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NOTES ON ENERGY FOR SPACE-TIME PROCESSES OVER LÉVY PROCESSES

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Dedicated to Professor Masanori Kishi on his 60th birthday

§ 1. Introduction

Let $X = (X_t, 0 \le t < \infty)$ be a Lévy process on the Euclidean space R^d, that is, a process on R^d with stationary independent increments which has right continuous paths with left limits. We denote by P^x the probability measure such that $P^x(X_0 = x) = 1$ and by E^x the expectation relative to P^x. The process is characterized by the exponent Ψ through

$$E^0(\exp i\langle z, X_t\rangle) = \exp(-t\Psi(z)).$$

The λ-energy $E^\lambda_X(\nu)$ of a measure ν on R^d for X is defined by

$$E^\lambda_X(\nu) = \int \mathrm{Re}([\lambda + \Psi(z)]^{-1})|\mathscr{F}\nu(z)|^2 dz,$$

where $\mathscr{F}$ denotes the Fourier tranform on R^d. A nice explanation of the reason why it is called the λ-energy is given in Rao [11]. Throughout the paper $\mathscr{F}\nu(z)$ is defined by $\int \exp i\langle z, x\rangle\ \nu(dx)$ and we write $\mathscr{F}u(z)$ in place of $\mathscr{F}udx(z)$ if $\nu(dx) = u(x)dx$. So our λ-energy differs from Rao's by a constant multiple.

The space-time process $Y = (Y_t, 0 \le t < \infty)$ over X is a Lévy process on $R^1 \times R^d$ defined on the probability space $(R^1 \times \Omega, P^{r,x})$, where Ω is the path space of X and $P^{r,x} = \delta_r \otimes P^x$, δ_r being the Dirac measure at $r \in R^1$. The trajectory $Y_t(r, \omega)$ is $(r + t, X_t(\omega))$ and the exponent of Y is $\Psi(z) - it$. So the λ-energy $E^\lambda_Y(\mu)$ of a measure μ on $R^1 \times R^d$ for Y is

$$E^\lambda_Y(\mu) = \iint \mathrm{Re}([\lambda + \Psi(z) - it]^{-1})|\mathscr{F}\mu(t, z)|^2 dt\, dz,$$

where $\mathscr{F}$ denotes the Fourier transform on $R^1 \times R^d$.

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If we assume the existence of a transition probability density $p(t, x)$ of X relative to the Lebesgue measure dx, that is, $P^0(X_t \in dx) = p(t, x)dx$, the λ-resolvent density $U^\lambda(x)$ of X is $\int_0^\infty \exp(-\lambda t)p(t, x)dt$ and the λ-resolvent density $W^\lambda(t, x)$ of Y relative to the Lebesgue measure $dtdx$ on $R^1 \times R^d$ is

$$\exp(-\lambda t) 1_{]0,\infty[}(t)p(t, x) .$$

In this paper we show

THEOREM. *Let X be a Lévy process on R^d with a transition probability density, and Y be the space-time process over X. Let μ be a bounded measure on $R^1 \times R^d$ of compact support.*

(I) *Assume that the λ-energy of μ for Y is finite. Then we have the following.*

(i) *The R^d-marginal μ_2 of μ* (i.e. $\mu_2(B) = \mu(R^1 \times B)$) *has finite λ-energy for X.*

(ii) *If the R^1-marginal μ_1 of μ* (i.e. $\mu_1(B) = \mu(B \times R^d)$) *is singular to the Lebesgue measure on R^1, then the R^d-marginal μ_2 does not charge any semipolar set.*

(II) *Consider the case that μ is of the direct product form $\eta \otimes \nu$.*

(i) *If μ has finite λ-energy for Y and ν is carried by a semipolar set for X, then η has a L^2-density relative to the Lebesgue measure on R^1.*

(ii) *If ν is a bounded measure of compact support on R^d with finite λ-energy for X and it does not charge any semipolar set for X, then we can find a singular measure η of compact support so that $\mu = \eta \otimes \nu$ has finite λ-energy for Y.*

Using Theorem, we can get a new characterization of semipolar sets, which is announced for a more general class of Markov processes with transition probability density [9].

COROLLARY. *Let X be a Lévy process on R^d which has a transition probability density. Then a closed set B in R^d is semipolar if and only if*

$$P^x(X_t \in B \text{ for some } t \in A) = 0$$

for every $x \in R^d$ and every set $A \subset]0, \infty[$ of Lebesgue measure 0.

