

**MUKAI-UMEMURA'S EXAMPLE OF
THE FANO THREEFOLD WITH GENUS 12
AS A COMPACTIFICATION OF \mathbf{C}^3**

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§0. Introduction

Let (X, Y) be a smooth projective compactification with the non-normal irreducible boundary Y , namely, X is a smooth projective algebraic threefold and Y a non-normal irreducible divisor on X such that $X - Y$ is isomorphic to \mathbf{C}^3 . Then Y is ample and the canonical divisor K_X on X can be written as $K_X = -rY$ ($1 \leq r \leq 4$). Thus X is a Fano threefold. In particular, $\text{Pic } X \cong \mathbf{Z} \mathcal{O}_X(Y)$. The non-normality of Y implies that $r \leq 2$ (cf. [4]). In the case of $r = 2$, such a (X, Y) is uniquely determined up to isomorphism, in fact, $(X, Y) \cong (V_5, H_5^\infty)$, where $X = V_5$ is a Fano threefold of degree 5 in \mathbf{P}^6 , and $Y = H_5^\infty$ is a ruled surface swept out by lines which intersect the line Σ with the normal bundle $N_{\Sigma|X} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$, in particular, Σ is the singular locus of Y . In the case of $r = 1$, there is an example of such a compactification of \mathbf{C}^3 , in fact, let $X = V'_{22}$ be a Fano threefold of genus $g = 12$ constructed by Mukai-Umemura [11] and $Y = H'_{22}$ be the ruled surface swept out by conics which intersect the line ℓ in V'_{22} with the normal bundle $N_{\ell|X} \cong \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell(1)$, then H'_{22} is a non-normal hyperplane section of V'_{22} such that $V'_{22} - H'_{22}$ is isomorphic to \mathbf{C}^3 , in particular, the line ℓ is the singular locus of H'_{22} (cf. [6]).

Now, in this paper, we will construct a birational map $\pi : V'_{22} \cdot \rightarrow V_5$ such that the restriction π_0 of π on $V'_{22} - H'_{22}$ gives an isomorphism $V'_{22} - H'_{22} \cong V_5 - H_5^\infty \cong \mathbf{C}^3$, via the resolution of indeterminacy of the double projection of V'_{22} from the singular locus $\text{Sing } H'_{22}$ of H'_{22} which is a line on V'_{22} (see Theorem 1). Furthermore, we will study the detailed structure of the desingularization and the normalization of the boundary divisor H'_{22} (see Theorem 2).

Recently, Mukai ([11_a]) proved that there is a 4-dimensional family of Fano threefolds of first kind with index one, genus 12 which are the compactifications of \mathbf{C}^3 with non-normal boundaries, in particular, our example (V'_{22}, H'_{22}) belongs

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to this Mukai's family.

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Notation

| | |
|---------------------|--|
| K_X | Canonical divisor on a variety X |
| ω_X | Canonical sheaf on X |
| $N_{C X}$ | Normal bundle of C in X |
| $ H $ | Complete linear system associated with a divisor H |
| $\text{Bs } H $ | Base locus of the linear system $ H $ |
| $\text{Sing } X$ | Singular locus of X |
| $\rho(X)$ | Picard number of X |
| E_{red} | Reduction of a scheme E |
| $\text{supp } D$ | Support of a divisor D |
| (i)-curve | Smooth rational curve with self-intersection number $-i$ |
| $b_i(X)$ | $:= \dim H^i(X; \mathbf{R})$ |
| $h^i(\mathcal{F})$ | $:= \dim H^i(*; \mathcal{F})$ |
| $\chi(\mathcal{F})$ | $:= \sum_{i=0} (-1)^i h^i(\mathcal{F})$ |

§1. Mukai-Umemura's example

Let $\mathbf{C}[x, y]$ be the polynomial ring of two complex variables x and y . The special linear group $SL(2, \mathbf{C})$ acts $\mathbf{C}(x, y)$ as follows:

$$\begin{cases} x^\sigma = ax + by \\ y^\sigma = cx + dy \end{cases} \quad \text{for } \sigma = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in SL(2, \mathbf{C}).$$

Let us denote by R_n a vector space of homogeneous polynomials of degree n in $\mathbf{C}[x, y]$. Let $f(x, y) = \sum_{i=0}^n a_i \binom{n}{i} x^{n-i} y^i \in R_n$ be a non-zero homogeneous polynomial of degree n . We take $(a_0 : a_1 : \dots : a_n)$ as homogeneous coordinates on the projective space $\mathbf{P}(R_n) \cong \mathbf{P}^n$, on which $SL(2, \mathbf{C})$ acts. Let us denote by $X(f)$ the closure of $SL(2, \mathbf{C})$ -orbit $SL(2, \mathbf{C}) \cdot f$ of f in $\mathbf{P}(R_n)$. Then $SL(2, \mathbf{C})$ acts on $X(f)$.

Now, we consider the following two polynomials:

$$f_6(x, y) = xy(x^4 - y^4), \text{ and}$$

$$h_{12}(x, y) = xy(x^{10} + 11x^5y^5 + y^{10}).$$

We put

$$V_5 := X(f_6) \hookrightarrow \mathbf{P}(R_6) \cong \mathbf{P}^6, \text{ and}$$

$$V'_{22} := X(h_{12}) \hookrightarrow \mathbf{P}(R_{12}) \cong \mathbf{P}^{12}.$$

Then we have

LEMMA 1 (Lemma 3.3 in [11]). (1) $V_5 \hookrightarrow \mathbf{P}^6$ is a Fano threefold of index 2, genus 21 and the hyperplane section of V_5 is the positive generator of $\text{Pic } V_5 \cong \mathbf{Z}$

(2) V'_{22} is a Fano threefold of index 1, genus 12 and the hyperplane section of V'_{22} is the positive generator of $\text{Pic } V'_{22} \cong \mathbf{Z}$.

The defining equations for V_5, V'_{22} are given as follows respectively:

$$(V_5) \quad \begin{cases} a_0 a_4 - 4a_1 a_3 + 3a_2^2 = 0 \\ a_0 a_5 - 3a_1 a_4 + 2a_2 a_3 = 0 \\ a_0 a_6 - 9a_2 a_4 + 8a_3^2 = 0 \\ a_1 a_6 - 3a_2 a_5 + 2a_3 a_4 = 0 \\ a_2 a_6 - 4a_3 a_5 + 3a_4^2 = 0 \end{cases}$$

$$(V'_{22}) \quad \sum_{\lambda=0}^{\rho} \binom{8}{\lambda} \binom{8}{\rho-\lambda} (a_\lambda a_{\rho+4-\lambda} - 4a_{\lambda+1} a_{\rho+3-\lambda} + 3a_{\lambda+2} a_{\rho+2-\lambda}) = 0$$

$$(0 \leq \rho \leq 16)$$

Now, we put

$$H_5^\infty := V_5 \cap \{a_0 = 0\} \hookrightarrow \mathbf{P}^5,$$

$$H'_{22} := V'_{22} \cap \{a_0 = 0\} \hookrightarrow \mathbf{P}^{11}.$$

Let us denote by $\text{Sing } H_5^\infty$ (resp. $\text{Sing } H'_{22}$) the singular locus of H_5^∞ (resp. H'_{22}). Then we have

PROPOSITION 1 ([5]). (1) $V_5 - H_5^\infty = V_5 \cap \{a_0 \neq 0\} \cong \mathbf{C}^3$,

(2) $\Sigma := \text{Sing } H_5^\infty = \{a_0 = a_1 = \cdots = a_4 = 0\} \cong \mathbf{P}^1(a_5 : a_6) \hookrightarrow \mathbf{P}^6$ is a line on V_5 . In particular, H_5^∞ is a non-normal hyperplane section of V_5 swept out by lines which intersect the line Σ .

PROPOSITION 2 ([6]). H'_{22} is a non-normal hyperplane section such that $V'_{22} - H'_{22} = V'_{22} \cap \{a_0 \neq 0\} \cong \mathbf{C}^3$.

We will study the detailed structure of H'_{22} below.

LEMMA 2. (1) $\ell := \text{Sing } H'_{22} = \{a_0 = \cdots = a_{10} = 0\} \cong \mathbf{P}^1(a_{11} : a_{12}) \hookrightarrow \mathbf{P}^{12}$ is a line on V'_{22} .

(2) The normal bundle $N_{\ell|V'_{22}} \cong O_{\ell}(-2) \oplus O_{\ell}(1)$, and there is no other line in V'_{22} which intersects the line ℓ .

