & (s > 0), \\
\Phi(0) = I_N.
\end{cases}
\]

In particular, \( g_a(t, s) \) are independent of \( a \).

**Proof.** We put \( R_a(t, s) = E[\mathcal{Y}(a + t) \cdot (\mathcal{Y}(a - s) - E(\mathcal{Y}(a - s) \mid F_a(s)))^*]. \) By Theorem 3.2,

\[
R_a(t, s) = E[\mathcal{Y}(a + t) \cdot (\mathcal{Y}(a - s) - E(\mathcal{Y}(a - s) \mid \partial F_X(a)))^*] + E(\mathcal{Y}(a + t) \int_0^s f^*(s, \xi) d\gamma^a(\xi)) = I - II.
\]

It is easy to see from Theorem 2.4 and Lemma 4.2 that \( II = \int_0^s g_a(t, \xi) f^*(s, \xi) d\xi \).

On the other hand, by Theorem 2.3 (iii) (ix), \( I = E[\mathcal{Y}(a + t) \cdot (\mathcal{Y}(a) - E(\mathcal{Y}(a) \mid \partial F_X(a))^*) \cdot e^{\Lambda^*}]. \) Since \( K_a(0) = K_X(0) \) and \( \partial F_X(a) = \sigma(Y_n(a); n_0 \leq n \leq N - 1) \), it can be seen that \( E(\mathcal{Y}(a) \mid \partial F_X(a)) = J \cdot (Y_n(a) \cdots Y_{N-1}(a))^* \), where \( J \) is the \( N \times (N - n_0) \)-matrix in (3.4). Therefore we find that \( I = E[\mathcal{Y}(t)(\mathcal{Y}(0) - E(\mathcal{Y}(0) \mid \partial F_X(0))^*)e^{\Lambda^*} = R(t)e^{\Lambda^*}. \) Consequently, we have

\[
R_a(t, s) = R(t)e^{\Lambda^*} - \int_0^s R_a(t, \xi)e^* \cdot f^*(s, \xi) d\xi.
\]

Since \( (\partial / \partial s) f^*(s, \xi) = f^*(s, \xi) A^* \) by (3.2), we obtain the following linear
differential equation

\[
\begin{align*}
\frac{\partial}{\partial s} R_{\alpha}(t, s) &= R_{\alpha}(t, s)(A - P(s)e^s e - \sqrt{2\pi}^{-1} b \cdot e)^s \quad (s > 0), \\
R_{\alpha}(t, 0) &= R(t). 
\end{align*}
\]

Thus, using the unique solution \( \Phi(t) \) of equation (4.4), we find that \( R_{\alpha}(t, s) = R(t)\Phi^*(s) \) and this completes the proof of Lemma 4.3. (Q.E.D.)

In the proof of Lemma 4.3, we have shown

\[
R(t) = K_\alpha(0)e^{tA} - E(\Phi(t) \cdot (Y_{n_0}(0) \cdots Y_{N-1}(0)))J^*,
\]

where \( J \) is the \( N \times (N - n_0) \)-matrix in (3.4). Furthermore we define for any \( t \in \mathbb{R} \) an \( N \times (N - n_0) \)-matrix \( J(t) \) by

\[
J(t) = (K_\alpha(t)_{m,n})_{n_0 \leq m \leq N - 1, n_0 \leq n \leq N - 1}^{-1}.
\]

Immediately from Lemmas 4.2 and 4.3, we have

\section*{Theorem 4.1}

For any \( a \in \mathbb{R} \), \( t \in (0, \infty) \) and \( T \in (0, \infty) \),

\[
E(\Phi(t) \cdot (Y_{n_0}(0) \cdots Y_{N-1}(0)))^s = J(t) \cdot (Y_{n_0}(a) \cdots Y_{N-1}(a))^s + \int_0^T R(t)\Phi^*(s)e^s dv_s^\alpha (s).
\]

Since \( K_\alpha(t) = K_\alpha(0) e^{tA} \) \( (t \geq 0) \), it is easy to see from (4.3) and (4.5) that \( E[\Phi(t) \cdot (Y_{n_0}(0) \cdots Y_{N-1}(0)))^s] = R(t) \) \( (t \geq 0) \). Therefore, similarly as in Theorem 4.1, we can show from Theorems 2.1, 2.2, 3.1 and (3.3) that