Remark. The above Corollary does not hold if we do not assume the existence of a transition probability density. Indeed, let X be the space-time Brownian motion on $R^1 \times R^d$ and let $B = \{(t_0, x),\ x \in R^d\}$. Then $P^0(X_{t_0} \in B) = 1$, but B is semipolar.

In § 2 we shall prepare some notations and several lemmas. The proof of Theorem and Corollary will be given in the subsequenct sections.

§ 2. Preliminaries

Throughout this section we assume that the Lévy process X has a λ-resolvent density $U^\lambda(x)$, that is,

$$\int_0^\infty \exp(-\lambda t) P^0(X_t \in dx) dt = U^\lambda(x) dx .$$

But *we do not assume the existence of a transition probability density.* So all the results in this section hold for the space-time process Y over X, if X has a transition probability density. We note that U^λ is always chosen to be *lower semicontinuous.* See Hawkes [4]. The convolution operation is written as "$*$". The symbol "$\sim$" is used to denote the reflection, that is, $\tilde{\mu}(dy) = \mu(-dy)$, $\tilde{f}(x) = f(-x)$. The symmetrized λ-resolvent density is written as U_S^λ:

$$U_S^\lambda(x) = \{U^\lambda(x) + U^\lambda(-x)\}/2$$

Then

$$\mathscr{F}(U_S^\lambda)(z) = \mathrm{Re}([\lambda + \Psi(z)]^{-1}) ,$$

where Ψ is the exponent of X.

The celebrated theorem of Bochner plays an important role in the proof of Theorem. So we repeat it here:

Let f be bounded in a neighborhood of the origin and belong to L^1. If $\mathscr{F}(f)$ is nonnegative, then $\mathscr{F}(f)$ belong to L^1 and $f = \mathscr{F}^{-1}(\mathscr{F}(f))$ almost surely.

Applying this theorem to our case, we have

LEMMA 2.1. *The λ-energy $E_X^\lambda(\mu)$ of a measure μ for X is finite if and only if $U_S^\lambda * \mu * \tilde{\mu}$ is bounded. If $E_X^\lambda(\mu)$ is finite, then*

$$U_S^\lambda * \mu * \tilde{\mu} = \mathscr{F}^{-1}[\mathrm{Re}([\lambda + \Psi]^{-1})|\mathscr{F}\mu|^2]$$

almost everywhere, and so

$$U_S^\lambda * \mu * \tilde{\mu}(0) \leq (2\pi)^{-d} E_X^\lambda(\mu) .$$

The last inequality follows from the lower semicontinuity of $U_S^\lambda * \mu * \tilde{\mu}$ and the continuity of the right-hand side of the equality. Using this lemma we can prove

COROLLARY OF LEMMA 2.1. *If $E_X^\lambda(\mu)$ is finite, then $E_X^\lambda(\mu)$ is monotone decreasing as λ increases. If $\mu = \mu_1 + \mu_2$, where μ_i, $i = 1, 2$, are measures. then $E_X^\lambda(\mu) \geq E_X^\lambda(\mu_i)$, $i = 1, 2$.*

The first assertion follows from the monotone decreasingness of $U_S^\lambda * \mu * \tilde{\mu}(x)$ in λ for every fixed x. The second statement follows from the inequality $U_S^\lambda * \mu * \tilde{\mu}(x) \geq U_S^\lambda * \mu_i * \tilde{\mu}_i(x)$ for every x.

Let $C^\lambda(K)$ be the *λ-capacity of a Borel set K*, that is, the total mass of the uniquely determined measure π on the closure of K such that $\tilde{U}^\lambda * \pi(x) = E^x(\exp(-\lambda T_K))$, where $T_K = \inf(t > 0, X_t \in K)$. The following lemma is proved essentially by Kanda [5] and Hawkes [4] without explicit mentioning. The explicit statement (proved from a very different point of view) is given by Rao.

LEMMA 2.2 Rao ([11]). *Let K be a compact set and ν be a bounded measure on K. Then*

$$E_X^\lambda(\nu) \geq (2\pi)^d |\nu(K)|^2 / 2C^\lambda(K) .$$

We say that a Borel set B is *thin* if $E^x(\exp(-\lambda T_B)) < 1$ for every $x \in R^d$. The set B is *semipolar* if B is a countable union of thin sets. The set B is called *polar* if $E^x(\exp(-\lambda T_B)) = 0$ for every x. Then we can give a characterization of polar sets using λ-energy.