(3) H'_{22} is a unique member of the linear system $|O_{V'_{22}}(1) \otimes I_{\ell}^3|$, where I_{ℓ} is the ideal sheaf of ℓ in $O_{V'_{22}}$. In particular, H'_{22} is a ruled surface swept out by conics in V'_{22} which intersect the line ℓ .

Proof. We shall rewrite the defining equation (V'_{22}) as follows:

$$\begin{array}{l}
 \text{(e.0)} \quad a_0 a_4 - 4a_1 a_3 + 3a_2^2 = 0 \\
 \text{(e.1)} \quad a_0 a_5 - 3a_1 a_4 + 2a_2 a_3 = 0 \\
 \text{(e.2)} \quad 7a_0 a_6 - 12a_1 a_5 - 15a_2 a_4 + 20a_3^2 = 0 \\
 \text{(e.3)} \quad a_0 a_7 - 6a_2 a_5 + 5a_3 a_4 = 0 \\
 \text{(e.4)} \quad 5a_0 a_8 + 12a_1 a_7 - 42a_2 a_6 - 20a_3 a_5 + 45a_4^2 = 0 \\
 \text{(e.5)} \quad a_0 a_9 + 6a_1 a_8 - 6a_2 a_7 - 28a_3 a_6 + 27a_4 a_5 = 0 \\
 \text{(e.6)} \quad a_0 a_{10} + 12a_1 a_9 + 12a_2 a_8 - 76a_3 a_7 - 21a_4 a_6 \\
 \quad \quad \quad + 72a_5^2 = 0 \\
 \text{(e.7)} \quad a_0 a_{11} + 24a_1 a_{10} + 90a_2 a_9 - 130a_3 a_8 - 405a_4 a_7 \\
 \quad \quad \quad + 420a_5 a_6 = 0 \\
 \text{(e.8)} \quad a_0 a_{12} + 60a_1 a_{11} + 534a_2 a_{10} - 380a_3 a_9 - 3195a_4 a_8 \\
 \quad \quad \quad - 720a_5 a_7 + 2940a_6^2 = 0 \\
 \text{(e.9)} \quad a_1 a_{12} + 24a_2 a_{11} + 90a_3 a_{10} - 130a_4 a_9 - 405a_5 a_8 \\
 \quad \quad \quad + 420a_6 a_7 = 0 \\
 \text{(e.10)} \quad a_2 a_{12} + 12a_3 a_{11} + 12a_4 a_{10} - 76a_5 a_9 - 21a_6 a_8 \\
 \quad \quad \quad + 72a_7^2 = 0 \\
 \text{(e.11)} \quad a_3 a_{12} + 6a_4 a_{11} - 6a_5 a_{10} - 28a_6 a_9 + 27a_7 a_8 = 0 \\
 \text{(e.12)} \quad 5a_4 a_{12} + 12a_5 a_{11} - 42a_6 a_{10} - 20a_7 a_9 + 45a_8^2 = 0 \\
 \text{(e.13)} \quad a_5 a_{12} - 6a_7 a_{10} + 5a_8 a_9 = 0 \\
 \text{(e.14)} \quad 7a_6 a_{12} - 12a_7 a_{11} - 15a_8 a_9 + 20a_9^2 = 0 \\
 \text{(e.15)} \quad a_7 a_{12} - 3a_8 a_{11} + 2a_9 a_{10} = 0 \\
 \text{(e.16)} \quad a_8 a_{12} - 4a_9 a_{11} + 3a_{10}^2 = 0
 \end{array}$$

For simplicity, let us denote by $\{a_j = 1\}$ the affine part $\{a_j \neq 0\}$ of $\mathbf{P}^{12}(a_0 : \dots : a_j : \dots : a_{12})$, namely, $\{a_j = 1\} \cong \mathbf{C}^{12}(a_0, \dots, a_{j-1}, a_{j+1}, \dots, a_{12})$.

CLAIM 1. $H'_{22} \cap \{a_1 = 1\} \cong \mathbf{C}^{12}(a_2, a_6)$

In fact, setting $a_0 = 0, a_1 = 1$ in the equations (e.0) – (e.9) in $(V'_{22})^*$, one can easily see that the coordinate functions $a_3, a_4, a_7, a_8, \dots, a_{12}$ are given by the polynomials of a_2 and a_6 . This proves the claim.

Now, we have $H'_{22} \cap \{a_1 = 0\} = V'_{22} \cap \{a_0 = a_1 = 0\} = \{a_0 = a_1 = \dots = a_{10} = 0\} \cong \mathbf{P}^1(a_{11} : a_{12})$ (a line in V'_{22}). Since $H'_{22} - H'_{22} \cap \{a_1 = 0\} \cong \mathbf{C}^2$ by the Claim 1, we have that H'_{22} is non-normal (cf. [5]) and hence $\text{Sing } H'_{22} = H'_{22} \cap \{a_1 = 0\}$. This proves (1).

Next, let us consider the affine part $H'_{22} \cap \{a_{12} = 1\} \hookrightarrow \mathbf{C}^{12}(a_1, \dots, a_{11})$ of H'_{22} . Setting $a_0 = 0, a_{12} = 1$ in the defining equation $(V'_{22})^*$, one can get the defining equation of $H'_{22} \cap \{a_{12} = 1\}$ in \mathbf{C}^{11} . More precisely, from (e.9) – (e.16) with $a_{12} = 1$, putting $x := a_9, y := a_{10}, z := a_{11}$, one can get the following:

$$\left\{ \begin{array}{l} \text{(e.16)'} \quad a_8 = 2^2xz - 3y^2 \\ \text{(e.15)'} \quad a_7 = 2^2 \cdot 3xz^2 - 3^2y^2z - 2xy \\ \text{(e.14)'} \quad 7a_6 = 2^4 \cdot 3^2xz^3 - 2^2 \cdot 3^3y^2z^2 + 2^2 \cdot 3^2xyz \\ \quad \quad \quad - 3^2 \cdot 5y^3 - 2^2 \cdot 5x^2 \\ \text{(e.13)'} \quad a_5 = 2^3 \cdot 3^2xyz^2 - 2 \cdot 3^3y^3z + 3xy^2 - 2^2 \cdot 5x^2z \\ \text{(e.12)'} \quad a_4 = -2^4 \cdot 3x^2z^2 - 2^5x^2y + 2^3 \cdot 3^3xy^2z - 3^3 \cdot 5y^4 \\ \text{(e.11)'} \quad a_3 = -2^4 \cdot 5x^3 - 2^4 \cdot 3^3x^2z^3 + 2^4 \cdot 3^3x^2yz \\ \quad \quad \quad + 2^3 \cdot 3^4xy^2z^2 - 2^2 \cdot 3^4xy^3 - 3^5y^4z \\ \text{(e.10)'} \quad a_2 = -2^5 \cdot 5^2x^3z - 2^7 \cdot 3^3x^2z^4 + 2^4 \cdot 3^3 \cdot 11x^2yz^2 \\ \quad \quad \quad + 2^6 \cdot 3^4xy^2z^3 - 2^3 \cdot 3^3 \cdot 29xy^3z - 2^3 \cdot 3^5y^4z^2 \\ \quad \quad \quad + 2^3 \cdot 3^2 \cdot 7x^2y^2 + 3^4 \cdot 5^2y_5 \\ \text{(e.9)'} \quad a_1 = -2^4 \cdot 3^2 \cdot 5 \cdot 7x^3z^2 - 2^8 \cdot 3^4x^2z^5 + 2^6 \cdot 3^3 \cdot 19x^2yz^3 \\ \quad \quad \quad + 2^7 \cdot 3^5xy^2z^4 - 2^5 \cdot 3^4 \cdot 17xy^3z^2 - 2^4 \cdot 3^6y^4z^3 \\ \quad \quad \quad - 2^3 \cdot 3^3x^2y^2z + 2^2 \cdot 3^6 \cdot 5y^5z + 2^7 \cdot 5x^3y \\ \quad \quad \quad + 3^3 \cdot 5 \cdot 19xy^4 \end{array} \right.$$

CLAIM 2. $H'_{22} \cap \{a_{12} \neq 0\} \cong V(f) := \{(x, y, z) \in \mathbf{C}^3; f(x, y, z) = 0\}$,
where

$$\begin{aligned}
(*) \quad f(x, y, z) &= b_0x^4 + (b_1yz + b_2z^3)x^3 + \\
&\quad (b_3y^3 + b_4y^2z^2 + b_5yz^4)x^2 + (b_6y^4z + b_7y^3z^3)x \\
&\quad + b_8y^6 + b_9y^5z^2, \\
(b_0 &= -2^8 \cdot 5^2, b_1 = 2^9 \cdot 3^3 \cdot 5, b_2 = -2^6 \cdot 3^4 \cdot 5, \\
b_3 &= -2^8 \cdot 3^3 \cdot 7, b_4 = -2^4 \cdot 3^4 \cdot 127, b_5 = 2^9 \cdot 3^5, \\
b_6 &= 2^2 \cdot 3^6 \cdot 89, b_7 = -2^8 \cdot 3^6, b_8 = -3^6 \cdot 5^3, b_9 = 2^5 \cdot 3^7).
\end{aligned}$$

In fact, putting a_1, \dots, a_8 in (e.k)' ($9 \leq k \leq 16$) into (e.8) with $a_{12} = 1$, one can get the equation $f(x, y, z) = 0$. It is easy to see that the polynomial $f(x, y, z)$ is irreducible. Hence, $V(f)$ is the defining equation of $H'_{22} \cap \{a_{12} \neq 0\}$ in \mathbf{C}^3 .