\section*{Theorem 4.2}

For any \( a \in \mathbb{R} \), \( t \in (0, \infty) \) and \( T \in (0, \infty) \),

\[
E(\Phi(t) \cdot (Y_{n_0}(a) \cdots Y_{N-1}(a)))^s = J(t) \cdot (Y_{n_0}(a) \cdots Y_{N-1}(a))^s + \int_0^T R(t)\Phi^*(s)e^s dv_s^\alpha (s).
\]

For any \( t \in [0, \infty) \) we define a \( 1 \times N \)-vector \( r(t) \) by

\[
r(t) = E[X(t) \cdot (Y_{n_0}(0) e^{tA})^s - E(\Phi(0) \cdot (Y_{n_0}(0) e^{tA}))^s].
\]

Since \( X(t) = (-2\pi)^{-1/2} X_{N-1}(t) = (-2\pi)^{-1/2} Y_{N-1}(t) \), it can be seen from Theorems 4.1 and 4.2 that

\section*{Theorem 4.3}

For any \( a \in \mathbb{R} \), \( t \in (0, \infty) \) and \( T \in (0, \infty) \),

\begin{enumerate}
\item \( E(X(t) \cdot (Y_{n_0}(a) e^{tA}))^s \)
\[ E(X(a + t) | \partial F_X(a)) + \int_0^T r(t) \Phi^*(s)e^*d\nu_*(s), \]

(ii) \[ E(X(a - t) | F_X((a, a + T))) = E(X(a - t) | \partial F_X(a)) + \int_0^T r(t) \Phi^*(s)e^*d\nu_*(s). \]

More generally, we define for any \( Y \in M_X \) two \( 1 \times N \)-vectors \( r_+(Y) \) by

\[ r_+(Y) = E\{ Y \cdot (\mathcal{X}(0) - E(\mathcal{X}(0) | \partial F_X(0))\} \]

and

\[ r_-(Y) = E\{ Y \cdot (\mathcal{Y}(0) - E(\mathcal{Y}(0) | \partial F_X(0))\} \].

Note that \( r(t) = r_+(X(-t)) = r_-(X(t)) \) (\( t \in [0, \infty) \)). Using the unitary operator \( U(-\alpha) \) in (2.12), we can prove, by Theorem 4.3,

**Theorem 4.4.** Let any \( a \in \mathbb{R} \) and \( T \in (0, \infty) \) be fixed.

(i) For any \( Y \in M_X(a) \)

\[ E(Y | F_X((a - T, a))) = E(Y | \partial F_X(a)) + \int_0^T r_-(U(-\alpha)Y)\Phi^*(s)e^*d\nu_*(s). \]

(ii) For any \( Y \in M_X(a) \)

\[ E(Y | F_X((a, a + T))) = E(Y | \partial F_X(a)) + \int_0^T r_+(U(-\alpha)Y)\Phi^*(s)e^*d\nu_*(s). \]

### § 5. Integral representations of the predictors (III)

In this section we shall obtain integral representations of \( E(X(a + t) | F_X((-a, a))) - E(X(a + t) | \partial F_X(0)) \) (\( a > 0, \ t > 0 \)). We define for any \( t \in [0, \infty) \) two \( N \times N \)-matrices \( S_\pm(t) \) by

\[ S_+(t) = E(\mathcal{Y}(t) \cdot \mathcal{X}^*(0)) \cdot K_\mathcal{X}(0)^{-1} \]

and

\[ S_-(t) = E(\mathcal{X}(-t) \cdot \mathcal{Y}^*(0)) \cdot K_\mathcal{X}(0)^{-1}. \]

As an application of Theorem 4.1, we shall prove

**Theorem 5.1.** For any \( a \in (0, \infty) \) and \( t \in (0, \infty) \),

\[ E(\mathcal{Y}(a + t) | F_X((-a, a))) = E(\mathcal{Y}(a + t) | \partial F_X(0)) + \int_0^a S_+(t)e^{(a-\tau)A}(P(s)e^* + \sqrt{2\pi^{-1}}b)d\nu_*(s). \]
\[ + \int_a^{\alpha} R(t)\phi^*(s)e^*d\nu_s(s). \]

