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ON THE COHEN-MACAULAY PROPERTY OF $A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ FOR SPACE MONOMIAL CURVES

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1. Introduction

Let $A = k[X, Y, Z]$ and $k[U]$ be polynomial rings over a field k and let l, m and n be positive integers with $\gcd(l, m, n) = 1$. We denote by $\mathfrak{p}$ the defining ideal of the space monomial curve $x = u^l$, $y = u^m$, and $z = u^n$. In other words, $\mathfrak{p}$ is the kernel of the k-algebra homomorphism $\varphi : A \to k[U]$ defined by $\varphi(X) = U^l$, $\varphi(Y) = U^m$, and $\varphi(Z) = U^n$. Let $R_s(\mathfrak{p})$ be a symbolic Rees algebra of $\mathfrak{p}$, i.e., $R_s(\mathfrak{p}) = \sum_{i \geq 0} \mathfrak{p}^{(i)} t^i$, where t is an indeterminate over A, and let S be an A-subalgebra of $R_s(\mathfrak{p})$ generated by $\mathfrak{p}t$ and $\mathfrak{p}^{(2)}t^2$, i.e., $S = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$. In this paper we are mainly interested in the Cohen-Macaulay or Gorenstein property of S.

The research on the ring-theoretic property of S was begun by Herzog and Ulrich [7], who show among many interesting results that, if $\mathfrak{p}$ is self-linked, that is $\mathfrak{p} = (x_1, x_2) : \mathfrak{p}$ for some $x_1, x_2 \in \mathfrak{p}$, then S is a Gorenstein ring. When $\mathfrak{p}$ is not self-linked, however, there are examples where S is Cohen-Macaulay but not Gorenstein (cf. [7, Example 2.4]), and examples where S is not Cohen-Macaulay (cf. [4, Example (3.8)]). The principal aim of this paper is to determine exactly when S is Cohen-Macaulay. To state our main result, we assume that $\mathfrak{p}$ is not a complete intersection and choose a matrix M of the form

$$M = \begin{bmatrix} X^{a_1} & Y^{b_1} & Z^{c_1} \\ Y^{b_2} & Z^{c_2} & X^{a_2} \end{bmatrix}$$

(here a_i, b_i and c_i are positive integers) so that the ideal $\mathfrak{p}$ is generated by the 2 by 2 minors of M. We note that this choice is possible, see [5]. Then as was shown in [7, Corollary 1.10], $\mathfrak{p}$ is *not* self-linked if and only if either $a_1 > a_2$, $b_1 > b_2$ and $c_1 > c_2$ or $a_1 < a_2$, $b_1 < b_2$ and $c_1 < c_2$. If for simplicity we assume that $a_1 > a_2$, $b_1 > b_2$, and $c_1 > c_2$, then our main result can be stated as follows.

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THEOREM 1.1. *With the above notation the following two conditions are equivalent.*

(1) $S = A[pt, p^{(2)}t^2]$ *is a Cohen-Macaulay ring.*
(2) $(a_1 - 2a_2)(b_1 - 2b_2)(c_1 - 2c_2) \geq 0$

When this is the case, the Cohen-Macaulay type of S *is equal to three.*

It follows from this theorem that S is never a Gorenstein ring, unless p is either a complete intersection or a self-linked ideal. Goto, Nishida and Shimoda have discovered that condition (2) in Theorem 1.1 implies condition (1) (cf. [4, Theorem (3.1)]). Thus our contribution is to show that condition (2) is also necessary for S to be a Cohen-Macaulay ring. We shall prove Theorem 1.1 in the next section.

In section 3 we shall study certain projective space monomial curves. Let $B = k[X, Y, Z, W]$ and $k[U, V]$ be polynomial rings over k and let $\Phi : B \to k[U, V]$ be the k-algebra homomorphism defined by $\Phi(X) = U^l$, $\Phi(Y) = U^m V^{l-m}$, $\Phi(Z) = U^n V^{l-n}$, and $\Phi(W) = V^l$, where $l > m$, $l > n$ and $m \neq n$. Let $P = \operatorname{Ker} \Phi$ and let $T = B[Pt, P^{(2)}t^2]$ be a B-subalgebra of $R_s(P)$. We shall also discuss the Cohen-Macaulay property of T and we get a result which is a projective analogy of Theorem 1.1 (see Theorem 3.7). The proof and some corollaries will be given in section 3.

2. Proof of Theorem 1.1

Let $A = k[X, Y, Z]$ and $k[U]$ be polynomial rings over a field k. Let $\varphi : A \to k[U]$ be the k-algebra homomorphism defined by $\varphi(X) = U^l$, $\varphi(Y) = U^m$, and $\varphi(Z) = U^n$, where l, m, n are positive integers with $\gcd(l, m, n) = 1$. We denote $\operatorname{Ker} \varphi$ by $p(l, m, n)$, then as is well-known, unless $p(l, m, n)$ is a complete intersection, $p(l, m, n)$ is generated by the maximal minors of a matrix M of the form

$$M = \begin{bmatrix} X^{a_1} & Y^{b_1} & Z^{c_1} \\ Y^{b_2} & Z^{c_2} & X^{a_2} \end{bmatrix},$$

with a_1, a_2, b_1, b_2, c_1, and c_2 positive integers (cf. [5]).