By the defining equation of $V(f)$, one can see the singular locus $\text{Sing } V(f) = \{x = y = 0\}$ and the multiplicity of $V(f)$ at a general point of $\text{Sing } V(f)$ is equal to three.

Thus $H'_{22} \in |\mathcal{O}_{V_{22}}(1) \otimes I_{\ell}^3|$. Since $h^0(\mathcal{O}_{V_{22}}(1) \otimes I_{\ell}^3) \leq 1$ by Iskovskih [7], H'_{22} is a unique member of $|\mathcal{O}_{V_{22}}(1) \otimes I_{\ell}^3|$. This implies that any conics in V'_{22} intersecting the line ℓ is always contained in H'_{22} . By Iskovskih [7], for every point $p \in V'_{22}$, there is a finite number of conics passing through p . Thus we have the assertion (3). The assertion (2) is proved in Mukai-Umemura [11].

Q.E.D.

§2. Double projection

We will study the double projection of V'_{22} from the line ℓ , which is the singular locus of H'_{22} . For simplicity, we put $X := V'_{22}$, $Y := H'_{22}$.

First, let us consider the linear system $|\mathcal{H}| := |\mathcal{O}_X(1) \otimes I_{\ell}^2|$ on X . Let $\sigma_1: X_1 \rightarrow X$ be the blowing up of X along the line ℓ in X . By Lemma 2-(2), we have $L_1 := \sigma_1^{-1}(\ell) \cong \mathbf{F}_3$ (Hirzebruch surface). We put $|\mathcal{H}_1| := |\sigma_1^*H - 2L_1|$, where $H \in |\mathcal{O}_X(1)|$. Let Y_1 be the proper transform of Y in X_1 . By Lemma 2-(3), we have a linear equivalence $Y_1 \sim \sigma_1^*H - 3L_1$. By Lemma 5.4 in Iskovskih [7], we have

- LEMMA 3. (1) $\dim |\mathcal{H}| = \dim |\mathcal{H}_1| = 6$,
(2) $\dim |\sigma_1^*H - 3L_1| = 0$, namely, Y_1 is the unique member of the linear system $|\sigma_1^*H - 3L_1|$,
(3) $(\sigma_1^*H - 2L_1)^3 = 2$,

(4) $Y_1 \cdot L_1 \sim 3\ell_1 + 7f_1$ in L_1 , where ℓ_1, f_1 is the negative section, a fiber of L_1 respectively.

Let K_{X_1} be a canonical divisor on X_1 . Then we have $K_{X_1} \sim -\sigma_1^*H + L_1$. Since $(L_1 \cdot \ell_1) = 1$, we have $(K_{X_1} \cdot \ell_1) = 0$. By the following exact sequence of normal bundles:

$$\begin{array}{ccccccc}
 0 & \rightarrow & N_{\ell_1|L_1} & \rightarrow & N_{\ell_1|X_1} & \rightarrow & N_{L_1|X_1|\ell_1} & \rightarrow & 0 \\
 & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & \mathcal{O}(-3) & & \mathcal{O}(a) \oplus \mathcal{O}(b) & & \mathcal{O}(1) & &
 \end{array}$$

where $a + b = 2$, we have

LEMMA 4.

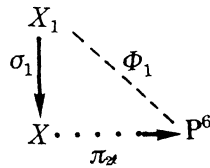
$$N_{\ell_1|X_1} \cong \begin{cases} \text{(a)} & \mathcal{O}(-1) \oplus \mathcal{O}(-1), \\ \text{(b)} & \mathcal{O}(-2) \oplus \mathcal{O} & , \text{ or} \\ \text{(c)} & \mathcal{O}(-3) \oplus \mathcal{O}(1). \end{cases}$$

LEMMA 5. $\text{Bs} |H_1| = \ell_1$, where $\text{Bs} |H_1|$ is the base locus of $|H_1|$.

Proof. Since $(\sigma_1^*H - 2L_1) \cdot \ell_1 = -1$, $\ell_1 \subseteq \text{Bs} |\mathcal{H}_1|$. By Lemma 2-(2), there is no other line in X which intersects ℓ . Thus, by the same argument as in the proof of Lemma 5.4-(ii) in [7], we have the claim.

Q.E.D.

Let us denote by $\pi_{2\ell}$ a rational map defined by the linear system $| \mathcal{O}_X(1) \otimes I_{\ell}^2 |$, which is called the "double projection from ℓ ". Then we have a diagram:



where $\Phi_1 := \Phi_{|\mathcal{H}_1|}$ is a rational map defined by the linear system $|\mathcal{H}_1|$.

Next, we will resolve the indeterminacy of the rational map $\Phi_1 : X_1 \dashrightarrow \mathbf{P}^6$

LEMMA 6. (1) $\text{Sing } Y_1 = 2\ell_1$, namely, ℓ_1 is the singular locus of Y_1 with the

multiplicity 2,

(2) $Y_1 \cap L_1 = A_1 + A_2 + A_3$, where A_i 's are non-singular rational curves and $A_1 \sim 2\ell_1$, $A_2 \sim \ell_1 + 4f_1$, $A_3 \sim 3f_1$ in L_1 .

Proof. Looking at the blowing up $\sigma_1: X_1 \rightarrow X$ locally, one may identify the Zariski open set $\sigma_1^{-1}(X_1 \cap \{a_{12} \neq 0\})$ with the blowing up $\mu: M \rightarrow \mathbf{C}^3(x, y, z)$ with the center $\text{Sing } V(f) = \{x = y = 0\}$. M is covered by two coordinate patches $U_0 = \mathbf{C}^3(r, s, t)$, $U_1 = \mathbf{C}^3(u, v, w)$, with $r \cdot v = 1$ on $U_0 \cap U_1$, and μ is given by

$$\mu: \begin{cases} x = rs = u \\ y = s = uv \\ z = t = w. \end{cases}$$

Let V_1 be the proper transform of $V(f)$ in M . Then we have

$$V_1 \cap U_0 = \{f_1^*(r, s, t) = 0\}, \text{ where}$$

$$f_1^* := b_0 r^4 s + (b_1 s t + b_2 t^3) r^3 + (b_3 s^2 + b_4 s t^2 + b_5 t^4) r^2 \\ + (b_6 s^2 t + b_7 s t^3) r + b_8 s^3 + b_9 s^2 t^2, \text{ and}$$

$$V_1 \cap \{s = 0\} = \{r^2 t^3 (b_2 r + b_5 t) = 0\}.$$

This shows that $\{r = s = 0\}$ is the singular locus of V_1 with the multiplicity 2 and $V_1 \cap \{s = 0\}$ consists of three irreducible non-singular rational curves. Since $Y_1 \cdot L_1 \sim 2\ell_1 + 7f_1$, we have the assertions (1) and (2).

Q.E.D.

Let $\sigma_i: X_i \rightarrow X_{i-1}$ be the blowing up of X_{i-1} along the section ℓ_{i-1} of L_{i-1} with $(\ell_{i-1}^2)_{L_{i-1}} \leq 0$, and put $L_i := \sigma_i^{-1}(\ell_{i-1})$ ($i \geq 2$). Let f_i be a fiber of L_i , Y_i the proper transform of Y_{i-1} in X_i , and put $\mathcal{H}_i := \sigma_i^* \mathcal{H}_{i-1} - L_i$.

LEMMA 7.