**Proof.** It is easy to see from Theorem 4.1 that
\[ E(\mathcal{F}(a + t)|F_X((-a,a))) = E(\mathcal{F}(a + t)|F_X((0,a))) + \int_a^{\alpha} R(t)\phi^*(s)e^*d\nu_s(s). \]

By Theorems 2.1 (iii) (iv) and 2.3 (iii) (iv), we can show that
\[ E(\mathcal{F}(a + t)|F_X(a)) = S_+(t)\mathcal{F}(a). \]

Therefore, it follows from Theorem 3.1 that
\[ E(\mathcal{F}(a + t)|F_X((0,a))) = S_+(t)\cdot E(\mathcal{F}(a)|\partial F_X(0)) + \int_0^{\alpha} f(a, s)d\nu_s(s) \]
\[ = E(\mathcal{F}(a + t)|\partial F_X(0)) + \int_0^{\alpha} S_+(t)f(a, s)d\nu_s(s), \]
where \( f(a, s) \) are \( N \times 1 \)-vectors in (3.2). Thus we have proved Theorem 5.1. (Q.E.D.)

Similarly, we find from Theorems 3.2 and 4.2 that

**Theorem 5.2.** For any \( a \in (0, \infty) \) and \( t \in (0, \infty) \),
\[ E(\mathcal{F}(-a - t)|F_X((-a,a))) \]
\[ = E(\mathcal{F}(-a - t)|\partial F_X(0)) + \int_0^{\alpha} S_-(t)e^{\alpha-tA}(0)e^* + \sqrt{2\pi}^{-1}b)\nu_s(s) \]
\[ + \int_a^{\alpha} R(t)\phi^*(s)e^*d\nu_s(s), \]

**Remark 5.1.** By Theorems 2.2 (ii) (iii) and 2.4 (ii) (iii) we note that the decompositions in Theorems 5.1 and 5.2 are orthogonal.

For any \( t \in [0, \infty) \) we define two \( 1 \times N \)-vectors \( S_+(t) \) by
\[ S_+(t) = E(X(t)\cdot \mathcal{F}(0)) \cdot K_x(0)^{-1} \]
and
\[ S_-(t) = E(X(-t)\cdot \mathcal{F}(0)) \cdot K_x(0)^{-1}. \]

Since \( X(t) = (-2\pi)c_\mathcal{F}X_{\mathcal{F}}^{-1}(t) = (-2\pi)c_\mathcal{F}^{-1}Y_{\mathcal{F}}^{-1}(t) \), we can show from Theorems 5.1 and 5.2 that

**Theorem 5.3.** For any \( a \in (0, \infty) \) and \( t \in (0, \infty) \),
(i) \[ E(X(a + t) | F_x((-a, a))) \]
\[ = E(X(a + t) | \partial F_x(0)) + \int_0^a S_+(t) e^{(a-t)A} \{P(s)e^* + \sqrt{2\pi}^{-1}b\} dv^+(s) \]
\[ + \int_\alpha^{2\pi} r(t)\Phi^+(s)e^* dv^+(s) \]
\[ = E(X(-a - t) | \partial F_x(0)) \]
\[ + \int_0^a S_-(t) e^{(a-s)A} \{P(s)e^* + \sqrt{2\pi}^{-1}b\} dv^-(s) \]
\[ + \int_\alpha^{2\pi} r(t)\Phi^-(s)e^* dv^-\tau(s) . \]

(ii) \[ E(X(-a - t) | F_x((-a, a))) \]
\[ = E(X(-a - t) | \partial F_x(0)) \]
\[ + \int_0^a S_-(t) e^{(a-s)A} \{P(s)e^* + \sqrt{2\pi}^{-1}b\} dv^-(s) \]
\[ + \int_\alpha^{2\pi} r(t)\Phi^-(s)e^* dv^-\tau(s) . \]

More generally, we define for any \( Y \in M_x \) two \( 1 \times N \)-vectors \( S_+(Y) \) by
\[ S_+(Y) = E(Y \cdot e^*(0)) \cdot K_x(0)^{-1} \]
and
\[ S_-(Y) = E(Y \cdot e^*(0)) \cdot K_x(0)^{-1} . \]

Using the unitary operators \( U(\pm a) \) in (2.12), we can generalize Theorem 5.3 as follows.

