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LOCAL RIGIDITY THEOREMS OF 2-TYPE HYPERSURFACES IN A HYPERSPHERE

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Dedicated to Professor Tadashi Nagano on his 60th birthday

1. Introduction

A submanifold M (connected but not necessary compact) of a Euclidean m-space E^m is said to be *of finite type* if each component of its position vector X can be written as a finite sum of eigenfunctions of the Laplacian Δ of M, that is,

$$X = X_0 + \sum_{t=1}^{k} X_t$$

where X_0 is a constant vector and $\Delta X_t = \lambda_t X_t$, $t = 1, 2, \cdots, k$. If in particular all eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ are mutually different, then M is said to be *of k-type* (cf. [3] for details).

In terms of finite type submanifolds, a well-known result of T. Takahashi [10] says that a submanifold M is S^m is of 1-type if and only if M is a minimal submanifold of S^m. The theory of minimal submanifolds has attracted many mathematicians for many years. Many interesting results concerning minimal submanifolds have been obtained. For instances, T. Otsuki investigated in [7, 8] minimal (i.e., 1-type) hypersurfaces M of a hypersphere S^{n+1} of a Euclidean $(n+2)$-space E^{n+2} such that M has exactly two distinct principal curvatures. Some interesting local classification theorems were obtained by him (cf. [7, 8]). On the other hand, the problem of classification of 2-type hypersurfaces of S^{n+1} was initiated in [3]. Several results in this respect were obtained in [1, 3, 4, 5, 6].

In this paper we consider the classification problem similar to Otsuki's for 2-type hypersurfaces in S^{n+1}. As a consequence the following two local rigidity theorems are obtained.

THEOREM 1. *Let M be a hypersurface of the hypersphere $S^{n+1}(1)$ in*

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E^{n+2} with at most two distinct principal curvatures. Then M is of 2-type if and only if M is an open portion of the product of two spheres $S^p(r_1) \times S^{n-p}(r_2)$ such that $r_1^2 + r_2^2 = 1$ and $(r_1, r_2) \neq (\sqrt{p/n}, \sqrt{(n-p)/n})$.

THEOREM 2. *Let M be a hypersurface of the hypersphere $S^{n+1}(1)$ in E^{n+2}. Then M is conformally flat and of 2-type if and only if M is an open portion of $S^1(r_1) \times S^{n-1}(r_2)$ where $r_1^2 + r_2^2 = 1$ and $(r_1, r_2) \neq (\sqrt{1/n}, \sqrt{(n-1)/n})$.*

Remark 1. Theorems 1 and 2 generalize the main results of [1, 6], Theorem 3 of [5] and also Theorem 4.5 of [3, p. 279].

2. Some basic formulas

Let M be a connected hypersurface of the unit hypersphere $S^{n+1}(1)$ centered at the origin of E^{n+2}. Then the position vector X of M in E^{n+2} is normal to M as well as to $S^{n+1}(1)$. Denote by ξ a unit local vector field normal to M and tangent to $S^{n+1}(1)$. Let A, h and H denote the Weingarten map, the second fundamental form, and the mean curvature vector of M in E^{n+2}, respectively, and A', h' and H' the corresponding invariants of M is $S^{n+1}(1)$. We put

$$\alpha^2 = \langle H, H\rangle\,, \quad \beta^2 = \langle H', H'\rangle$$

where $\langle\ ,\ \rangle$ denotes the inner product of E^{n+2}. We have

$$H = H' - X\,, \quad H' = \beta\xi\,, \quad \alpha^2 = \beta^2 + 1\,. \tag{2.1}$$

For simplicity we put $B = A_\xi (= A'_\xi)$. From [3, 4] we have

$$\Delta H = (\Delta\beta)\xi + \|h\|^2 H' - n\alpha^2 X + (\Delta H)^T\,, \tag{2.2}$$

where $\|h\|$ is the length of h and $(\Delta H)^T$, the tangential component of ΔH, satisfies [4]

$$(\Delta H)^T = \frac{n}{2}\operatorname{grad}\beta^2 + 2B(\operatorname{grad}\beta)\,. \tag{2.3}$$