- (1) $Y_2 \cap L_2 = B_1 + B_2$, where $B_1 \sim 2\ell_2$, $B_2 \sim 2f_2$ in L_2 ,
- (2) $\text{Sing } Y_2 = 2\ell_2$,
- (3) $\text{Bs } |\mathcal{H}_2| = \ell_2$.

Proof. By Lemma 4, we have the following three cases:

$$L_2 \cong \begin{cases} \text{(a)} & \mathbf{P}^1 \times \mathbf{P}^1, \\ \text{(b)} & \mathbf{F}_2, \\ \text{(c)} & \mathbf{F}_4. \end{cases}$$

Since $Y_2 \sim \sigma_2^* Y_1 - 2L_2$, we have

$$Y_2 \cdot L_2 \sim \begin{cases} \text{(a)} & 2\ell_2 & \text{if } L_2 \cong \mathbf{P}^1 \times \mathbf{P}^1, \\ \text{(b)} & 2\ell_2 + 2f_2 & \text{if } L_2 \cong \mathbf{F}_2, \\ \text{(c)} & 2\ell_2 + 4f_2 & \text{if } L_2 \cong \mathbf{F}_4. \end{cases}$$

On the other hand, by blowing up $U_0 = \mathbf{C}^3(r, s, t)$ along $\{r = s = 0\}$, one can get the local equation for Y_2 . From this, one can show that $\text{Sing } Y_2 = 2\ell_2$, and $Y_2 \cap L_2 = B_1 + B_2$, where $B_1 \sim 2\ell_2$, $B_2 \sim 2f_2$ in L_2 . Thus we have $L_2 \cong \mathbf{F}_2$. Since $(H_2, \ell_2) = -1$, $\ell_2 \subseteq \text{Bs } |\mathcal{H}_2|$. On the other hand, since $|\mathcal{H}_2| \cap L_2 \subseteq |\mathcal{H}_{2|L_2}| = |\ell_2 + f_2|$, we have the claim.

Q.E.D.

COROLLARY 8. $L_2 \cong \mathbf{F}_2$, namely, $N_{\ell_2|X_1} \cong \mathcal{O}(-2) \oplus \mathcal{O}$.

Similarly, one can show the following

LEMMA 9.

- (1) $Y_3 \cap L_3 = C_1 + C_2$, where $C_1 \sim 2\ell_3$, $C_2 \sim 2f_3$ in L_3 .
- (2) $\text{Sing } Y_3 = 2\ell_3 + 2f_3$,
- (3) $\text{Bs } |\mathcal{H}_3| = \ell_3$,
- (4) $L_3 \cong \mathbf{F}_2$, namely, $N_{\ell_3|X_3} \cong \mathcal{O}(-2) \oplus \mathcal{O}$,
- (5) $Y_4 \cap L_4 = D$, where $D \sim 2\ell_4$ in L_4 ,
- (6) $L_4 \cong \mathbf{P}^1 \times \mathbf{P}^1$, namely, $N_{\ell_4|X_4} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Let $L_j^{(4)}$ ($1 \leq j \leq 3$) be the proper transform of L_j in X_4 and $A_i^{(4)}$ ($1 \leq i \leq 3$), $f_1^{(4)}$ be the proper transforms of A_i , a fiber f_1 in X_4 respectively. Then we have easily

$$(2.1) \quad \mathcal{H}_4 = \sigma_4^* \mathcal{H}_3 - L_4 \sim Y_4 + L_1^{(1)} + 2L_2^{(4)} + 3L_3^{(4)} + 4L_4$$

$$(2.2) \quad K_{X_4} \sim -(Y_4 + 2L_1^{(4)} + 3L_2^{(4)} + 4L_3^{(4)} + 5L_4)$$

$$(2.3) \quad (L_4 \cdot \ell_4) = (L_4 \cdot f_4) = -1, (\ell_4 \cdot \ell_4) = 0$$

$$(2.4) \quad (\mathcal{H}_4^3) = (\mathcal{H}_4 \cdot \mathcal{H}_4 \cdot \mathcal{H}_4) = 5$$

$$(2.5) \quad |\mathcal{H}_4| \cap L_4 = |\ell_4|$$

$$(2.6) \quad (\mathcal{H}_4 \cdot \mathcal{H}_4 \cdot Y_4) = 0$$

$$(2.7) \quad (\mathcal{H}_4 \cdot \mathcal{H}_4 \cdot L_1^{(4)}) = 5$$

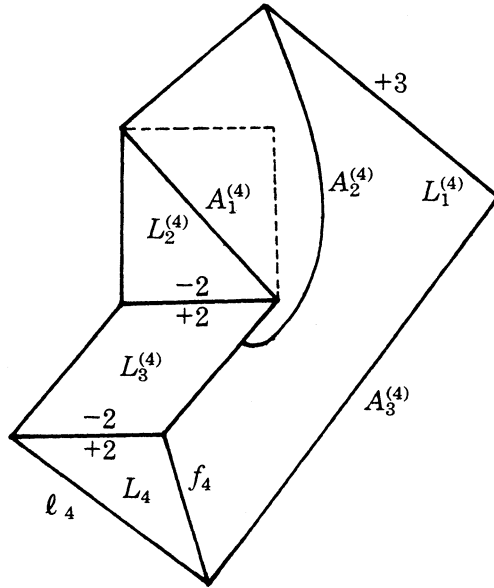
$$(2.8) \quad (\mathcal{H}_4 \cdot \mathcal{H}_4 \cdot L_j) = (\mathcal{H}_4 \cdot \mathcal{H}_4 \cdot L_j^{(4)}) = 0 \quad (j = 2, 3)$$

$$(2.9) \quad (\mathcal{H}_4 \cdot A_2^{(4)}) = 5, \quad (\mathcal{H}_4 \cdot f_1^{(4)}) = 1.$$

By (2.5), we have

LEMMA 10. $Bs|\mathcal{H}_4| = \phi$.

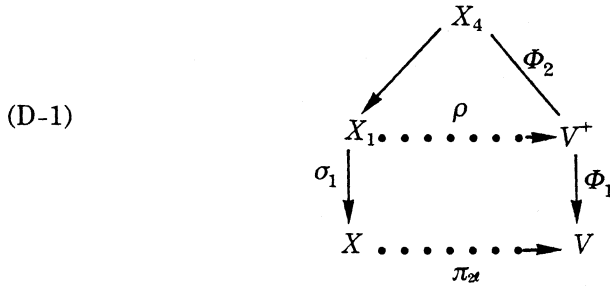
Let $\Phi : X_4 \rightarrow \mathbf{P}^6$ be a morphism defined by the linear system $|\mathcal{H}_4|$. We put $V := \Phi(X_4)$. By (2.4), $\deg V = 5$. By (2.6), (2.7), (2.8), $X - Y \cong X_4 - (Y_4 \cup L_4 \cup L_1^{(4)} \cup L_2^{(4)} \cup L_3^{(4)}) \cong V - \Phi(L_1^{(4)}) \cong \mathbf{C}^3$.



By (2.3), L_4 can be blown down along ℓ_4 , and then blowing downs can be done step by step (cf. Reid [12]). Finally we have a smooth projective threefold V^+ with

$b_2(V^+) = 2$, and morphisms $\Phi_2: X_4 \rightarrow V^+$, $\Phi_1: V^+ \rightarrow V$, a birational map $\rho: X \dashrightarrow V^+$ (which is called a flop) such that

- (i) $\Phi = \Phi_1 \circ \rho \circ \Phi_2$
- (ii) $X_1 - \ell_1 \cong V^+ - \Sigma_1$, where $\Sigma_1 := \Phi_2(L_4 \cap L_1^{(4)})$
- (iii) $V^+ - \rho(Y_1) \cong V - \Phi(Y_4)$



Let Y_1^+, L_1^\dagger be the proper transforms of Y_1, L_1 in V^+ respectively. We put $\Gamma := \Phi(Y_4) = \Phi_1(Y_1^+)$ and $Z := \phi(L_1^{(4)}) = \Phi_1(L_1^\dagger)$. Then, by (2.6), (2.7), (2.9), Γ is a smooth rational curve of degree 5 in \mathbf{P}^6 and Z is a ruled surface swept out by lines which intersect the line $\Sigma := \Phi_1(\Sigma_1)$ on V . In particular, $\Gamma \hookrightarrow Z$ and $\Gamma \cap \Sigma = \{\text{one point}\}$. Let γ be a conic in X which intersect the line ℓ . Then $\gamma \hookrightarrow Y$. Let γ_1 be the proper transform of γ in X_1 and $\gamma_1^+ := \rho(\gamma_1) \hookrightarrow Y_1^+$. Since $K_{V^+} = \rho_*(K_{X_1}) = -Y_1^+ - 2L_1^\dagger$, we have $(K_{V^+} \cdot \gamma_1^+) = -1$. Thus, $\Phi_1: V^+ \rightarrow V$ be the contraction of an extremal ray by K.M.M. [9]. Since Y_1^+ is contracted to the smooth curve Γ by Φ_1 , V is smooth by Mori [10]. By (2.4), we have $\text{deg } V = 5$. Moreover, we have $K_V \sim -2Z$. Since $V - Z \cong \mathbf{C}^3$ by construction, Z is ample, thus, V is a Fano threefold of first kind with index 2, genus 21. Since Z is swept out by lines in V , Z is non-normal. In fact, the singular locus of Z is just the line $\Sigma := \Phi_1(\Sigma_1)$. Therefore we have $(V, Z) \cong (V_5, H_5^\infty)$ (see §1), namely