**Theorem 5.4.** Let any \( a \in (0, \infty) \) be fixed.

(i) For any \( Y \in M_x(a) \)
\[ E(Y | F_x((-a, a))) \]
\[ = E(Y | \partial F_x(0)) + \int_0^a S_+(U(-a)Y) e^{(a-s)A} \{P(s)e^* + \sqrt{2\pi}^{-1}b\} dv^+(s) \]
\[ + \int_\alpha^{2\pi} r_-(U(-a)Y)\Phi^+(s)e^* dv^+(s) . \]

(ii) For any \( Y \in M_x(-a) \)
\[ E(Y | F_x((-a, a))) \]
\[ = E(Y | \partial F_x(0)) + \int_0^a S_-(U(a)Y) e^{(a-s)A} \{P(s)e^* + \sqrt{2\pi}^{-1}b\} dv^-(s) \]
\[ + \int_\alpha^{2\pi} r_+(U(a)Y)\Phi^-(s)e^* dv^-\tau(s) . \]

**Remark 5.2.** We note that the decompositions in Theorems 5.3 and 5.4 are orthogonal.
§ 6. Prediction errors

In [6] we have obtained the following commutative diagram

\[
\begin{array}{ccc}
M_x & \xrightarrow{U} & Z_d \\
\downarrow{U_1} & & \downarrow{V} \\
K & \xrightarrow{\hat{K}} & L^2(R)
\end{array}
\]

(6.1)

\[U_1(k(\cdot-t)) = X(t) \quad \hat{K}(k(\cdot-t)) = \sqrt{2\pi^{-1}}E(t-\cdot) \quad \text{and} \quad U(X(t)) = e^{it\cdot}.
\]

Similarly we have the following commutative diagram

\[
\begin{array}{ccc}
M_x & \xrightarrow{U} & Z_d \\
\downarrow{U_1} & & \downarrow{\hat{V}} \\
K & \xrightarrow{\hat{K}} & L^2(R)
\end{array}
\]

(6.2)

We note that

\[f = \sqrt{2\pi^{-1}}(Vf\cdot h)^* = \sqrt{2\pi^{-1}}(\hat{V}f\cdot h)^* \quad (f \in L^2(R)).
\]

Using \(L^2\)-functions \(E_n\) in (2.4) we define \(N\) functions \(\varphi_n\) in \(Z_d\) \((0 \leq n \leq N-1)\) by

\[\varphi_n = \sqrt{2\pi^{-1}}V(E_n).
\]

(6.4)

It is easy to see from (6.3) that

\[V(\sqrt{2\pi^{-1}}E_n(t-\cdot)) = e^{it\cdot}\varphi_n \quad \text{and} \quad \hat{V}(\sqrt{2\pi^{-1}}E_n(\cdot-t)) = e^{it\cdot}\hat{\varphi}_n.
\]

(6.5)

Therefore it can be shown from (2.1), (2.6), (2.9) and (2.11) that

\[U(X_n(t)) = e^{it\cdot}\varphi_n \quad \text{and} \quad U(Y_n(t)) = e^{it\cdot}\hat{\varphi}_n.
\]

(6.6)

Furthermore we are able to prove by Theorems 2.2 (iv) and 2.4 (iv) that

\[U(\nu^+(t))(\lambda) = \sqrt{2\pi} \int_{a-t}^{a+t} \{b_{n\lambda}^{-1}\varphi_{n\lambda}(\lambda)i\lambda e^{is} - P_{Z_{d(a,t)}}(e\cdot U\varphi(s))(\lambda)\}ds
\]

(6.7)

and

\[U(\nu^-(t))(\lambda) = \sqrt{2\pi} \int_{a-t}^{a} \{b_{n\lambda}^{-1}\varphi_{n\lambda}(-\lambda)(-i\lambda)e^{is} - P_{Z_{d(a,t)}}(e\cdot U\varphi(s))(\lambda)\}ds,
\]

(6.8)

where \(U\varphi(s) = (UX_0(s), \ldots, UX_{N-1}(s))^*\) and \(U\varphi(s) = (UY_0(s), \ldots, UY_{N-1}(s))^*\).