If M is of 2-type, then there exist constants b, c and a constant vector X_0 such that (cf. [3])

$$\Delta H = bH + c(X - X_0)\,. \tag{2.4}$$

From (2.1)–(2.4) we may obtain

$$\langle \Delta H, X\rangle = -n\alpha^2 = -b + c - c\langle X, X_0\rangle\,, \tag{2.5}$$

$$\frac{n}{2}\operatorname{grad}\beta^2 + 2B(\operatorname{grad}\beta) = -c(X_0)^T, \tag{2.6}$$

where $(X_0)^T$ is the tangential component of X_0 and

$$\langle \Delta H, H\rangle = \beta\Delta\beta + \beta^2\|h\|^2 + n\alpha^2 = b\alpha^2 - c - c\langle X_0, H\rangle. \tag{2.7}$$

On the other hand, the last equality of (2.5) yields

$$-n\Delta\alpha^2 = -c\Delta(\langle X, X_0\rangle) = nc\langle H, X_0\rangle. \tag{2.8}$$

Thus, by combining (2.7) and (2.8), we have

$$\Delta\alpha^2 = \beta\Delta\beta + \beta^2\|h\|^2 + (n-b)\alpha^2 + c. \tag{2.9}$$

From (2.9) and the equation of Gauss we have the following [4]

LEMMA 1. *Let M be a 2-type hypersurface of $S^{n+1}(1)$. If M has constant mean curvature β, then M has constant length of the second fundamental form and constant scalar curvature.*

Also from (2.5) we may obtain

$$c(X_0)^T = n\operatorname{grad}\alpha^2 = n\operatorname{grad}\beta^2. \tag{2.10}$$

Therefore (2.6) and (2.10) imply [5]

LEMMA 2. *Let M be a 2-type hypersurface of $S^{n+1}(1)$. Then* grad β^2 *is an eigenvector of B with eigenvalue $-(3n/2)\beta$ on the open subset $U = \{u \in M \mid \operatorname{grad}\beta^2 \neq 0 \text{ at } u\}$.*

Let $e_1, \cdots, e_n$ be an orthonormal local frame field tangent to M. Denote by $\omega^1, \cdots, \omega^n$ the field of dual frames. Let (ω^A_B), $A, B = 1, \cdots, n+2$, be the connection forms associated with the orthonormal frame $\{e_1, \cdots, e_n, e_{n+1}, e_{n+2}\}$, where $e_{n+1} = \xi$ and $e_{n+2} = X$. Then the structure equations of M in E^{n+2} are given by

$$d\omega^i = -\sum_{j=1}^n \omega^i_j \wedge \omega^j, \quad \omega^i_j = -\omega^j_i, \tag{2.11}$$

$$d\omega^i_j = \sum_{k=1}^n \omega^i_k \wedge \omega^k_j + \omega^i_{n+1} \wedge \omega^j_{n+1} + \omega^i \wedge \omega^j, \tag{2.12}$$

$$d\omega^{n+1}_i = \sum_{j=1}^n \omega^{n+1}_j \wedge \omega^j_i, \quad i, j, k = 1, \cdots, n. \tag{2.13}$$

Moreover, if we put $h^{n+1}_{ij} = \langle h(e_i, e_j), e_{n+1}\rangle$, then we have

$$\omega^{n+1}_i = \sum_{j=1}^n h^{n+1}_{ij}\omega^j, \quad h^{n+1}_{ij} = \langle Be_i, e_j\rangle. \tag{2.14}$$

In particular, if $e_1, \cdots, e_n$ diagonalize $B = A_\xi$ such that

$$h_{ij}^{n+1} = \mu_i \delta_{ij} . \tag{2.15}$$

Then from (2.11)–(2.15) we have

$$e_i \mu_j = (\mu_i - \mu_j)\omega_i^j(e_j) , \tag{2.16}$$

$$(\mu_j - \mu_k)\omega_j^k(e_i) = (\mu_i - \mu_k)\omega_i^k(e_j) \tag{2.17}$$

for distinct i, j, k.