THEOREM 1. *Let (V'_{22}, H'_{22}) , $\ell := \text{Sing } H'_{22}$, (V_5, H_5^∞) be as before. Then the double projection $\pi_{2\ell}: V'_{22} \rightarrow V_5$ of V'_{22} from the line ℓ gives an isomorphism $V'_{22} - \mathcal{H}'_{22} \xrightarrow{\sim} V_5 - H_5^\infty (\cong \mathbf{C}^3)$.*

Remark 1. Let $\Sigma := \text{Sing } H_5^\infty$ be the singular locus of H_5^∞ . Then, Σ is a line on V_5 with the normal bundle $N_{\Sigma|V_5} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. The set $\{x \in \Sigma; \text{ there is a unique line passing through the point } x\}$ consists of the only point p (cf. [5]). One can easily see that there is a smooth rational curve Γ of degree 5 in V_5 such that

$\Gamma \cap \Sigma \{p\}$ and $\Gamma \hookrightarrow H_5^\infty$. Then the linear system $|\mathcal{O}_{V_5}(3) \otimes I^2|$ defines the inverse birational map $\pi_2^{-1} : V_5 \dashrightarrow V'_{22}$ with $V_5 - H_5^\infty \simeq V'_{22} - H'_{22}$ (cf. [7]).

§3. Normalization and resolution of the boundary divisor

First, we will prepare some general results on a non-normal hyperplane section of a Fano threefold of special series.

Let X be a Fano threefold of special series, namely, X is a smooth threefold $V_{2g-2} \hookrightarrow \mathbf{P}^{g+1}$ of degree $2g - 2$. Then the anticanonical line bundle $-K_X$ is an ample generator of $\text{Pic } X \cong \mathbf{Z}$. Let Y be a non-normal member of the linear system $|-K_X|$. Since $\text{Pic } X \cong \mathbf{Z}[Y]$, Y is irreducible. Let $\sigma : S \rightarrow Y$ be the normalization, and let $I \hookrightarrow \mathcal{O}_Y$ be the conductor of σ . We put $E := \text{loc } I$ (the locus of I) and $D := \sigma^{-1}(E)$. Since Y is Cohen-Macaulay, E and D are Cohen-Macaulay. Since $Y \sim -K_X$, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ and $H^i(X, \mathcal{O}_X(-Y)) = 0$ for $i < 3$, we have

$$\begin{aligned} (3.1) \quad & \omega_Y \cong \mathcal{O}_Y \\ (3.2) \quad & H^1(Y, \mathcal{O}_Y) = 0, H^2(Y, \mathcal{O}_Y) \cong \mathbf{C} \\ (3.3) \quad & \omega_S \cong I \otimes \sigma^* \omega_Y \cong I \text{ (i.e. } K_S \sim -D \text{ as a Weil divisor)}. \end{aligned}$$

By (3.34.2), (3.34.3) in Mori [10], we have exact sequences:

$$(3.4) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow \sigma_* \mathcal{O}_S \rightarrow \omega_E \rightarrow 0$$

$$(3.5) \quad 0 \rightarrow \sigma_* \omega_S \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_E \rightarrow 0$$

Taking σ^* in (3.5), we have

$$(3.6) \quad 0 \rightarrow \omega_S \rightarrow \mathcal{O}_S \rightarrow \sigma^* \mathcal{O}_E \cong \mathcal{O}_D \rightarrow 0.$$

By (3.2), (3.3), (3.3), we have

LEMMA 11 ([14]). $h^0(\mathcal{O}_E) = 1$ and $h^1(\mathcal{O}_E) = 0$, namely E_{red} is connected and each irreducible component E_i of E_{red} is a smooth rational curve.

Take a general hyperplane section H of X . From (3.4), we get

$$(3.7) \quad 0 \rightarrow \mathcal{O}_Y(H) \rightarrow \sigma_* \mathcal{O}_S \otimes \mathcal{O}_Y(H) \rightarrow \omega_E(H) \rightarrow 0.$$

Since $H^1(Y, \mathcal{O}_Y(H)) = 0$, we have

$$(3.8) \quad h^0(\sigma_* \mathcal{O}_S \otimes \mathcal{O}_Y(H)) = h^0(\mathcal{O}_Y(H)) + h^0(\omega_E(H)).$$

We put $\delta := (H \cdot E)_X$.

CLAIM (3.9). $h^0(S, \sigma^*H) = g + \delta$.

In fact, since E is Cohen-Macaulay, $h^0(\omega_Y(H)) = h^1(\mathcal{O}_E(-H))$. By the following exact sequence:

$$0 \rightarrow \mathcal{O}_E(-H) \rightarrow \mathcal{O}_E \rightarrow \mathcal{O}_{E \cap H} \rightarrow 0,$$

we have $h^1(\mathcal{O}_E(H)) = h^0(\mathcal{O}_{E \cap H}) - h^0(\mathcal{O}_E) = \delta - 1$. Since $h^0(\sigma^*H) = h^0(\sigma_*\mathcal{O}_S(\sigma^*H)) = h^0(\sigma_*\mathcal{O}_S \otimes \mathcal{O}_Y(H))$ and $h^0(\mathcal{O}_Y(H)) = g + 1$, we have $h^0(S, \sigma^*H) = g + \delta$.

Let $\Delta(S, \sigma^*H) := \dim S + \deg \sigma^*H - h^0(S, \sigma^*H)$ be the Δ -genus of the polarized variety (S, σ^*H) (cf. [3]). Since $\dim S = 2$ and $\deg \sigma^*H = (H^3)_X = 2g - 2$, we have

LEMMA 12. $\Delta(S, \sigma^*H) = g - \delta$.

LEMMA 13. $(D \cdot \sigma^*H) = 2(E \cdot H) = 2\delta$.

Proof. By (3.36.2) in Mori [10], we have

$$0 \rightarrow \mathcal{O}_E \rightarrow \sigma_*\mathcal{O}_D \rightarrow \omega_E \rightarrow 0.$$

Thus we have $\chi(\sigma_*\mathcal{O}_D \otimes H) = \chi(\mathcal{O}_E(H)) + \chi(\omega_E(H)) = 2\delta + \chi(\mathcal{O}_E) + \chi(\omega_E) = 2$. On the other hand, $\chi(\sigma_*\mathcal{O}_D \otimes H) = \chi(\mathcal{O}_D \otimes \sigma^*H) = (D \cdot \sigma^*H) + \chi(\mathcal{O}_D)$. Since $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_S) - \chi(\omega_S) = 0$, we have $(D \cdot \sigma^*H) = 2\delta$.

Q.E.D.

Let $C \in |\sigma^*H|$ be a smooth member. By Bertini's theorem, such a member C exists. Let us denote by $g(C)$ the genus of C .

LEMMA 14. $g(C) = g - \delta$.

Proof. By the adjunction theorem, $2g(C) - 2 = C(\omega_S + C)$. Since $(C^2) = 2g - 2$ and $(C \cdot \omega_S) = 2\delta$ by Lemma 13, we have $g(C) = g - \delta$.

Q.E.D.

Let $\mu : M \rightarrow S$ be the minimal resolution, and put $\phi := \mu \circ \sigma : M \rightarrow Y$. Since $K_S \sim -D$ (as a Weil divisor), we have $K_M \sim -\widehat{D} - \sum_i m_i \Delta_i$ ($m_i > 0$, $m_i \in \mathbf{Z}$), where \widehat{D} is the proper transform of D in M and $\cup_i \Delta_i$ is the exceptional set of μ .

LEMMA 15. *M is rational or ruled.*

Proof. Since $H^0(M, \mathcal{O}_M(mK_M)) = 0$ for $m > 0$, by the classification of surfaces, we have the lemma.