We define functions \(D^+_a(t, \lambda)\) by
By (6.7), (6.8) and (6.9), it follows from Theorems 3.1 and 3.2 that

\[
\begin{cases}
D^+_{a}(t, \lambda) = \sqrt{2\pi} \left[ b^{-1}_{\text{ns}} \varphi_{\text{ns}}(\lambda) \cdot i \lambda e^{i(t+a)} - P_{Z_{a,t}, a}(e \cdot U\mathcal{W}(t + a))(\lambda) \right], \\
D^-_{a}(t, \lambda) = \sqrt{2\pi} \left[ b^{-1}_{\text{ns}} \varphi_{\text{ns}}(-\lambda)(-i\lambda) e^{i(a-t)} - P_{Z_{a,t}, a}(e \cdot U\mathcal{W}(a - t))(\lambda) \right].
\end{cases}
\]

In [6] we have introduced the function \( P(\lambda, \varphi)(\lambda \in C, \varphi \in \mathcal{C}(\{0\})) \) defined by

\[
P(\lambda, \varphi) = (2\pi)^{-1} \sum_{n=0}^{N-1} \left( \sum_{k=0}^{n-1} c_{n+k+1} \varphi(k)(0) \right)(-i\lambda)^n.
\]

We then define \( N \) functions \( P_n \) \((0 \leq n \leq N - 1)\) by

\[
P_n(\lambda) = (n!)^{-1} P(\lambda, x^n).
\]

Firstly we shall prove

**Lemma 6.1.** \( \varphi_n = P_n \) for any \( n \in \{n_0, n_0 + 1, \ldots, N - 1\} \).

**Proof.** We define \( N \) real \( L^2 \)-functions \( F_n \) \((0 \leq n \leq N - 1)\) by

\[
F_n(t) = \sqrt{2\pi^{-1}} \chi_{(0, \infty)}(t) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \lambda \end{pmatrix}.
\]

By Lemma 8.2 in [6] and (2.3), (6.8) in [7], we have

\[
F_n = (n!)^{-1} \left[ P(-\cdot, x^n) \cdot P(-\cdot)^{-1} \right] = \left( \hat{P}_n \hat{P}^{-1} \right)^{\wedge}.
\]

Since \( P_n \) \((n_0 \leq n \leq N - 1)\), \( P \) and \( Q \) are polynomials of at most order \( N - n_0 - 1 \), \( N \) and \( n_0 \), respectively, we see from (2.7) in [7] and Lemma 4.1 in [7] that

\[
E_n = \left( \hat{P}_n \cdot \hat{Q} \cdot \hat{P}^{-1} \right)^{\wedge} = \left( \hat{P}_n \cdot h \right)^{\wedge} \quad (n_0 \leq n \leq N - 1).
\]

This implies Lemma 6.1 by (6.3) and (6.5). (Q.E.D.)

Noting that \( P_{N-1} = -(2\pi)^{-1} c_N \), we note by (6.11), (6.12) and Lemma 6.1 that
Moreover, by Theorem 3.1 and (6.6),
\begin{equation}
P_{3Z_{a}(e)}(e \cdot UZ(t + a))(\lambda) = e^{i\alpha \lambda} e^{iA \cdot (\varphi_{a}(\lambda) \cdots \varphi_{N_{a}}(\lambda))^{*}}.
\end{equation}