3. Proof of Theorem 1

Let M be a 2-type hypersurface of $S^{n+1}(1)$ with at most 2 distinct principal curvatures. Assume M has non-constant mean curvature. We put

$$W = \{u \in M \mid \beta^2(u) \neq 0 \text{ and } (\operatorname{grad} \beta^2)(u) \neq 0\} . \tag{3.1}$$

Then W is nonempty. From Lemma 2 we may choose e_1 in the direction of grad β^2 and hence we have

$$e_2 \mu_1 = \cdots = e_n \mu_1 = 0 , \quad Be_1 = \mu_1 e_1 , \quad \mu_1 = -(3n/2)\beta . \tag{3.2}$$

Let $T_1 = \{Y \in TU \mid B(Y) = \mu_1 Y\}$. If T_1 is of dimension ≥ 2 on some subset Z of W, then we may choose $e_2 \in T_1$ on Z. From (2.16) we obtain $e_1 \mu_2 = e_1 \mu_1 = 0$. This implies that β^2 is constant on Z, since grad β^2 is parallel to e_1. However, this is impossible from the definition of W. Therefore, we see that T_1 is 1-dimensional on W. Since M has at most two distinct principal curvatures, (3.2) implies that the remaining principal curvatures are given by

$$\mu_2 = \cdots = \mu_n = \frac{5n}{2(n-1)}\beta , \quad \text{on } W . \tag{3.3}$$

From (2.17) and (3.3) we obtain

$$\omega_1^k(e_i) = 0 , \quad i \neq k , \quad i, k = 2 , \cdots, n . \tag{3.4}$$

Moreover, from (2.16), (3.2) and (3.3) we find

$$\omega_i^1(e_1) = 0 . \tag{3.5}$$

From (3.2), (3.3) and (3.4) we have

$$\omega_1^{n+1} = -\left(\frac{3n}{2}\right)\beta\omega^1 , \quad \omega_i^{n+2} = \frac{5n}{2(n-1)}\beta\omega^i , \quad i = 2, \cdots, n , \tag{3.6}$$

$$d\beta = (e_1\beta)\omega^1 . \tag{3.7}$$

Thus, by taking the exterior differentiation of the first equation of (3.6) and applying (2.13), (3.6) and (3.7), we obtain $d\omega^1 = 0$. Therefore, there exists locally a function u such that

$$\omega^1 = du . \tag{3.8}$$

Equations (3.7) and (3.8) imply that β is a function of u. Denote by β' and β'' the first and the second derivatives of β with respect to u, respectively. From (2.16), (3.2) and (3.3), we obtain

$$\beta\omega_1^k(e_k) = -\left(\frac{5}{3n+2}\right)\beta', \quad k = 2, \cdots, n . \tag{3.9}$$

Combining (3.4) and (3.9) we get

$$\omega_1^k = -\left(\frac{5}{3n+2}\right)\left(\frac{\beta'}{\beta}\right)\omega^k, \quad k = 2, \cdots, n . \tag{3.10}$$

By taking exterior differentiation of ω_1^2 and applying (2.11), (2.12), (3.6) and (3.10) we may obtain

$$\left(\frac{5}{3n+2}\right)^2\left(\frac{\beta'}{\beta}\right)^2 - \left(\frac{5}{3n+2}\right)\left(\frac{\beta'}{\beta}\right)' = \frac{15n^2\beta^2}{4(n-1)} - 1 , \tag{3.11}$$

from which we have

$$0 = (3n+2)\beta\beta'' - (3n+7)(\beta')^2 + \frac{3n^2(3n+2)^2}{4(n-1)}\beta^4 - \frac{(3n+2)^2}{5}\beta^2 . \tag{3.12}$$