Q.E.D.

LEMMA 16. *If $h^1(\mathcal{O}_M) = 0$, then $\text{Sing } S$ consists of at worst rational singularities, in particular, S is rational.*

Proof. Let us consider the following exact sequence:

$$0 \rightarrow H^1(S, \mathcal{O}_S) \rightarrow H^1(M, \mathcal{O}_M) \rightarrow H^0(S, R^1\mu_*\mathcal{O}_M) \rightarrow H^2(S, \mathcal{O}_S) \rightarrow 0.$$

By assumption, we have $H^1(M, \mathcal{O}_M) = 0$. Since $H^2(S, \mathcal{O}_S) \cong H^0(S, \omega_S) = 0$, we have the claim.

Q.E.D.

Now, Mukai-Umemura's example V'_{22} is a special class of Fano threefolds of special series with the genus $g = 12$, and H'_{22} is a non-normal hyperplane section of V'_{22} such that $V'_{22} - H'_{22} \cong \mathbf{C}^3$. We can apply the above lemmas to these $X := V'_{22}$ and $Y := H'_{22}$.

LEMMA 17. *Assume that $(X, Y) = (V'_{22}, H'_{22})$. Then we have*

- (1) $E_{\text{red}} \cong \mathbf{P}^1$,
- (2) $Y - E_{\text{red}} \cong \mathbf{C}^2$,
- (3) $H^1(Y; \mathbf{Z}) = 0$, $H^2(Y; \mathbf{Z}) \cong \mathbf{Z}$, $H^3(Y; \mathbf{Z}) = 0$,
- (4) S is a rational surface and $\text{Sing } S$ consists of at worst rational singularities.
- (5) $g(C) = 12 - \delta$ for a general smooth member $C \in |\sigma^*H|$.

Proof. By Lemma 2 and its proof, we have (1) and (2). Since $X - Y \cong \mathbf{C}^3$, we have $H^i(X; \mathbf{Z}) \cong H^i(Y; \mathbf{Z})$ for $i \geq 0$. It is known that $H^i(V'_{22}; \mathbf{Z}) = H^i(\mathbf{P}^3; \mathbf{Z})$ for $i \geq 0$, that is, V'_{22} has the same cohomology as \mathbf{P}^3 . This proves (3). Let us consider the following exact sequence (cf. [1]):

$$(*) \quad 0 \rightarrow H^2(Y; \mathbf{Z}) \rightarrow H^2(S; \mathbf{Z}) \oplus H^2(E; \mathbf{Z}) \rightarrow H^2(D; \mathbf{Z}) \rightarrow 0$$

$$\rightarrow H^3(Y; \mathbf{Z}) \rightarrow H^3(S; \mathbf{Z}) \rightarrow 0$$

Since $H^3(Y; \mathbf{Z}) = 0$, we have $H^3(S; \mathbf{Z}) = 0$. Since $b_3(M) = b_3(S) = 0$ (cf. [2]), $b_1(M) = 0$, hence, $h^1(\mathcal{O}_M) = h^1(\mathcal{O}_S) = 0$. By Lemma 16, we have (4). Since $g = 12$, by Lemma 14, we have (5).

Q.E.D.

LEMMA 18. $K_M + \phi^*H$ is nef.

Proof. Assume that $K_M + \phi^*H$ is not nef. Then, by Cone theorem and Contraction theorem in [8] (cf. [9]), there is a contraction $\pi : M \rightarrow Z$ of the extremal ray, where Z is normal and $\pi^{-1}(z)$ is connected for any $z \in Z$.

Case (a). $\dim Z = 2$. Then there is a curve R such that $\pi(R)$ is a point and $R^2 < 0$, $(K_M + \phi^*H) \cdot R < 0$. Since $(\phi^*H \cdot R) \geq 0$ and $R^2 < 0$, we have $R \cong \mathbf{P}^1$ and $R^2 = -1$, hence, $(\phi^*H \cdot R) = 0$. Thus R is an exceptional curve of μ . Since $\mu : M \rightarrow S$ is the minimal resolution, this is a contradiction.

Case (b). $\dim Z = 1$. Since M is rational, we have $Z \cong \mathbf{P}^1$. Since $\rho(M) = \rho(Z) + 1 = 2$, M is isomorphic to \mathbf{F}_n (Hirzebruch surface), namely, $\pi : M \rightarrow Z \cong \mathbf{P}^1$ is a \mathbf{P}^1 -bundle over \mathbf{P}^1 . For a fiber f , we have $(K_M + \phi^*H) \cdot f < 0$. Hence, $(\phi^*H \cdot f) = (H \cdot \phi(f)) = 1$ since $(K_M \cdot f) = -2$. Thus, Y is a ruled surface swept out by lines on X . By Lemma 2-(2), E_{red} is a line on X and $E_{\text{red}} \cap \phi(f) = \emptyset$ for a general fiber f . This shows that $\phi(f) \subset Y - E_{\text{red}} \cong \mathbf{C}^2$. This is a contradiction.

Case (c). $\dim Z = 0$. In this case, $M \cong \mathbf{P}^2$. For a smooth member $C \in |\phi^*H|$, we put $\deg C = d$. Then, $C^2 = d^2 = 22$, this is a contradiction.

Q.E.D.

By Lemma 2-(3), $Y := H'_{22}$ is a ruled surface swept out by conics which intersect the line $\ell := \text{Sing } Y$ in $X := V'_{22}$, where $\ell = E_{\text{red}}$. Take a general conic γ in Y . Then, $\gamma \cap E_{\text{red}} \neq \emptyset$. Let $\hat{\gamma}$ be the proper transform of γ in M . Then we have $(\phi^*H \cdot \hat{\gamma}) = (H \cdot \gamma) = 2$. Since $K_M + \phi^*H$ is nef by Lemma 18, we have $(K_M + \phi^*H) \cdot \hat{\gamma} \geq 0$, hence, $(K_M \cdot \hat{\gamma}) \geq -2$. On the other hand, since $K_M \sim -\hat{D} - \sum_i m_i \Delta_i$ ($m_i \geq 0, m_i \in \mathbf{Z}$), we have $(K_M \cdot \hat{\gamma}) \leq 0$.

CLAIM (1). $(K_M \cdot \hat{\gamma}) \neq 0$.

In fact, if $(K_M \cdot \hat{\gamma}) = 0$, then $(\hat{D} \cdot \hat{\gamma}) = 0, (\Delta_i \cdot \hat{\gamma}) = 0$ for each i . We take a general γ . Thus $\mathbf{P}^1 \cong \hat{\gamma} \hookrightarrow M - \hat{D} - \cup \Delta_i Y - E_{\text{red}} \cong \mathbf{C}^2$. This is a contradiction.

CLAIM (2). There is an irreducible conic γ_0 in Y such that $(K_M \cdot \hat{\gamma}_0) = -2$ (that is, $\hat{\gamma}_0 \cong \mathbf{P}^1$ with the self-intersection number $\hat{\gamma}_0^2 = 0$).

In fact, by Claim (1), we have $(K_M \cdot \hat{\gamma}) = -1$ or -2 for any conic γ in Y . If $(K_M \cdot \hat{\gamma}) = -1$, then $\hat{\gamma}$ is a (-1) -curve. Thus, M contains a continuous family of (-1) -curves. This is a contradiction.

Let $\tau : M \rightarrow \mathbf{P}^1$ be a morphism defined by the linear system $|\hat{\gamma}_0|$. For a general p in \mathbf{P}^1 , $\tau^{-1}(p) \sim \hat{\gamma}_0$.

LEMMA 19. $K_M + \phi^*H \sim (11 - \delta)\hat{\gamma}_0$.

Proof. By Basepoint-free Theorem of Kawamata [7], we have $\text{Bs } |m(K_M + \phi^*H)| = \emptyset$ for $m \gg 0$. We put $\hat{f} := \tau^{-1}(p)$ (a general fiber of τ). By Claim (2), $(K_M + \phi^*H)\hat{f} = 0$. Let $\tau_m : M \rightarrow Z_0$ be a morphism defined by the linear system $|m(K_M + \phi^*H)|$. Since M is rational and since $\tau_m(\hat{f})$ is a point, we have $Z_0 \cong \mathbf{P}^1$, in particular, we have $m(K_M + \phi^*H) \sim k\hat{\gamma}_0$. Since $(\phi^*H \cdot K_M) = -2\delta$, $(\phi^*H \cdot \phi^*H) = 22$ and $(\phi^*H \cdot \hat{\gamma}_0) = 2$, we have $(22 - 2\delta)m = 2k$, hence, $k = (11 - \delta)m$. Since $\text{Pic } M$ has no torsion, we have $(K_M + \phi^*H) \sim (11 - \delta)\hat{\gamma}_0$.
Q.E.D.