LEMMA 2.3 (Kanda [6], Hawkes [4] and Rao [11]). *A Borel set B is non-polar if and only if there exists a bounded measure whose support is in B with finite λ-energy for X.*

The next lemmas show some peculiarity for sets which are non-polar but semipolar.

LEMMA 2.4 (Kanda [6], Rao [11]). *Let K be a compact set such that $K \subset \{x; E^x(\exp(-\lambda T_K)) < \delta\}$ for some $\delta < 1$. Then $C^\lambda(K) \uparrow C$ as $\lambda \uparrow \infty$ for some finite constant C.*

LEMMA 2.5 (Kanda [8], Fitzsimmons [3]). *Let K be a closed set such that $K \subset \{x; E^x(\exp(-\lambda T_K)) \leq \delta, \hat{E}^x(\exp(-\lambda \hat{T}_K)) \leq \delta\}$ for some $\delta < 1$. Then a subset B of K is polar if and only if $\pi(B) = 0$, where π is the λ-capacitary* *measure of K for X, that is, the uniquely determined measure on K such that $\tilde{U}^{\lambda} * \pi(x) = E^{x}(\exp(-\lambda T_K))$.*

In the above we used the dual process of X with the symbol "$\wedge$" attached. But recently Fitzsimmons noted that $K \subset \{x; E^{x}(\exp(-\lambda T_K)) \leq \delta\}$ is sufficient for the statement [3].

The following lemma gives a relation between a measure which does not charge semipolar sets and its energy.

LEMMA 2.6 (Rao [12], Kanda [7]). *If ν is a bounded measure which charges no semipolar sets and $E_X^{\lambda}(\nu) < \infty$, then $E_X^{\lambda}(\nu) \downarrow 0$ as $\lambda \uparrow \infty$.*

Finally we give a lemma which is essential in the proof of (II) of Theorem.

LEMMA 2.7 (Zabczyk [14]). *Let U be a real function on R^d of class L^1. Then there exists a singular measure η (relative to the Lebesgue measure) such that $U * \eta$ equals a continuous function on R^d except on a set of Lebesgue measure 0.*

§ 3. Proof of Theorem (I)

In the subsequent sections, the process X is a Lévy process on R^d with the exponent Ψ which has a transition probability density. Hence the space-time process Y over X is a Lévy process on $R^1 \times R^d$ with the λ-resolvent density $W^{\lambda}(t, x)$ as is explained in § 1. We denote by $\mathscr{F}$ the Fourier transform on $R^1 \times R^d$. We add the suffixes x and t for the Fourier transforms on the variable x of R^d and on the variable t of R^1, respectively. Thus

$$\mathscr{F}_x(U_S^{\lambda} * \nu * \tilde{\nu})(z) = \mathrm{Re}([\lambda + \Psi(z)]^{-1}|\mathscr{F}_x\nu(z)|^2 ,$$
$$\mathscr{F}(W_S^{\lambda} * \mu * \mu)(t, z) = \mathrm{Re}([\lambda + \Psi(z) - it]^{-1})|\mathscr{F}\mu(t, s)|^2 .$$

In what follows, we assume for simplicity that

$$\mu \text{ is a probability measure on } R^1 \times R^d .$$

Then μ is disintegrated as

$$\mu(dsdx) = \mu_2(dx)\mu_1(ds, x) ,$$

where $\mu_2(dx)(= \mu(R^1 \times dx)$, *the R^d-marginal of μ*) and $\mu_1(ds, x)$ are probability measures on R^d and R^1, respectively.

Proof of i) *of the part* (I). Set

$$f(t, x) = \mathscr{F}_t(\mu_1(\circ, x))(t).$$

Then $\mathscr{F}(\mu)(t, z) = \mathscr{F}_x(f(t, x)\mu_2(dx))(z)$. By the assumption, the λ-energy of μ for Y is finite. So $\int \mathrm{Re}([\lambda + \Psi(z) - it]^{-1})|\mathscr{F}(\mu)(t, z)|^2 dz < \infty$ for almost all t. Since $E_X^\lambda(f(t, x)\mu_2(dx)) = \int \mathrm{Re}([\lambda + \Psi(z)]^{-1})|\mathscr{F}(\mu)(t, z)|^2 dz$, it follows from the estimate $\mathrm{Re}([\lambda + \Psi(z) - it]^{-1}) \geq C\mathrm{Re}([\lambda + \Psi(z)]^{-1})$ for every z, where C is a positive constant (independent of z but dependent on t), that $E_X^\lambda(f(t, x)\mu_2(dx)) < \infty$ for almost all t. But