Solving differential equation (3.12) for β' we get

$$(\beta')^2 = -\left(\frac{3n+2}{5}\right)^2\beta^2 - \left(\frac{n(3n+2)}{2(n-1)}\right)^2\beta^4 + c_1\beta^{2(3n+7)/(3n+2)} \tag{3.13}$$

for some constant c_1. Also from (2.1), (3.2) and (3.9) we have

$$\Delta\alpha^2 = -2\beta\beta'' + \frac{2(2n-7)}{3n+2}(\beta')^2 , \tag{3.14}$$

$$\Delta\beta = \frac{5(n-1)(\beta')^2}{(3n+2)\beta} - \beta'' . \tag{3.15}$$

From (2.9), (3.2)–(3.4), (3.14), and (3.15) we obtain

$$0 = \beta\beta'' + \left(\frac{n+9}{3n+2}\right)(\beta')^2 + n\beta^2 + \frac{n^2(9n+16)}{4(n-1)}\beta^4 + (n-b)(1+\beta^2) + c. \tag{3.16}$$

Combining (3.12) and (3.16) we find

$$4(n+4)(\beta')^2+\frac{10n^2(3n+2)}{4(n-1)}\beta^4+\frac{2(3n+2)(4n+1)}{5}\beta^2 \\ +(3n+2)\{(n-b)(1+\beta^?)+c\}=0\,. \tag{3.17}$$

From (3.13) and (3.17) we conclude that β is constant on W which is a contradiction. Therefore, W must be empty. Hence, by continuity, we conclude that M has constant non-zero mean curvature in $S^{n+1}(1)$. Hence, by Lemma 1, $\|h\|$ is also constant. Since M has at most two distinct principal curvatures, the constancy of β and of $\|h\|$ implies that M has exactly two constant principal curvatures because M is assumed to be of 2-type. Thus, by Theorem 2.5 of [9], M is locally the product of two spheres $S^p(r_1)\times S^{n-p}(r_2)$ such that $r_1^2+r_2^2=1$. Moreover, since M is not minimal in $S^{n+1}(1)$, we have $(r_1, r_2)\neq(\sqrt{p/n}, \sqrt{(n-p)/n})$.

The converse of this is easy to verify. (Q.E.D.)

4. Proof of Theorem 2

If M is an open portion of the product $S^1(r_1)\times S^{n-1}(r_2)$ with $r_1^2+r_2^2=1$ and $(r_1, r_2)\neq(\sqrt{1/n}, \sqrt{(n-1)/n})$, then it is easy to verify that M is a 2-type conformally flat hypersurface of $S^{n+1}(1)\subset E^{n+2}$.

Conversely, assume M is a 2-type conformally flat hypersurface of $S^{n+1}(1)$. If either $n=2$ or $n\geq 4$, then M is quasi-umbilical, that is, M has at most two distinct principal curvatures such that one of them is of multiplicity $\geq n-1$, according to a result of E. Cartan and J. A. Schouten (cf. [2, p. 154]). In these two cases, Theorem 1 implies that M is an open portion of the product of a circle and an $(n-1)$-sphere with the appropriate radii mentioned above.

In the remaining part of this section we will prove that the same result also holds when $n=3$. Now, assume $n=3$. Denote the Ricci tensor and the scalar curvature of M respectively by R and r. Put

$$L=-R+\frac{r}{4}g, \tag{4.1}$$

where g denotes the metric tensor of M. Sicne M is conformally flat, a result of H. Weyl (cf. [2, p. 26]) yields

$$(\nabla_Y L)(Z, W)=(\nabla_Z L)(Y, W) \tag{4.2}$$

for vectors Y, Z, W tangent to M.