COROLLARY 20. $\text{Bs } |K_M + \phi^*H| = \emptyset$.

Let \hat{f} be a regular fiber of τ . Then $\phi(\hat{f}) = \gamma \hookrightarrow Y \hookrightarrow X$ is a conic in X .

LEMMA 20. Each Δ_i is contained in a singular fiber of τ .

Proof. Assume that Δ_1 not contained in any singular fiber of τ . Then $\tau|_{\Delta_1} : \Delta_1 \rightarrow \mathbf{P}^1$ is a surjective morphism, hence, $(\Delta_1 \cdot \hat{f}) \neq 0$ for a regular fiber \hat{f} . Since $\phi(\Delta_1)$ is a point and since $\phi(\hat{f}) = \gamma$ is a conic in $Y \hookrightarrow X$, we have an infinite number of conics in X passing through the point $\phi(\Delta_1) \in X$. On the other hand, for each point $x \in X$, the number of conics passing through the point x is finite by Iskovskih [7]. Thus we have a contradiction.

Q.E.D.

LEMMA 21. Let B be an irreducible component of a singular fiber of $\tau : M \rightarrow \mathbf{P}^1$. Then $B^2 = -1$ or -2 . Furthermore,

(i) $B^2 = -1 \Leftrightarrow \phi(B) = E_{\text{red}} \cong \mathbf{P}^1$

(ii) $B^2 = -2 \Leftrightarrow B = \Delta_i$ for some i .

Proof. Since $(K_M + \phi^*H) \cdot B = (11 - \delta) \cdot (\hat{\gamma} \cdot B) = 0$, we get $(K_M \cdot B) = -(\phi^*H \cdot B) \leq 0$. Since $B \cong \mathbf{P}^1$ and $B^2 < 0$, we have $B^2 = -1$ or $B^2 = -2$. (i): $B^2 = -1 \Leftrightarrow (K_M \cdot B) = -1 \Leftrightarrow (\phi^*H \cdot B) = 1 \Leftrightarrow (H \cdot \phi(B)) = 1 \Leftrightarrow \phi(B)$ is a line in $Y \Leftrightarrow \phi(B) = E_{\text{red}}$ (because $E_{\text{red}} = \text{Sing } Y$ is a unique line in Y by Lemma 2-(2)). (ii): $B^2 = -2 \Leftrightarrow \phi(B)$ is a point of $Y \Leftrightarrow B$ is a component of the exceptional set of $\mu \Leftrightarrow B = \Delta_i$ for some i .

Q.E.D.

COROLLARY 22. *Sing S consists of (at worst) rational double points.*

Proof. For each Δ_i , one has $(\Delta_i \cdot \Delta_i) = -2$. This proves the corollary.

LEMMA 23. $\delta = 4$.

Proof. Let $C \in |\sigma^*H|$ be a smooth member. By Bertini theorem, such a member C exists. We put $C_0 := \sigma(C)$. Then $\sigma : C \rightarrow C_0$ is the normalization. We may assume that C_0 is contained in a $K3$ surface H_0 , which is a hyperplane section of $X := V'_{22}$. Since $\text{Sing } Y =: E_{\text{red}}$ is a line in X , $\text{Sing } C_0$ consists of only one point p_0 . On the other hand, from the defining equation (*) in Lemma 2, the local equation of C_0 around p_0 in H_0 can be written as $u_0x^3 + u_1x^2y + u_2xy^3 + u_3y^5 = 0$, where $p_0 = (0, 0)$. Thus C_0 has two singular points p_0 and p'_0 (infinitely near singular point lying over p_0) with the multiplicity three and two respectively. Since H_0 is a $K3$ surface, the arithmetic genus $p_a(C_0) = \frac{1}{2}(C_0 \cdot C_0) + 1 = 12$, hence, the genus $g(C) = p_a(C_0) - 4 = 8$. Since $g(C) = 12 - \delta$ by Lemma 12, we have $\delta = 4$.

Q.E.D.

LEMMA 24. $K_M^2 = -6$ and $b_2(M) = 16$.

Proof. Since $(K_M + \phi^*H)^2 = K_M^2 - 4\delta + 22 = 0$ and $\delta = 4$, we have $K_M^2 = -6$. By Noether formula, we have $b_2(M) = 16$.

Q.E.D.

LEMMA 24. *The number of the singular fiber of $\tau : M \rightarrow \mathbf{P}^1$ is equal to one.*

Proof. Let F_i ($1 \leq i \leq t$) be a singular fiber of τ , $1 + \alpha_i$ the number of the irreducible components of F_i , and e_i the number of the irreducible components of $\overline{F_i - \Delta}$, where $\Delta := \cup \Delta_i$. By Lemma 21, $e_i =$ the number of irreducible components of $\widehat{D} \cap F_i =$ the number of (-1) -curves in F_i . Since M is rational, we have $b_2(M) = 2 + \sum_i \alpha_i$. Since $b_2(M) = b_2(S) + b_2(\Delta)$ and $b_2(\Delta) = \sum_i (1 + \alpha_i - e_i)$, we have $b_2(S) = 2 - \sum (1 - e_i)$. On the other hand by the following exact sequence (cf. [1]):

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^2(Y; \mathbf{Z}) & \rightarrow & H^2(S; \mathbf{Z}) \oplus H^2(E; \mathbf{Z}) & \rightarrow & H^2(D; \mathbf{Z}) \rightarrow 0, \\
 & & \parallel & & \parallel & & \\
 & & \mathbf{Z} & & \mathbf{Z} & &
 \end{array}$$

we have $b_2(S) = b_2(D)$. Since $K_M \sim -\widehat{D} - \sum m_i \Delta_i$ and $(K_M \cdot \hat{f}) = -2$ for a regular fiber \hat{f} of τ , we have $(\widehat{D} \cdot \hat{f}) = 2$. This shows that $b_2(\widehat{D}) > \sum e_i$. Thus we have $2 - \sum(1 - e_i) = b_2(S) = b_2(D) = b_2(\widehat{D}) > \sum e_i$, that is, $2 > t \geq 1$. Therefore we have $t = 1$.

Q.E.D.

LEMMA 25. $\widehat{D} = 2\widehat{D}_1 + 3\widehat{D}_2 + 3\widehat{D}_3$, where \widehat{D}_1 is a section of $\tau : M \rightarrow \mathbf{P}^1$ and \widehat{D}_i 's are the (-1) -curves in the singular fiber of τ for $i = 2, 3$.

Proof. Let $\sigma_1 : X_1 \rightarrow X$, Y_1, L_1, A_i ($1 \leq i \leq 3$), ℓ_1, f_1 be as in Lemma 6. Since $Y_1 \sim \sigma^*H - 3L_1$, by the adjunction formula, we have $K_{Y_1} \sim -2L_1|_{Y_1} \sim -2(A_1 + A_2 + A_3)$ as a Weil divisor. Let $\nu : \bar{S}_1 \rightarrow Y_1$ be the normalization and \bar{A}'_1 (resp. \bar{A}_1) be the closed subscheme in Y_1 (resp. \bar{S}_1) defined by the conductor of ν . Since $\text{Sing } Y_1 = A_1$, $\text{supp } A_1 = \text{supp } \bar{A}'_1$.

CLAIM (1). There is a morphism $\eta : \bar{S}_1 \rightarrow S$ such that $\sigma \circ \eta = \sigma_1 \circ \nu$ (see D-2).

$$\begin{array}{ccccc}
 & & \bar{S}_1 & \xrightarrow{\nu} & Y_1 \\
 & & \eta \downarrow & & \downarrow \sigma_1 \\
 \text{(D-2)} & M & \xrightarrow{\mu} & S & \xrightarrow{\sigma} & Y \\
 & & & \searrow \psi & &
 \end{array}$$

In fact, let \bar{A}_i be the proper transform of A_i in \bar{S}_1 . Since $\bar{S}_1 - \cup \text{supp } \bar{A}_i \cong Y_1 - \cup \text{supp } A_i \cong Y - \text{supp } E$, we have the claim. In particular, $\eta(\text{supp } \bar{A}_3)$ is a point on S , $\bar{S}_1 - \text{supp } \bar{A}_3 \cong S - \eta(\text{supp } \bar{A}_3)$ and $\eta(\text{supp } \bar{A}_1 \cup \text{supp } \bar{A}_2) = \text{supp } D$.