Therefore, defining functions \(\psi(t, \lambda)\) by
\begin{equation}
\psi(t, \lambda) = \sqrt{2\pi} \{\beta_{4}^{z} e^{i\alpha \lambda} \phi_{a}(\lambda) i \hat{\lambda} - e \cdot e^{iA \cdot (\varphi_{a}(\lambda) \cdots \varphi_{N_{a}}(\lambda))^{*}}\}
\end{equation}
we find from (6.10) that
\begin{equation}
D_{+}^{z}(t, \lambda) + \sqrt{2\pi} \int_{0}^{t} e \cdot f(t, s) D_{+}^{z}(s, \lambda) ds = e^{i\alpha \lambda} \psi(t, \lambda).
\end{equation}

By the uniqueness of solution of Volterra equation (6.19), we have
\begin{equation}
D_{-}^{z}(t, \lambda) = e^{i\alpha \lambda} D_{-}^{z}(t, \lambda).
\end{equation}

Similarly, by Theorem 3.2, (6.6) and (6.10),
\begin{equation}
D_{-}^{z}(t, \lambda) + \sqrt{2\pi} \int_{0}^{t} e \cdot f(t, s) D_{-}^{z}(s, \lambda) ds = e^{i\alpha \lambda} \psi(t, -\lambda)
\end{equation}
and
\begin{equation}
D_{-}^{z}(t, \lambda) = e^{i\alpha \lambda} D_{-}^{z}(t, \lambda).
\end{equation}

In particular, we see from (6.19) and (6.21) that
\begin{equation}
D_{+}^{z}(t, \lambda) = D_{+}^{z}(t, -\lambda).
\end{equation}

Next we shall obtain explicit representations of functions \(f(t, s)\) and \(g(t, s)\) in (3.2) and (4.1), respectively.

**Lemma 6.2.** (i) \(f(t, s) = (D_{+}^{z}(s, \cdot), e^{it \cdot \varphi})_{d},\)
(ii) \(g(t, s) = R(t) \Phi^{*}(s) e^{*} = (D_{+}^{z}(s, \cdot), e^{it \cdot \tilde{\varphi}})_{d},\)
where \(\varphi = (\varphi_{0} \cdots \varphi_{N_{a}})^{*}.\)

**Proof.** By (6.7), (6.8) and (6.9),
\begin{equation}
U(\nu_{a}^{z}(s))(\lambda) = \pm \int_{a}^{a+s} D_{+}^{z}(\pm(t - a), \lambda) d\alpha.
\end{equation}

Therefore it follows from Lemma 3.1 and (6.6) that
\begin{equation}
f(t, s) = \frac{\partial}{\partial s} E(\nu_{a}^{z}(s) \cdot \mathcal{X}(t)) = (D_{+}^{z}(s, \cdot), e^{it \cdot \varphi})_{d}.
\end{equation}
Similarly we find from Lemmas 4.1, 4.3 and (6.6) that
\[ g(t, s) = R(t) \Phi^*(s) e^* = \frac{\partial}{\partial s} E(v_0^-(s) \Phi(t)) = (D_5^-(s, \cdot), e^{it\cdot\varphi})_d. \]

(Q.E.D.)

**Lemma 6.3.** \( S_+(t)f(a, s) = (D_5^+(s, \cdot), e^{i(a+t)\cdot\varphi})_d. \)

**Proof.** By (5.3), (5.4), Theorems 2.1 (ix) and 2.3 (ix),
\[ S_+(t) = (0 \cdots 0 (-c_N)^{-1}2\pi) e^{it}. \]

Therefore it follows from (3.2) and Lemma 6.2 that
\[ S_+(t)f(a, s) = (0 \cdots 0 (-c_N)^{-1}2\pi) f(a + t, s) = (0 \cdots 0 (-c_N)^{-1}2\pi)(D_5^+(s, \cdot), e^{i(a+t)\cdot\varphi})_d. \]

Noting that \( \varphi_{N+1} = P_{N+1} = -(2\pi)^{-1}c_N, \) we have Lemma 6.3. (Q.E.D.)

**Lemma 6.4.** \( r(t) \Phi^*(s) e^* = (D_5^+(s, \cdot), e^{i(t-t)\cdot\varphi})_d. \)

**Proof.** By (4.3), (4.7) and Lemma 6.2,
\[ r(t) \Phi^*(s) e^* = (0 \cdots 0 (-c_N)^{-1}2\pi) R(t) \Phi^*(s) e^* = (0 \cdots 0 (-c_N)^{-1}2\pi)(D_5^+(s, \cdot), e^{i(t-t)\cdot\varphi})_d. \]

(Q.E.D.)