Throughout this section we assume that $p = p(l, m, n)$ is not a complete intersection. The purpose is to investigate the ring $S = A[pt, p^{(2)}t^2]$. To begin with we put

$$e_1 = Z^{c_1+c_2} - X^{a_2}Y^{b_1},\ e_2 = X^{a_1+a_2} - Y^{b_2}Z^{c_1},\ \text{and}\ e_3 = Y^{b_1+b_2} - X^{a_2}Z^{c_2},$$

(hence $p = (e_1, e_2, e_3)A$) and

$$\begin{aligned} a &= \min\{a_1, a_2\}, & a_3 &= \max\{a_1 - a_2, 0\}, & a_3' &= \max\{0, a_2 - a_1\}, \\ b &= \min\{b_1, b_2\}, & b_3 &= \max\{b_1 - b_2, 0\}, & b_3' &= \max\{0, b_2 - b_1\}, \\ c &= \min\{c_1, a_2\}, & c_3 &= \max\{c_1 - c_2, 0\}, & c_3' &= \max\{0, c_2 - c_1\}. \end{aligned}$$

Then we have the following

LEMMA 2.1 ([3], [10], [12]). *There exists an element Δ of $p^{(2)}$ such that*

$$\begin{aligned} X^a\Delta - Y^{b_3}Z^{c_3'}e_2^2 + Y^{b_3'}Z^{c_3}e_1e_3 &= 0, \\ Y^b\Delta - X^{a_3'}Z^{c_3}e_3^2 + X^{a_3}Z^{c_3'}e_1e_2 &= 0, \\ Z^c\Delta - X^{a_3}Y^{b_3'}e_1^2 + X^{a_3'}Y^{b_3}e_2e_3 &= 0, \end{aligned}$$

and we have $p^{(2)} = p^2 + (\Delta)$.

Proof. See [3, Proposition 2.4], [3, Corollary 2.5], or [10, Lemma 2.3]. □

Let $R = A[T_1, T_2, T_3, T_4]$ and $A[t]$ be polynomial rings and let $\phi : R \to A[t]$ be the A-algebra homomorphism such that $\phi(T_i) = e_it$ for $i = 1,2,3$, and $\phi(T_4) = \Delta t^2$. Then $J = \operatorname{Ker}\phi$ is a prime ideal in R with $\operatorname{ht}_R J = 3$, and contains the following five elements

$$\begin{aligned} f_1 &= X^{a_1}T_1 + Y^{b_1}T_2 + Z^{c_1}T_3, \\ f_2 &= Y^{b_2}T_1 + Z^{c_2}T_2 + X^{a_2}T_3, \\ g_1 &= X^aT_4 - Y^{b_3}Z^{c_3'}T_2^2 + Y^{b_3'}Z^{c_3}T_1T_3, \\ g_2 &= Y^bT_4 - X^{a_3'}Z^{c_3}T_3^2 + X^{a_3}Z^{c_3'}T_1T_2, \\ g_3 &= Z^cT_4 - X^{a_3}Y^{b_3'}T_1^2 + X^{a_3'}Y^{b_3}T_2T_3. \end{aligned}$$

We put $I = (f_1, f_2, g_1, g_2, g_3)R$. These f_1, f_2, g_1, g_2, g_3 are pfaffians with degree four in the skew symmetric matrix.

$$\begin{bmatrix} 0 & Z^{c_3'}T_2 & X^{a_3'}T_3 & Y^{b_3'}T_1 & T_4 \\ -Z^{c_3'}T_2 & 0 & Y^b & -X^a & Z^{c_3}T_3 \\ -X^{a_3'}T_3 & -Y^b & 0 & Z^c & X^{a_3}T_1 \\ -Y^{b_3'}T_1 & X^a & -Z^c & 0 & Y^{b_3}T_2 \\ -T_4 & -Z^{c_3}T_3 & -X^{a_3}T_1 & -Y^{b_3}T_2 & 0 \end{bmatrix}$$

Since $A[T_1, T_2, T_3]/(f_1, f_2) \cong R(p)$ (the Rees algebra of p), that is an integra domain (cf. [14, Theorem 3.6]), we get that f_1, f_2, and g_1 forms an R-regular sequence. Hence by [1, Theorem 2.1] we have the following

LEMMA 2.2 ([4, Lemma (3.2)], [7]). *R/I is a Gorenstein ring of dimension four.*

We say that $\mathfrak{p}$ is self-linked if there exist elements x_1, x_2 in $\mathfrak{p}$ such that $\mathfrak{p} = (x_1, x_2) : \mathfrak{p}$. In [7, Corollary 1.10] it is proved on the local ring $\hat{A} = [[X, Y, Z]]$ that conditions (1) and (2) of the following lemma are equivalent. But we need the equivalence of these on $A = k[X, Y, Z]$.

LEMMA 2.3 ([7, Corollary1.10]). *The following conditions are equivalent.*

(1) *$\mathfrak{p}$ is not a self-linked ideal.*

(2) *The matrix M satisfies one of the following conditions.*

(a) $a_1 > a_2$, $b_1 > b_2$, *and* $c_1 > c_2$.

(b) $a_1 < a_2$, $b_1 < b_2$, *and* $c_1 < c_2$.

Proof. If $\mathfrak{p} = I_2(M)$ is self-linked, then so is $\mathfrak{p}\hat{A} = I_2(M)\hat{A}$. By [7, Corollary 1.10] we have that condition (2) implies condition (1).

Next we assume that condition (2) is not satisfied. After elementary row and column operations on M, we may assume that the components of the first column of M are part of a minimal system of generators of $I_1(M)$. So we suppose $a_1 \leq a_2$, $b_1 \geq b_2$ and construct a 2 by 3 matrix

$$N = M \begin{bmatrix} Y^{b_1-b_2} & X^{a_2-a_1} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then the matrix obtained by deleting the last column of N is symmetric and $\mathfrak{p} = I_2(N)$. We have by [15, Theorem 2.1] that $\mathfrak{p}$ is self-linked. □

Let $\mathfrak{m} = (X, Y, Z)R$. Observing the generators f_1, f_2, g_1, g_2, g_3 of I, we see by Lemma 2.3 that $\mathfrak{p}$ is not self-linked if and only if $I \subset \mathfrak{m}$. Therefore we have by the lemma stated below that $\mathfrak{p}$ is self-linked if and only if $I = J$. Although the following lemma is proved in [4], we show the proof for the completeness of this paper.

LEMMA 2.4 ([4, Lemma 3.3]). $\mathrm{Ass}_R\, R/I \subset \{J, \mathfrak{m}\}$ *and* $IR_J = JR_J$.