On the other hand, from the equation of Gauss, we have

$$R(Y, Z) = 2\langle Y, Z\rangle + 3\beta\langle BY, Z\rangle - \langle B^2Y, Z\rangle . \tag{4.3}$$

From (4.1) and (4.3) we find

$$L(Y, Z) = \left(\frac{r}{4} - 2\right)\langle Y, Z\rangle - 3\beta\langle BY, Z\rangle + \langle B^2Y, Z\rangle . \tag{4.4}$$

Therefore, by applying (4.2), (4.3), (4.4) and the equation of Codazzi, we obtain

$$(Yr)Z - (Zr)Y = 12\{(Y\beta)BZ - (Z\beta)BY\} - 4\{(\nabla_Y B^2)Z - (\nabla_Z B^2)Y\} , \tag{4.5}$$

$$r = 6 + 9\beta^2 - \|B\|^2 . \tag{4.6}$$

Let e_1, e_2, e_3 be orthonormal eigenvectors of B such that

$$Be_i = \mu_i e_i , \qquad i = 1, 2, 3 . \tag{4.7}$$

From (4.5) and (4.7) we may get

$$(\mu_j^2 - \mu_i^2)\omega_i^j(e_j) = 3(e_i\beta)\mu_j - \frac{1}{4}(e_i r) - e_i(\mu_j^2) , \tag{4.8}$$

$$(\mu_j^2 - \mu_k^2)\omega_j^k(e_i) = (\mu_i^2 - \mu_k^2)\omega_i^k(e_j) \tag{4.9}$$

for distinct i, j, k $(i, j, k = 1, 2, 3)$.

Let V be open subset of M on which V has three distinct principal curvatures in $S^4(1)$. If V is empty, then Theorem 2 follows from Theorem 1. So, from now on we may assume that V is non-empty and we work on V only.

Since the three principal curvatures μ_1, μ_2, μ_3 are distinct on V, formulas (2.17) and (4.9) give

$$\omega_i^j(e_k) = 0 \tag{4.10}$$

for distinct i, j and k. If the mean curvature β is constant on V, then from Lemma 1 and formula (4.6), $\|h\|$, $\|B\|$ and r are all constant on V. Thus (4.8) yields

$$e_i\mu_j^2 = (\mu_i^2 - \mu_j^2)\omega_i^j(e_j) \tag{4.11}$$

for distinct i and j. Combining (2.16) and (4.11) we find

$$e_i\mu_j = 0 \tag{4.12}$$

for distinct i and j. Since $3\beta = \mu_1 + \mu_2 + \mu_3$, (4.12) implies that V is an isoparametric hypersurface in $S^4(1)$ with three distinct principal curvatures. Furthermore, from (4.11), we have $\omega_i^j(e_j) = 0$. Therefore, from (4.10), we get $\omega_i^j = 0$. Thus V is flat and also the product of any two of the three principal curvatures is equal to -1. But this is a contradiction, since the later condition implies V is totally umbilical. Consequently, we know that the mean curvature of V in $S^4(1)$ is nonwhere constant. Hence, by applying Lemma 2, we may choose e_1 in the direction of grad β^2. In this case we have

$$(4.13) \qquad \mu_1 = -\frac{9}{2}\beta, \quad \mu_2 = \frac{15}{4}\beta + \delta, \quad \mu_3 = \frac{15}{4}\beta - \delta$$

for some function δ and from (2.16) and (4.13) that

$$(4.14) \quad e_2\beta = e_3\beta = 0, \ e_1\mu_2 = -\left(\delta + \frac{33}{4}\beta\right)\omega_1^2(e_2), \ e_1\mu_3 = \left(\delta - \frac{33}{4}\beta\right)\omega_1^3(e_3).$$

From (2.16), (4.13), and (4.14) we get

$$(4.15) \qquad \omega_1^2(e_1) = \omega_1^3(e_1) = 0.$$

Therefore, we obtain from (4.10) and (4.15) that

$$(4.16) \qquad \omega_1^2 = \phi\omega^2, \quad \omega_1^3 = \eta\omega^3,$$

where

$$(4.17) \qquad \phi = -\frac{4e_1\delta + 15e_1\beta}{33\beta + 4\delta}, \quad \eta = \frac{4e_1\delta - 15e_1\beta}{33\beta - 4\delta}.$$