Now we shall obtain explicit integral representations of prediction errors.

**Theorem 6.1.** For any \( a \in \mathbb{R}, \) \( t \in (0, \infty) \) and \( T \in (0, \infty), \)
\[ ||X(a + t) - E(X(a + t) \mid F_X((a - T, a)))||^2 = ||X(a + t) - E(Y(a + t) \mid F_X((a, a + T)))||^2 = \int_0^T (D_5^+(s, \cdot), e^{it\cdot\varphi})_d ds. \]

**Proof.** Using Lemmas 6.4 and 6.2, we see from Theorems 4.3 and 3.4 that
\[ ||X(a + t) - E(Y(a + t) \mid F_X((a - T, a)))||^2 = \int_0^T (D_5^+(s, \cdot), e^{it\cdot\varphi})_d ds = \int_0^T (D_5^+(s, \cdot), e^{it\cdot\varphi})_d ds. \]
The rest is similarly proved. (Q.E.D.)

THEOREM 6.2. For any \( a \in (0, \infty) \) and \( t \in (0, \infty) \),

\[
\|X(\pm (a + t)) - E(X(\pm (a + t)) \mid F_X((-a, a)))\|^2
= \int_a^{a+t} (D^+_a(s, \cdot), e^{it(s+\cdot)})^2 ds - \int_a^{a-t} (D^-_a(s, \cdot), e^{it(s-\cdot)})^2 ds .
\]

Proof. Using Lemmas 6.3, 6.4 and 6.2, we find from Theorems 5.3 and 3.4 that

\[
\|X(\pm (a + t)) - E(X(\pm (a + t)) \mid F_X((-a, a)))\|^2
= \|X(\pm (a + t))\|^2 - \|E(X(\pm (a + t)) \mid F_X((-a, a)))\|^2
- \int_a^{a+t} (D^+_a(s, \cdot), e^{it(s+\cdot)})^2 ds - \int_a^{a-t} (D^-_a(s, \cdot), e^{it(s-\cdot)})^2 ds
= \|X(\pm (a + t)) - E(X(\pm (a + t)) \mid F_X(0))\|^2
- \int_a^{a+t} (D^+_a(s, \cdot), e^{it(s+\cdot)})^2 ds - \int_a^{a-t} (D^-_a(s, \cdot), e^{it(s-\cdot)})^2 ds
= \int_a^{a+t} (D^+_a(s, \cdot), e^{it(s+\cdot)})^2 ds - \int_a^{a-t} (D^-_a(s, \cdot), e^{it(s-\cdot)})^2 ds . \quad (Q.E.D.)
\]

§7. Integral representations of the predictors (IV)

In this section we shall give explicit integral representations of prediction formulas in §3, §4 and §5 using the section of the previous section. By Theorem 3.3 and Lemma 6.2,

THEOREM 7.1. For any \( a \in \mathbb{R} \), \( t \in (0, \infty) \) and \( T \in (0, \infty) \),

(i) \( E(X(a + T + t) \mid F_X((a, a + T))) = E(X(a + T + t) \mid \partial F_X(a)) + \int_0^T (D^+_a(s, \cdot), e^{it(s+\cdot)}) d\nu^+_a(s) \)

(ii) \( E(X(a - t - T) \mid F_X((a - T, a))) = E(X(a - t - T) \mid \partial F_X(a)) + \int_0^T (D^+_a(s, \cdot), e^{it(s-\cdot)}) d\nu^-_a(s) . \)

Similarly, we see from Theorem 3.4 and Lemma 6.2 that

THEOREM 7.2. For any \( a \in \mathbb{R} \) and \( t \in (0, \infty) \),

\( X(a \pm t) = E(X(a \pm t) \mid \partial F_X(a)) + \int_0^t (D^+_a(s, \cdot), e^{it}) d\nu^+_a(s) \).
By Theorem 4.3 and Lemma 6.4,