Proof. By Lemma 2.2, we have $J \in \mathrm{Min}_R\, R/I = \mathrm{Ass}_R\, R/I$. Choose $\xi \in$

$(X^a, Y^b, Z^c)A \setminus \cup_{Q \in \mathrm{Ass}_R R/I \setminus \{\mathfrak{m}\}} Q$ and write $\xi = \lambda_1 X^a + \lambda_2 Y^b + \lambda_3 Z^c$, where $\lambda_i \in A$. We put $g = \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3$, then $g = \xi T_4 - \eta$ with $\eta \in A([T_1, T_2, T_3]$ and $(f_1, f_2, T_4 - \eta/\xi)R[1/\xi] \subset IR[1/\xi] \subset JR[1/\xi]$. Note that $A[T_1, T_2, T_3]/(f_1, f_2)$ is an integral domain of dimension four and so $(f_1, f_2, T_4 - \eta/\xi) \cdot R[1/\xi]$ is a prime ideal in $R[1/\xi]$ of height three. Therefore $IR[1/\xi] = JR[1/\xi]$ and this implies the assertions of Lemma 2.4. □

Here we note that, if $\mathfrak{p}$ is not self-linked, then we have a primary decomposition of I of the form $I = J \cap Q$, where Q is an $\mathfrak{m}$-primary ideal.

Let $\mathfrak{M} = (X, Y, Z, T_1, T_2, T_3, T_4)R$. The invariant $\dim_k \mathrm{Ext}_S^4(S/\mathfrak{M}S, S)$ with respect to S, we denote by $\mathrm{r}(S)$, is called Cohen-Macaulay type of S. It is known that $\mathrm{r}(S) = \mu_S(K_S)$, where K_S is the canonical module of S and $\mu_S(\ )$ denotes the minimal number of generators (cf. [6]). The following proposition is the key in our proof of Theorem 1.1.

PROPOSITION 2.5. *Suppose that $\mathfrak{p}$ is not a self-linked ideal. Then the following conditions are equivalent.*

(1) $S = A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ *is a Cohen-Macaulay ring.*

(2) $IR_{\mathfrak{m}} \cap R = (X^\alpha, Y^\beta, Z^\gamma)R$ *for some* $\alpha, \beta, \gamma \geq 1$.

When this is the case, $\mathrm{r}(S) = 3$.

Proof. Let $I = J \cap Q$ be the primary decomposition of I, where Q is an $\mathfrak{m}$-primary ideal. Note that $[I :_R J] = Q$ and $[I :_R Q] = J$, and we have by [9, Proposition 3.1] that $S = R/J$ is a Cohen-Macaulay ring if and only if R/Q is a Cohen-Macaulay ring. Thus condition (2) implies condition (1). Now $K_S = \mathrm{Hom}_{R/I}(R/J, R/I) \cong [I :_R J]/I = Q/I$ as R-modules and $l_R(Q/I + \mathfrak{M}Q) = l_R(Q/\mathfrak{M}Q) = 3$. Hence we get $\mathrm{r}(S) = 3$.

Next we assume assertion (1), then R/Q is a Cohen-Macaulay ring. We may assume that $a_1 > a_2$, $b_1 > b_2$, and $c_1 > c_2$ by Lemma 2.3. We put positive integers $\alpha = \min\{a_2, a_3\}$, $\beta = \min\{b_2, b_3\}$, and $\gamma = \min\{c_2, c_3\}$. $Q = IR_{\mathfrak{m}} \cap R$ is contained in $(X^\alpha, Y^\beta, Z^\gamma)R$, since $(X^\alpha, Y^\beta, Z^\gamma)R$ is an $\mathfrak{m}$-primary ideal and contains I. We shall show the opposite inclusion. T_1, T_2, T_3, T_4 is a system of parameters of $(R/Q)_{\mathfrak{M}}$ and T_1, T_2, T_3, T_4 forms an $(R/Q)_{\mathfrak{M}}$-regular sequence, because $\mathrm{rad}(Q + (T_1, T_2, T_3, T_4)R) = \mathfrak{M}$. Thus we have $(T_1, T_2, T_3, T_4)R_{\mathfrak{M}} \cap Q_{\mathfrak{M}} = (T_1, T_2, T_3, T_4) = Q_{\mathfrak{M}}$, and this implies $(T_1, T_2, T_3, T_4)R \cap Q = (T_1, T_2, T_3, T_4)Q$.

We regard R as a graded ring with $\deg X = \deg Y = \deg Z = 0$, $\deg T_1 = \deg T_2 = \deg T_3 = 1$, and $\deg T_4 = 2$. Then Q is a graded ideal, since I is generated by homogeneous elements. We can choose homogeneous elements u_1, $u_2, \dots, u_s \in Q \cap R_0$ and $v_1, v_2, \dots, v_t \in Q \cap \bigoplus_{i \geq 1} R_i$ which generate Q. Then $q = (u_1, u_2, \dots, u_s)A$ is an ideal of A and each v_i belongs to $Q \cap (T_1, T_2, T_3, T_4)R = (T_1, T_2, T_3, T_4)Q$ and hence $Q = qR + (T_1, T_2, T_3, T_4)Q$. By Nakayama's lemma we have $Q = qR$.

We have $X^{a_1}, Y^{b_1}, Z^{c_1} \in qR$, since $f_1 = X^{a_1}T_1 + Y^{b_1}T_2 + Z^{c_1}T_3 \in qR$. Similarly

$$(X^{a_1}, Y^{b_1}, Z^{c_1}, X^{a_2}, Y^{b_2}, Z^{c_2}, X^{a_3}, Y^{b_3}, Z^{c_3})A \subset q.$$

Therefore $(X^\alpha, Y^\beta, Z^\gamma)R \subset qR = Q$, and thus we have $Q = (X^\alpha, Y^\beta, Z^\gamma)R$, as required. $\square$

Theorem 1.1 means that the Cohen-Macaulay property of S is determined by the matrix M. In order to prove this theorem, we assume that p is not self-linked and $a_1 > a_2$, $b_1 > b_2$, and $c_1 > c_2$. We put positive integers $\alpha = \min\{a_2, a_3\}$, $\beta = \min\{b_2, b_3\}$, $\gamma = \min\{c_2, c_3\}$, and a matrix

$$U = \begin{bmatrix} X^{a_2-\alpha}T_4 & X^{a_3-\alpha}T_1T_2 & -X^{a_3-\alpha}T_1^2 \\ -Y^{b_3-\beta}T_2^2 & Y^{b_2-\beta}T_4 & Y^{b_3-\beta}T_2T_3 \\ Z^{c_3-\gamma}T_1T_3 & -Z^{c_2-\gamma}T_3^2 & Z^{c_2-\gamma}T_4 \end{bmatrix}.$$

LEMMA 2.6. *The inequality* $(a_1 - 2a_2)(b_1 - 2b_2)(c_1 - 2c_2) \geq 0$ *holds if and only if* $\det U \notin \mathfrak{m}$.