By taking exterior differentiation of $\omega_1^4 = \mu_1\omega^1$ and applying (2.13), (4.14) and (4.16) we may obtain $d\omega^1 = 0$. Thus, there is a local function u such that

$$(4.18) \qquad \omega^1 = du.$$

From (4.14) and (4.18) we see that β is a function of u. From (2.16) and (4.8) we may obtain

$$(4.19) \qquad (\mu_j - \mu_i)e_i\mu_j = 3(e_i\beta)\mu_j - \frac{1}{4}(e_ir), \quad i \neq j,$$

Letting $i = 1$, $j = 2$ for (4.9) and using (4.6) we find

$$(4.20) \qquad (\mu_2 - \mu_3)\mu_1' + (\mu_1 - \mu_3)\mu_2' + (\mu_2 - \mu_1)\mu_3' = 0.$$

From (4.13) and (4.20) we obtain $\delta\beta' + 11\beta\delta' = 0$. Hence we get

$$\beta = a\delta^{-11}, \tag{4.21}$$

for some non-zero constant a. In particular, (4.21) implies that both δ and r are functions of u. Combining (4.13), (4.16), (4.17), and (4.21), we find

$$\mu_1 = -\frac{9}{2}a\delta^{-11}, \quad \mu_2 = \delta + \frac{15}{4}a\delta^{-11}, \quad \mu_3 = -\delta + \frac{15}{4}a\delta^{-11}, \tag{4.22}$$

$$\omega_1^2 = \frac{(165a\delta^{-12} - 4)\delta'}{33a\delta^{-11} + 4\delta}\omega^2, \quad \omega_1^3 = \frac{(165a\delta^{-12} + 4)\delta'}{33a\delta^{-11} - 4\delta}\omega^3. \tag{4.23}$$

From (4.8), (4.10) and the fact $e_i\delta = e_i r$, $i = 2, 3$, we have

$$\omega_2^3 = 0. \tag{4.24}$$

Taking exterior differentiation of the first equation of (4.23) and applying (2.11), (2.12), (4.22), (4.23) and (4.24), we may obtain

$$\begin{aligned}(165a\delta^{-12} - 4)\delta'' + (33a\delta^{-11} + 4\delta)^{-1}(32 - 11352a\delta^{-12} + 21780a^2\delta^{-24})(\delta')^2 \\ = (33a\delta^{-11} + 4\delta)\left(\frac{135}{8}a^2\delta^{-22} + \frac{9}{2}a\delta^{-10} - 1\right).\end{aligned} \tag{4.25}$$

Similarly, by taking exterior differentiation of the second equation of (4.23) we may obtain

$$\begin{aligned}(165a\delta^{-12} + 4)\delta'' + (33a\delta^{-11} - 4\delta)^{-1}(32 + 11352a\delta^{-12} + 21780a^2\delta^{-24})(\delta')^2 \\ = (33a\delta^{-11} - 4\delta)\left(\frac{135}{8}a^2\delta^{-22} - \frac{9}{2}a\delta^{-10} - 1\right).\end{aligned} \tag{4.26}$$

From (4.25) and (4.26) we get

$$\begin{aligned}176(\delta')^2(208 - 92565a^2\delta^{-24}) \\ = (1089a^2\delta^{-20} - 16\delta^4)(8415a^2\delta^{-24} - 176\delta^{-2} + 16).\end{aligned} \tag{4.27}$$

On the other hand, by taking the exterior differentiation of (4.24) and applying (2.12), (4.22) and (4.23), we obtain

$$\begin{aligned}16(\delta')^2(16 - 27225a^2\delta^{-24}) \\ = (1089a^2\delta^{-22} - 16\delta^2)(225a^2\delta^{-22} - 16\delta^2 + 16).\end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28) we know that both δ and β are constant on V. This is a contradiction. Consequently, V is empty. (Q.E.D.)

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