**THEOREM 7.3.** For any $a \in \mathbb{R}$, $t \in (0, \infty)$ and $T \in (0, \infty)$,

(i) $E(X(a + t) | F_X((a - T, a)))$

\[ = E(X(a + t) | \partial F_X(a)) + \int_0^T (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s) \]

(ii) $E(X(a - t) | F_X((a, a + T)))$

\[ = E(X(a - t) | \partial F_X(a)) + \int_0^T (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s). \]

From Theorem 5.3, Lemmas 6.3 and 6.4, it follows that

**THEOREM 7.4.** For any $a \in (0, \infty)$ and $t \in (0, \infty)$,

\[ E(X(\pm(a + t)) | F_X((-a, a))) \]

\[ = E(X(\pm(a + t)) | \partial F_X(0)) + \int_0^a (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s) \]

\[ + \int_a^{2a} (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s). \]

By (6.6), (6.20), (6.22) and (6.24), we have

\[ \frac{\partial}{\partial s} E(\mathcal{X}(a + t)\nu^\gamma(s)) = \frac{\partial}{\partial s} E(\mathcal{Y}(a - t)\nu^\gamma(s)) = (D_\gamma(s, \cdot), e^{it\varphi}). \]

Therefore, similarly as in Lemma 3.2, we obtain the following Theorem 7.5 as a supplement of Theorems 4.1 and 4.2.

**THEOREM 7.5.** For any $a \in \mathbb{R}$, $t \in (0, \infty)$ and $T \in (0, \infty)$,

(i) $E(\mathcal{X}(a + t) | F_X((a - T, a)))$

\[ = E(\mathcal{X}(a + t) | \partial F_X(a)) + \int_0^T (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s), \]

(ii) $E(\mathcal{Y}(a - t) | F_X((a, a + T)))$

\[ = E(\mathcal{Y}(a - t) | \partial F_X(a)) + \int_0^T (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s). \]

Furthermore, as a supplement of Theorems 5.1 and 5.2, we shall prove

**THEOREM 7.6.** For any $a \in (0, \infty)$ and $t \in (0, \infty)$,

(i) $E(\mathcal{X}(a + t) | F_X((-a, a)))$

\[ = E(\mathcal{X}(a + t) | \partial F_X(0)) + \int_0^a (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s) \]

\[ + \int_a^{2a} (D_\gamma(s, \cdot), e^{it\varphi})_d\nu^\gamma(s), \]

(ii) $E(\mathcal{Y}(-a - t) | F_X((-a, a)))$
\[ = E(\mathcal{X}(-a - t)|\partial F_{X}(0)) + \int_{0}^{a} (D_{t}^{\gamma}(s, \cdot), e^{(a+t)\varphi})_{a} \nu_{\cdot}(s) + \int_{a}^{2a} (D_{t}^{\gamma}(s, \cdot), e^{ut\varphi})_{a} \nu_{\cdot}(s). \]

**Proof.** By Theorems 7.5 (i), 2.1 (ix), 3.1 and Lemma 6.2,

\[ E(\mathcal{X}(a + t)|F_{X}((-a, a))) = E(\mathcal{X}(a + t)|\partial F_{X}(a)) + \int_{0}^{a} (D_{t}^{\gamma}(s, \cdot), e^{ut\varphi})_{a} \nu_{\cdot}(s) = E(\mathcal{X}(a + t)|F_{X}((0, a)) + \int_{a}^{2a} (D_{t}^{\gamma}(s, \cdot), e^{ut\varphi})_{a} \nu_{\cdot}(s) = E(\mathcal{X}(a + t)|\partial F_{X}(0)) + \int_{0}^{a} (D_{t}^{\gamma}(s, \cdot), e^{(a+t)\varphi})_{a} \nu_{\cdot}(s). \]

Similarly, we have (ii) from Theorems 7.5 (ii), 2.3 (ix), 3.2 and Lemma 6.2.

(Q.E.D.)

**Remark 7.1.** Theorem 7.3 (resp. Theorem 7.4) follows immediately from Theorem 7.5 (resp. Theorem 7.6).

**Remark 7.2.** The decompositions in Theorems 7.4 and 7.6 are orthogonal.

**REFERENCES**


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