Proof. We have

$$\begin{aligned} \det U = {} & X^{a_2-\alpha}Y^{b_2-\beta}Z^{c_2-\gamma}T_4^3 + X^{a_3-\alpha}Y^{b_2-\beta}Z^{c_3-\gamma}T_1^3T_3T_4 \\ & + X^{a_3-\alpha}Y^{b_3-\beta}Z^{c_2-\gamma}T_1T_2^3T_4 + X^{a_2-\alpha}Y^{b_3-\beta}Z^{c_3-\gamma}T_2T_3^3T_4, \end{aligned}$$

hence $\det U \notin \mathfrak{m}$ if and only if one of the following conditions is satisfied.

$$(1)\ X^{a_2-\alpha}Y^{b_2-\beta}Z^{c_2-\gamma} = 1, \quad (2)\ X^{a_3-\alpha}Y^{b_2-\beta}Z^{c_3-\gamma} = 1,$$
$$(3)\ X^{a_3-\alpha}Y^{b_3-\beta}Z^{c_2-\gamma} = 1, \quad (4)\ X^{a_2-\alpha}Y^{b_3-\beta}Z^{c_3-\gamma} = 1.$$

By the definition of α, β, γ, condition (1) is equivalent to saying that $a_2 \leq a_3$, $b_2 \leq b_3$, and $c_2 \leq c_3$. Further by the definition of a_3, b_3, c_3, we have that condition (1) and the following condition (1)$'$ are equivalent.

(1)$'$ $a_1 - 2a_2 \geq 0$, $b_1 - 2b_2 \geq 0$, and $c_1 - 2c_2 \geq 0$.

Similarly (2), (3), (4) are equivalent to the following conditions (2)', (3)', (4)', respectively.

(2)' $a_1 - 2a_2 \leq 0$, $b_1 - 2b_2 \geq 0$, and $c_1 - 2c_2 \leq 0$.
(3)' $a_1 - 2a_2 \leq 0$, $b_1 - 2b_2 \leq 0$, and $c_1 - 2c_2 \geq 0$.
(4)' $a_1 - 2a_2 \geq 0$, $b_1 - 2b_2 \leq 0$, and $c_1 - 2c_2 \leq 0$.

This implies that the inequality $(a_1 - 2a_2)(b_1 - 2b_2)(c_1 - 2c_2) \geq 0$ holds. □

Proof of Theorem 1.1. By Proposition 2.5 and Lemma 2.6 it is sufficient to prove that $\det U \notin \mathfrak{m}$ if and only if $IR_{\mathfrak{m}} = (X^{\alpha}, Y^{\beta}, Z^{\gamma})R_{\mathfrak{m}}$. Note that by Nakayama's lemma $IR_{\mathfrak{m}} = (X^{\alpha}, Y^{\beta}, Z^{\gamma})R_{\mathfrak{m}}$ if and only if $I \otimes_R K = (X^{\alpha}, Y^{\beta}, Z^{\gamma}) \otimes_R K$, where $K = R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ the residue field of $\mathfrak{m}$. We put a matrix

$$V = \begin{bmatrix} X^{a_1-\alpha}T_1 & X^{a_2-\alpha}T_3 & X^{a_2-\alpha}T_4 & X^{a_3-\alpha}T_1T_2 & -X^{a_3-\alpha}T_1^2 \\ Y^{b_1-\beta}T_2 & Y^{b_2-\beta}T_1 & -Y^{b_3-\beta}T_2^2 & Y^{b_2-\beta}T_4 & Y^{b_3-\beta}T_2T_3 \\ Z^{c_1-\gamma}T_3 & Z^{c_2-\gamma}T_2 & Z^{c_3-\gamma}T_1T_3 & -Z^{c_3-\gamma}T_3^2 & Z^{c_2-\gamma}T_4 \end{bmatrix},$$

then

$$[f_1, f_2, g_1, g_2, g_3] = [X^{\alpha}, Y^{\beta}, Z^{\gamma}]V.$$

We denote the i-th column vector of V by v_i. We have $v_1 \in \mathfrak{m}R^3$ and $T_4v_2 = T_3v_3 + T_1v_4 + T_2v_5$. We have $I \otimes_R K = (g_1, g_2, g_3) \otimes_R K$, since $[g_1, g_2, g_3] = [X^{\alpha}, Y^{\beta}, Z^{\gamma}]\, U$ and $U = [v_3, v_4, v_5]$. Therefore $\det U \notin \mathfrak{m} \Leftrightarrow (g_1, g_2, g_3) \otimes_R K = (X^{\alpha}, Y^{\beta}, Z^{\gamma}) \otimes_R K \Leftrightarrow I \otimes_R K = (X^{\alpha}, Y^{\beta}, Z^{\gamma}) \otimes_R K$. □

EXAMPLE 2.7 Let

$\mathfrak{p}_1 = \mathfrak{p}(n^2 + 2n + 2, n^2 + 2n + 1, n^2 + n + 1)$, where $n \geq 2$,
$\mathfrak{p}_2 = \mathfrak{p}(n^2, n^2 + 1, n^2 + n + 1)$, where $n \geq 3$,
$\mathfrak{p}_3 = \mathfrak{p}(n^2 + n + 1, n^2 + 2n - 1, 2n^2 - 1)$, where $n \geq 3$.

(1) ([4, Example (3.7)]) S is a Cohen-Macaulay ring of $\mathrm{r}(S) = 3$ for $\mathfrak{p} = \mathfrak{p}_1$ or $\mathfrak{p} = \mathfrak{p}_2$.