We put $D_1 := \eta_*(\bar{A}_2)$. Since $A_2 \sim \ell_1 + 4f_1$ in L_1 , A_2 is reduced, hence, D_1 is reduced. Let \widehat{D}_1 is the proper transform of D_1 in M .

CLAIM (2). \widehat{D}_1 is a section of $\tau : M \rightarrow \mathbf{P}^1$, and $(\phi^*H \cdot \widehat{D}_1) = 1$.

In fact, let γ be a general conic $Y \hookrightarrow X$, and $\bar{\gamma}$ the proper transform of γ in $Y_1 \hookrightarrow X_1$. Then we have $(L_1 \bar{\gamma}) = 1$. Since $Y_1 \cdot L_1 \sim (2\ell_1) + (\ell_1 + 4f_1) + (3f_1)$ and $\bar{\gamma} \hookrightarrow Y_1$, we have $(A_2 \cdot \bar{\gamma}) = 1$, hence, $(\widehat{D}_1 \cdot \bar{\gamma}) = 1$, where $\bar{\gamma}$ is the proper transform of γ in M . Thus \widehat{D}_1 is a section of $\tau : M \rightarrow \mathbf{P}^1$. Since $(\sigma_1^*H \cdot A_2) = (\sigma_1^*H \cdot \ell_1 + 4f_1) = 1$, we have $(\phi^*H \cdot \widehat{D}_1) = 1$.

CLAIM (3). $\widehat{D} \sim 2\widehat{D}_1 + 3\widehat{D}_2 + 3\widehat{D}_3$, where $\widehat{D}_2, \widehat{D}_3$ are the (-1) -curves in the singular fiber of τ .

In fact, since $-K_M \sim \widehat{D} + \sum m_i \Delta_i$, we have $2 = (\widehat{D} \cdot \hat{f}) + \sum m_i (\Delta_i \cdot \hat{f})$ for a regular fiber \hat{f} of τ . Since Δ_i 's are contained in the singular fiber of τ , we have $(\widehat{D} \cdot \hat{f}) = 2$. Since $\eta(\text{supp } \bar{A}_1 \cup \bar{A}_2) = \text{supp } D$ and $(A_2 \cdot \bar{\gamma}) = 1$, we have $\widehat{D} = 2\widehat{D}_1 + \sum n_i \widehat{D}_i$ ($i \geq 2$, $n_i \in \mathbf{Z}$, $n_i > 0$). We note that the proper transform of γ in M is linearly equivalent to a regular fiber \hat{f} of $\tau : M \rightarrow \mathbf{P}^1$. Since \widehat{D}_i 's ($i \geq 2$) are contained in the singular fiber of τ , by Lemma 21, \widehat{D}_i 's are the (-1) -curves in the singular fiber of τ . Hence $(\phi^*H \cdot \widehat{D}_i) = 1$ ($i \geq 2$).

Let us recall the normalization $\sigma : C \rightarrow C_0$ (see the proof of Lemma 6). From the local defining equation of C_0 in H_0 there, one can see that $\sigma^{-1}(p_0)$ consists of three distinct points, where $p_0 := \text{Sing } C_0$. This shows that $\widehat{D} = 2\widehat{D}_1 + a\widehat{D}_2 + b\widehat{D}_3$, where $a + b = 6$, since $(\phi^*H \cdot \widehat{D}) = 8$. On the other hand, since $\bar{K}_{S_1} \sim -2\nu^*(A_1 + A_2 + A_3) - \bar{A}_1$ as a Weil divisor, we have $D \sim -K_S \sim 2\eta_*\nu^*A_2 + (2\eta_*\nu^*A_1 + \eta_*\bar{A}_1)$. Since $\text{supp } D_1 = \text{supp } \eta_*\nu^*A_2$ and $\text{supp } \eta_*\bar{A}_1 = \text{supp } \eta_*\bar{A}_1 \hookrightarrow \text{supp } D$, we have $a = b = 3$. This completes the proof.

Q.E.D.

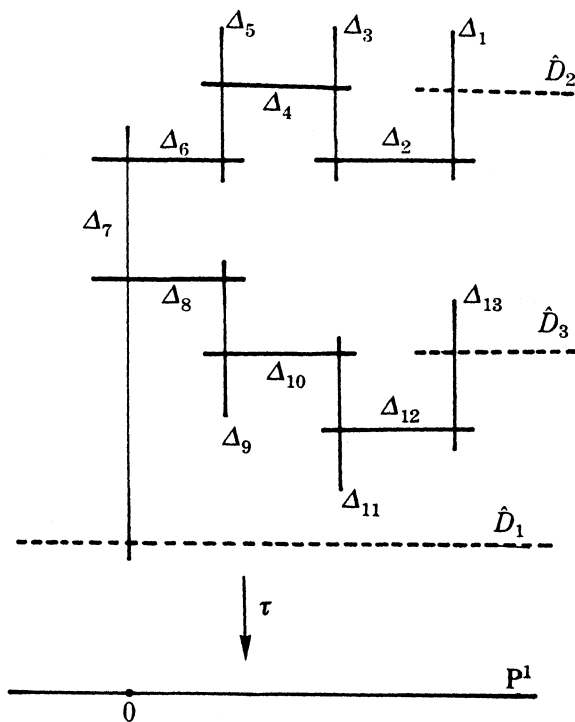
THEOREM 2. Let $(X, Y) := (V'_{22}, H'_{22})$ be as in §1. Let $\sigma : S \rightarrow Y := H'_{22}$ be the normalization, and E the non-normal locus defined by the conductor of σ , and D the analytic inverse image of E . Let $\mu : M \rightarrow S$ be the minimal resolution and $\mu^{-1}(\text{Sing } S) = \cup \Delta_i$, where Δ_i 's are irreducible. Then,

- (1) E is non-reduced and $E_{\text{red}} \cong \mathbf{P}^1$,
- (2) $\text{Sing } S = p_0$, p_0 is a rational double point of A_{13} -type,

(3) $D \sim 2D_1 + 3D_2 + 3D_3$ as a Weil divisor on S , where D_i 's are irreducible reduced Weil divisors on S such that $D_i \cong \mathbf{P}^1$ and $D_1 \cap D_2 \cap D_3 = \{p_0\}$,

(4) there is a fibering $\tau : M \rightarrow \mathbf{P}^1$ with exactly one singular fiber $\tau^{-1}(0)$ such that $\tau^{-1}(0) = \cup \Delta_i \cup \hat{D}_2 \cup \hat{D}_3$, $(\hat{D}_i \cdot \hat{D}_i) = -1$, $(\Delta_j \cdot \Delta_j) = -2$ for $i \geq 2$, $j \geq 1$, in particular, \hat{D}_1 is a section of τ (see Figure 2 below), where \hat{D}_i is the proper transform of \hat{D}_i in M , and

(5) $K_M \sim -2\hat{D}_1 - 3\hat{D}_2 - 3\hat{D}_3 - \sum_{i=1}^7 (3+i) \Delta_i - \sum_{i=1}^6 (3+i) \Delta_{14-i}$, where $(\hat{D}_1 \cdot \Delta_7) = (\hat{D}_2 \cdot \Delta_1) = (\hat{D}_3 \cdot \Delta_{13}) = 1$, $(\hat{D}_i \cdot \hat{D}_j) = 0$ ($i \neq j$), $(\Delta_i \cdot \Delta_{i+1}) = 1$, $(\Delta_i \cdot \Delta_j) = 0$ ($|i-j| > 1$).



Proof. By Lemma 2-(1), $E_{\text{red}} \cong \mathbf{P}^1$. By Lemma 23, $(E \cdot H) = 4$ for a hyperplane section H of $X := V'_{22}$. This proves (1). By Lemma 24, $\tau : M \rightarrow \mathbf{P}^1$ has exactly one singular fiber and the self-intersection number of each irreducible component the singular fiber is equal to -1 or -2 . By Lemma 21 and Lemma 25, \hat{D}_2 and \hat{D}_3 are the (-1) -curves in the singular fiber of τ , and other components of the singular fiber are the exceptional divisor of μ . This enables us to determine the type of the singular fiber of τ (see Figure 2). This proves (2), (3), (4).

Since $K_M \sim -\hat{D} - \sum m_i \Delta_i$, by the adjunction formula, we have (5)

Q.E.D.

Remark 2. Our example (V'_{22}, H'_{22}) of a compactification of \mathbf{C}^3 gives a counter example to Theorem (3.16) in the paper of Peternell [14].

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