(2) S is not a Cohen-Macaulay ring for $\mathfrak{p} = \mathfrak{p}_3$.

Proof. The prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ are respectively generated by the maximal minors of the matrices

$$\begin{bmatrix} X^n & Y^n & Z^{n+1} \\ Y & Z & X \end{bmatrix}, \begin{bmatrix} X^n & Y^n & Z^{n-1} \\ Y & Z & X \end{bmatrix} \text{ and } \begin{bmatrix} X^n & Y^n & Z^n \\ Y & Z & X^{n-1} \end{bmatrix}.$$

Since each p_i is not self-linked, by Theorem 1.1 we get conclusions (1) and (2). □

3. The projective cases

In this section we study a projective analogy of Theorem 1.1. For this purpose we need preliminaries, which are arguments on relations between non-homogeneous and homogeneous elements (cf. [16, Chap. VII §5]).

Let $A = k[X_1, X_2, \ldots, X_n]$ and $B = k[Y_0, Y_1, \ldots, Y_n]$ be polynomial rings. We regard A and B as graded rings with the grading

$$\eta_i = \deg X_i = \deg Y_i \quad \text{for} \quad i = 1,2,\ldots, n \quad \text{and} \quad \eta_0 = \deg Y_0,$$

where $\eta_0 > 0$ and η_0 divides η_i for any i $(i = 1,2,\ldots, n)$. For any polynomial $g = g(Y_0, Y_1, \ldots, Y_n)$ in B, we associate the polynomial $i_{Y_0}(g)$ in A defined by

$$i_{Y_0}(g) = g(1, X_1, X_2, \ldots, X_n).$$

Then $i_{Y_0} : B \to A$ is a k-algebra homomorphism.

Conversely for any non-zero polynomial $f = f(X_1, X_2, \ldots, X_n)$ in A, we define its homogenized polynomial ${}^hf$ in B as follows:

$${}^hf = Y_0^{\deg f/\eta_0} f\left(\frac{Y_1}{Y_0^{\zeta_1}}, \ldots, \frac{Y_n}{Y_0^{\zeta_n}}\right),$$

where $\zeta_i = \eta_i/\eta_0$. Note that $i_{Y_0}({}^hf) = f$ for $0 \neq f \in A$. When $\mathfrak{a}$ is an ideal in A, we denote by ${}^h\mathfrak{a}$ the ideal in B which is generated by $\{{}^hf \mid f \in \mathfrak{a}\}$. We can check that, if $\mathfrak{b}$ is a graded ideal in B and if Y_0 is a $B/\mathfrak{b}$-regular element, then $\mathfrak{b} = {}^h(i_{Y_0}(\mathfrak{b})A)$.

LEMMA 3.1. *Let $C = k[U_1, U_2, \ldots, U_m]$ and $D = k[V_0, V_1, \ldots, V_m]$ be polynomial rings. We regard D as a graded ring with $\deg V_0 > 0$, and let $i_{V_0} : D \to C$ be the k-algebra homomorphism as above. Suppose that $\Phi : B \to D$ is a homomorphism of graded rings and that $\varphi : A \to C$ is a ring homomorphism such that $\varphi \circ i_{Y_0} = i_{V_0} \circ \Phi$. Then $i_{Y_0}(\mathrm{Ker}\ \Phi)A = \mathrm{Ker}\ \varphi$.*

Proof. Obviously, $i_{Y_0}(\mathrm{Ker}\ \Phi)A \subset \mathrm{Ker}\ \varphi$. Conversely, for any $\xi \in \mathrm{Ker}\ \varphi$, $\Phi({}^h\xi)$ is a homogeneous element in D and $\Phi({}^h\xi) \in \mathrm{Ker}\ i_{V_0} = (V_0 - 1)D$. Hence we have $\Phi({}^h\xi) = 0$ and $\xi = i_{Y_0}({}^h\xi) \in i_{Y_0}(\mathrm{Ker}\ \Phi)A$. □

The purpose of this section is to give an analogy of Theorem 1.1 for the defining ideal P of a projective space monomial curve. The ideal P is given as follows:

Let $B=k[X, Y, Z, W]$ and $k[U, V]$ be polynomial rings over a field k. Let $\Phi : B \to k[U, V]$ be the k-algebra homomorphism such that $\Phi(X) = U^l$, $\Phi(Y) = U^m V^{l-m}$, $\Phi(Z) = U^n V^{l-n}$, and $\Phi(W) = V^l$, where l, m, n are positive integers with $\gcd(l, m, n) = 1$, $l > m$, $l > n$, and with $m \neq n$. We denote by $P(l, m, n)$ the prime ideal $\operatorname{Ker} \Phi$ in B. Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \to & P(l, m, n) & \to & B = k[X, Y, Z, W] & \xrightarrow{\Phi} & k[U, V] \\ & & \downarrow & & i_W \downarrow & & \downarrow i_V \\ 0 & \to & p(l, m, n) & \to & A = k[X, Y, Z] & \xrightarrow{\varphi} & k[U] \end{array},$$

where φ is the map we defined in section 2. Moreover we regard B and $k[U, V]$ as graded rings with $\deg X = \deg Y = \deg Z = \deg W = l$ and $\deg U = \deg V = 1$. Then we get the following corollary of Lemma 3.1.

COROLLARY 3.2. $i_W(P(l, m, n))A = p(l, m, n)$.

For the prime ideal $P = P(l, m, n)$, we assume that B/P is not a complete intersection but a Cohen-Macaulay ring. Then P is generated by the maximal minors of a matrix M' of the form

$$M' = \begin{bmatrix} X^{a_1}W^{d_1} & Y^{b_1} & Z^{c_1} \\ Y^{b_2} & Z^{c_2} & X^{a_2}W^{d_2} \end{bmatrix},$$

where $a_1 + d_1$, b_1, b_2, c_1, c_2, and $a_2 + d_2$ are positive integers (cf. [8], [13]). We put $\varepsilon_1 = Z^{c_1+c_2} - X^{a_2}W^{d_2}Y^{b_1}$, $\varepsilon_2 = X^{a_1+a_2} W^{d_1+d_2} - Y^{b_2}Z^{c_1}$, and $\varepsilon_3 = Y^{b_1+b_2} - X^{a_1}W^{d_1}Z^{c_2}$, then P is generated by ε_1, ε_2, and ε_3.

Corollary 3.2 means that $p = p(l, m, n)$ is generated by $i_W(\varepsilon_1)$, $i_W(\varepsilon_2)$, and $i_W(\varepsilon_3)$. Hence the matrices M corresponding to $p(l, m, n)$ and M' corresponding to $P(l, m, n)$ have the same exponents a_i, b_i, and c_i for $i = 1,2$.

We put

$$d = \min\{d_1, d_2\},\ d_3 = \max\{d_1 - d_2, 0\},\ d_3' = \max\{0, d_2 - d_1\},$$

as is in section 2. Then there exists an element Γ of $P^{(2)}$ and we have the following three relations by the same method as is in Proposition 2.1.

$$X^a W^d \Gamma - Y^{b_3} Z^{c_3'} \varepsilon_2^2 + Y^{b_3'} Z^{c_3} \varepsilon_1 \varepsilon_3 = 0,$$
$$Y^b \Gamma - X^{a_3'} W^{d_3'} Z^{c_3} \varepsilon_3^2 + X^{a_3} W^{d_3} Z^{c_3'} \varepsilon_1 \varepsilon_2 = 0,$$
$$Z^c \Gamma - X^{a_3} W^{d_3} Y^{b_3'} \varepsilon_1^2 + X^{a_3'} W^{d_3'} Y^{b_3} \varepsilon_2 \varepsilon_3 = 0.$$

Moralés and Simis gave the free resolution of $B/P^2 + (\Gamma)$ and proved the following lemma.

LEMMA 3.3 ([11, (2.1.2) Lemma]). $P^{(2)} = P^2 + (\Gamma)$.

From now on we regard B as a graded ring with $\deg X = \deg Y = \deg Z = \deg W = 1$, so that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and Γ are homogeneous elements. Let $R' = B[T_1, T_2, T_3, T_4]$ and $B[t]$ be polynomial rings and let $\Psi : R' \to B[t]$ be the B-algebra homomorphism such that

$$\Psi(T_i) = \varepsilon_i t \text{ for } i = 1,2,3 \quad \Psi(T_4) = \Gamma t^2.$$

We also regard R' and $B[t]$ as graded rings so that Ψ is graded, i.e.,

$$\deg T_i = \deg \varepsilon_i \text{ for } i = 1,2,3, \ \deg T_4 = \deg \Gamma, \text{ and } \deg t = 0.$$

LEMMA 3.4. *Suppose Ψ is the map defined above corresponding to $P = P(l, m, n)$ and ϕ is the map defined in section 2 corresponding to $p = p(l, m, n)$. Then $i_W(\text{Ker } \Psi)R = \text{Ker } \Psi$.*

Proof. For the k-algebra homomorphisms $i_W : R' \to R$ and $i_W : B[t] \to A[t]$, we have $\phi \circ i_W = i_W \circ \Psi$. By Lemma 3.1 we get the proof of Lemma 3.4. □

In the following section, we discuss the Cohen-Macaulay property of the algebra $T = \text{Im } \Psi = B[Pt, P^{(2)}t^2]$.

LEMMA 3.5. *Let $P = P(l, m, n)$ and $p = p(l, m, n)$. If $T = B[Pt, P^{(2)}t^2]$ is a Cohen-Macaulay ring, then so is $S = A[pt, p^{(2)}t^2]$.*

Proof. Since T is Cohen-Macaulay, we have $\text{proj.dim}_{R'} T = 3$ and an R'-graded free resolution $\mathbf{F}$.

$$0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\Psi} T \to 0,$$

where $F_0 = R'$. Since there is a natural identification $(R'[1/W])_0 \cong R$ and since $\text{Ker } \Psi$ is a graded ideal in R', we have

$$i_W(\text{Ker } \Psi) R \cong ((\text{Ker } \Psi) \otimes_{R'} R'[1/W])_0.$$

We put $\mathbf{G}. = (\mathbf{F}. \otimes_{R'} R'[1/W])_0$ and let ∂_i be the differential map of $\mathbf{G}.$ induced by d_i. By Lemma 3.4 we have

$$\mathrm{Ker}\ \phi = i_W(\mathrm{Ker}\ \Psi)\ R \cong ((\mathrm{Im}\ d_1) \otimes_{R'} R'[1/W])_0 = \mathrm{Im}\ \partial_1.$$

Hence the following sequence is exact and $S = \mathrm{Im}\ \phi$ is Cohen-Macaulay.

$$0 \to G_3 \xrightarrow{\partial_3} G_2 \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} G_0 \xrightarrow{\phi} S \to 0 \qquad \square$$

We remark that a prime ideal Ker Ψ in R', which is of hight three, contains the following five elements

$$\begin{aligned}
F_1 &= X^{a_1}W^{d_1}T_1 + Y^{b_1}T_2 + Z^{c_1}T_3,\\
F_2 &= Y^{b_2}T_1 + Z^{c_2}T_2 + X^{a_2}W^{d_2}T_3,\\
G_1 &= X^{a}W^{d}T_4 - Y^{b_3}Z^{c'_3}T_2^2 + Y^{b'_3}Z^{c_3}T_1T_3,\\
G_2 &= Y^{b}T_4 - X^{a'_3}W^{d'_3}Z^{c_3}T_3^2 + X^{a_3}W^{d_3}Z^{c'_3}T_1T_2,\\
G_3 &= Z^{c}T_4 - X^{a_3}W^{d_3}Y^{b'_3}T_1^2 + X^{a'_3}W^{d'_3}Y^{b_3}T_2T_3.
\end{aligned}$$

We put $J' = \mathrm{Ker}\ \Psi$ and $I' = (F_1, F_2, G_1, G_2, G_3)R'$. The following lemma means that I' and J' have similar properties as we stated in Lemma 2.2 and Lemma 2.4.

Although the proof of this lemma is given among the proofs of many other results of [11, (2.2.1) Theorem], we show it briefly for the completeness of this paper. We put $\mathfrak{m}_1 = (X, Y, Z)R'$ and $\mathfrak{m}_2 = (Y, Z, W)R'$.

LEMMA 3.6 ([11, (2.2.1) Theorem]).
(1) *R'/I' is a Gorenstein ring of dimension five.*
(2) $\mathrm{Ass}_{R'} R'/I' \subset \{J', \mathfrak{m}_1, \mathfrak{m}_2\}$ *and* $I' R'_{J'} = J' R'_{J'}$.

Proof. (1) An ideal I' is with $\mathrm{ht}_{R'} I' = 3$ and generated by pfaffians of degree four in the skew symmetric matrix

$$\begin{bmatrix}
0 & Z^{c'_3}T_2 & X^{a'_3}W^{d'_3}T_3 & Y^{b'_3}T_1 & T_4\\
-Z^{c'_3}T_2 & 0 & Y^{b} & -X^{a}W^{d} & Z^{c_3}T_3\\
-X^{a'_3}W^{d'_3}T_3 & -Y^{b} & 0 & Z^{c} & X^{a_3}W^{d_3}T_1\\
-Y^{b'_3}T_1 & X^{\alpha}W^{d} & -Z^{c} & 0 & Y^{b_3}T_2\\
-T_4 & -Z^{c_3}T_3 & -X^{a_3}W^{d_3}T_1 & -Y^{b_3}T_2 & 0
\end{bmatrix}.$$

(2) Choose $\xi \in (X^aW^d, Y^b, Z^c)B \setminus \cup_{Q \in \mathrm{Ass}_{R'}R'/I' \setminus \{\mathfrak{m}_1, \mathfrak{m}_2\}} Q$, and by the same method of Lemma 2.4 we get $I'R'[1/\xi] = J'R'[1/\xi]$. This implies $I'R'_{J'} = J'R'_{J'}$ and $\mathrm{Ass}_{R'} R'/I' \subset \{J', \mathfrak{m}_1, \mathfrak{m}_2\}$. $\square$

Remark. As can be seen from Lemma 3.6, $I' \neq J'$ if and only if either $I' \subset \mathfrak{m}_1$ or $I'/\mathfrak{m}_2$ is satisfied. Furthermore by observing the generators of I', we can check that $\mathfrak{m}_1 \in \mathrm{Ass}_{R'} R'/I'$ if and only if the matrix M' satisfies one of the following conditions.

(1) $a_1 > a_2 > 0$, $b_1 > b_2$, and $c_1 > c_2$.
(2) $a_2 > a_1 > 0$, $b_2 > b_1$, and $c_2 > c_1$.

Similarly $\mathfrak{m}_2 \in \mathrm{Ass}_{R'} R'/I'$ if and only if the matrix M' satisfies one of the following conditions.

(1) $d_1 > d_2 > 0$, $b_1 > b_2$, and $c_1 > c_2$.
(2) $d_2 > d_1 > 0$, $b_2 > b_1$, and $c_2 > c_1$.

Note that b_1, b_2, c_1, and c_2 are always positive because P is not a complete intersection. Now we prove the converse of Lemma 3.5.

THEOREM 3.7. *The following conditions are equivalent.*

(1) $T = B[Pt, P^{(2)}t^2]$ *is a Cohen-Macaulay ring for* $P = P(l, m, n)$.
(2) $A[p_1t, p_1^{(2)}t^2]$ *and* $A[p_2t, p_2^{(2)}t^2]$ *are Cohen-Macaulay rings, where* $p_1 = p(l, m, n)$ *and* $p_2 = p(l, l-m, l-n)$.

When this is the case, the Cohen-Macaulay type of T *is given by*

$$\begin{aligned} \mathrm{r}(T) &= 1 \text{ if } I' = J' \\ &= 3 \text{ if } I' \neq J'. \end{aligned}$$

Proof. We assume condition (1), then $B[Pt, P^{(2)}t^2]$ is also Cohen-Macaulay for $P = P(l, l-m, l-n)$. By Lemma 3.5 we get assertion (2).

Next assume condition (2). If $I' = J'$, then T is Gorenstein by Lemma 3.6.

When $\mathrm{Ass}_{R'} R'/I' = \{J', \mathfrak{m}_1\}$, we have a primary decomposition of I' of the form $I' = J' \cap Q$, where Q is a graded $\mathfrak{m}_1$-primary ideal. Since $J' = [I' :_{R'} Q]$, by [9, Proposition 3.1] it is sufficient to prove that R'/Q is Cohen-Macaulay. Let I and J be ideals in $R = A[T_1, T_2, T_3, T_4]$ defined by p_1 as is in section 2. By Corollary 3.2 we have $i_W(I') = I$ and by Lemma 3.4 we have $i_W(J') = J$. Note that $i_W(Q)R$ is an (X, Y, Z) R-primary ideal. Now $(I'R'[1/W])_0 = (J'R'[1/W])_0 \cap (QR'[1/W])_0$ and there is a natural identification $(R'[1/W])_0 \cong R$, thus we have $I = J \cap i_W(Q)R$. By Proposition 2.5 we have $i_W(Q)R = (X^\alpha, Y^\beta, Z^\gamma)R$ for some $\alpha, \beta, \gamma \geq 1$, since $A[p_1t, p_1^{(2)}t^2]$ is Cohen-Macaulay. Further, W is an R'/Q-regular element, hence

$$Q = {}^h(i_W(Q)R) = {}^h((X^\alpha, Y^\beta, Z^\gamma)R) = (X^\alpha, Y^\beta, Z^\gamma)R',$$

therefore R'/Q is Cohen-Macaulay. When this is the case, we have

$$K_T = \mathrm{Hom}_{R'/I'}(R'/J', R'/I') \cong [I' :_{R'} J']/I' = Q/I',$$

as R'-modules. Hence $\mathrm{r}(T) = \mu_{R'}(Q/I') = \mu_{R'}(Q) = 3$.

When $\mathrm{Ass}_{R'} R'/I' = \{J, \mathfrak{m}_2\}$, the proof follows from the above discussion by replacing X for W of B because $A[p_2t, p_2^{(2)}t^2]$ is Cohen-Macaulay.

When $\mathrm{Ass}_{R'} R'/I' = \{J, \mathfrak{m}_1, \mathfrak{m}_2\}$, then we have the primary decomposition of I' of the form $I' = J' \cap Q_1 \cap Q_2$, where each Q_i is a graded $\mathfrak{m}_i$-primary ideal. Since $i_W(I') = i_W(J') \cap i_W(Q_1)$, as can be seen from the above discussion, we get $Q_1 = (X^\alpha, Y^\beta, Z^\gamma)R'$ for some $\alpha, \beta, \gamma \geq 1$. On the other hand, since $i_X(I') = i_X(J') \cap i_X(Q_2)$, we have $Q_2 = (W^\delta, Y^{\beta'}, Z^{\gamma'})R'$ for some $\delta, \beta', \gamma' \geq 1$. Note from the above remark one of the following conditions occurs.

(i) $a_1 > a_2$, $b_1 > b_2$, $c_1 > c_2$, and $d_1 > d_2$.
(ii) $a_1 < a_2$, $b_1 < b_2$, $c_1 > c_2$, and $d_1 < d_2$.

If assertion (i) is satisfied, as can be seen from the proof of Proposition 2.5, we have $\beta = \min\{b_2, b_3\} = \beta'$ and $\gamma = \min\{c_2, c_3\} = \gamma'$. Therefore $Q_1 \cap Q_2 = (X^\alpha W^\delta, Y^\beta, Z^\gamma)R'$ and $R'/Q_1 \cap Q_2$ is Cohen-Macaulay. It follows that R'/J' is Cohen-Macaulay, since $[I' :_{R'} J'] = Q_1 \cap Q_2$. When this is the case, we have

$$K_T = \mathrm{Hom}_{R'/I'}(R'/J', R'/I') \cong [I' :_{R'} J']/I' = Q_1 \cap Q_2/I'$$

and $\mathrm{r}(T) = \mu_{R'}(Q_1 \cap Q_2) = 3$. □

By the proof of Theorem 3.7, we can determine the Cohen-Macaulay type of T in terms of the matrix M'.

COROLLARY 3.8. *Suppose T is a Cohen-Macaulay ring and the matrix M' satisfies $b_1 \geq b_2$. Then*

$$\begin{aligned} \mathrm{r}(T) &= 3 \text{ if } a_1 > a_2 > 0 \text{ and } b_1 > b_2 \text{ and } c_1 > c_2, \text{ or} \\ &\qquad d_1 > d_2 > 0 \text{ and } b_1 > b_2 \text{ and } c_1 > c_2, \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

Finally we consider the self-linked property of P.

COROLLARY 3.9. *If P is a self-linked ideal, then T is a Gorenstein ring.*

Proof. Let $P = P(l, m, n)$, $p_1 = p(l, m, n)$, and $p_2 = p(l, l-m, l-n)$.

Now there exist $\beta_1, \beta_2 \in P$ such that $P^2 \subset (\beta_1, \beta_2)$. We put $\alpha_i = i_W(\beta_i)$ for $i = 1,2$, then $\mathfrak{p}_1^2 \subset (\alpha_1, \alpha_2)$ in $A = k[X, Y, Z]$ and $\alpha_1, \alpha_2 \in \mathfrak{p}_1$. Now $\mathfrak{p}_1$ is a prime ideal, therefore it follows that $\mathfrak{p}_1 = (\alpha_1, \alpha_2) : \mathfrak{p}_1$ or $\mathfrak{p}_1 = (\alpha_1, \alpha_2)$. Hence $A[\mathfrak{p}_1 t, \mathfrak{p}_1^{(2)} t^2]$ is Gorenstein. Similarly, $A[\mathfrak{p}_2 t, \mathfrak{p}_2^{(2)} t^2]$ is Gorenstein. Hence by Lemma 2.3, Theorem 3.7, and Corollary 3.8, we have the proof. □

The converse of Corollary 3.9 does not hold in general.

EXAMPLE 3.10. Let $P = P(11, 5, 2)$. Then T is a Gorenstein ring but P is not a self-linked ideal.

Proof. The defining ideal P is generated by the maximal minors of the matrix

$$M' = \begin{bmatrix} X & Y^2 & Z^3 \\ Y & Z^2 & W^3 \end{bmatrix}.$$

Since $a = d = 0$, we get that $I' = J'$ and $\mathrm{r}(T) = 1$.

We put $\mathfrak{n} = (X, Y, Z, W)B$ and assume that $P = I_2(M')$ is self-linked. Note that the statement of [7, Theorem 1.1] is true for the ring $B_\mathfrak{n}$ and the ideal $PB_\mathfrak{n}$, even if $\dim B_\mathfrak{n} = 4$. Thus there exists a 2 by 3 matrix $N = (n_{ij})$ $(n_{ij} \in B_\mathfrak{n})$ such that $I_2(N) = B_\mathfrak{n}$ and $\sum_{i,j} m_{ij} n_{ij} = 0$, where $M' = (m_{ij})$. Hence

$$Xn_{11} + Y^2 n_{12} + Z^3 n_{13} + Yn_{21} + Z^2 n_{22} + W^3 n_{23} = 0,$$

and

$$Y(Yn_{12} + n_{21}) + Z^2(Zn_{13} + n_{23}) = -(Xn_{11} + W^3 n_{23}).$$

Since X, Y, Z, W is a $B_\mathfrak{n}$-regular sequence, we have $Yn_{12} + n_{21} \in (X, Z, W)$, $Zn_{13} + n_{22} \in (X, Y, W)$ and $Xn_{11} + W^3 n_{23} \in (Y, Z)$. Hence n_{11}, n_{21}, n_{22} and n_{23} $\in \mathfrak{n}B_\mathfrak{n}$, and $I_2(N) \in \mathfrak{n}B_\mathfrak{n}$, which is a contradiction. □

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