$$|\mathscr{F}_x(f(t, x)\mu_2(dx))(z)|^2 = G_1(t, z) + G_2(t, z),$$

where $G_1(t, z) = |\mathscr{F}_x(\mathrm{Re} f(t, x)\mu_2(dx))(z)|^2 + |\mathscr{F}_x(\mathrm{Im} f(t, x)\mu_2(dx))(z)|^2$ and

$$\begin{aligned} G_2(t, z) = 2\int \cos\langle z, x\rangle \mathrm{Im} f(t, x)\mu_2(dx) \int \sin\langle z, x\rangle \mathrm{Re} f(t, x)\mu_2(dx) \\ - 2\int \cos\langle z, x\rangle \mathrm{Re} f(t, x)\mu_2(dx) \int \sin\langle z, x\rangle \mathrm{Im} f(t, x)\mu_2(dx). \end{aligned}$$

Since $\mathrm{Re}([\lambda + \Psi(z)]^{-1}) = \mathrm{Re}([\lambda + \Psi(-z)]^{-1})$, $G_1(t, z) = G_1(t, -z)$ and $G_2(t, z) = -G_2(t, -z)$, we have

$$\begin{aligned} &\int_{|z|>R} \mathrm{Re}([\lambda + \Psi(z)]^{-1}) G_1(t, z)\, dz \\ &= \int_{|z|<R} \mathrm{Re}([\lambda + \Psi(z)]^{-1})[G_1(t, z) + G_2(t, z)]\, dz \leq E_X^\lambda(f(t, x)\mu_2(dx)) < \infty \end{aligned}$$

for every R. Thus $E_X^\lambda(\mathrm{Re} f(t, x)\mu_2(dx)) < \infty$. Now note that, by compactness of the support of the measure μ, there exist constants $c > 0$ and $\varepsilon > 0$ such that $\mathrm{Re} f(t, x) > c$ for every $|t| < \varepsilon$ and every x. Hence, using Corollary of Lemma 2.1, we see $E_X^\lambda(\mu_2) < \infty$. The proof of i) is finished.

Proof of ii) *of the part* (I). Assume that the R^1-marginal μ_1 of μ is singular to the Lebesgue measure (we choose a set E of Lebesgue measure 0 such that $\mu_1(R^1 - E) = 0$). Suppose that R^d-marginal μ_2 of μ charges a semipolar set. Then there exist a constant δ, $0 < \delta < 1$, and a compact set B such that $B \subset \{x; E^x(\exp(-\lambda T_B) \leq \delta, \hat{E}^x(\exp(\exp(-\lambda \hat{T}_B) \leq \delta\}$ and $\mu_2(B) > 0$. Note that B is non-polar for X. Indeed, for the restriction $\mu_2|_B$ of μ_2 to the set B, $E_X^\lambda(\mu_2|_B) < \infty$ by $E_X^\lambda(\mu_2) < \infty$ and by Corollary of Lemma 2.1. So B must be non-polar by Lemma 2.3. Let π_B be the λ-capacitary measure of the set B for X. Then $dt \otimes \pi_B$ is the λ-capacitary measure of the set $R^1 \times B$ for the space-time process Y over X. Indeed,

$$\iint W^\lambda(t-s, y-x)\,dt\pi_B(dy) = \int U^\lambda(y-x)\pi_B(dy)$$
$$= E^x(\exp(-\lambda T_B))$$
$$= E^{t,x}(\exp(-\lambda T_{R^1\times B}))\,,$$

where $T_{R^1\times B} = \inf(t>0,\ Y_t \in R^1 \times B)$. Clearly $(dt\otimes\pi_B)(E\times B)=0$. So, applying Lemma 2.5 for Y, the set $E\times B$ must be polar for Y. But, disintegrating μ as $\mu_1(ds)\mu_2(s, dx)$,

$$\mu(E\times B) = \iint_{E\times B} \mu_1(ds)\mu_2(s, dx)$$
$$= \iint_{R^1\times B} \mu_1(ds)\mu_2(s, dx) = \mu(R^1\times B) = \mu_2(B) > 0\,.$$

Since the λ-energy of μ for Y is finite by the assumption, the set $E\times B$ must be non-polar for Y by Lemma 2.3. Thus the R^d-marginal μ_2 does not charge a semipolar set. The proof of ii) is finished.

§ 4. Proof of Theorem (II)

We use the same symbols as in § 3. In the case of $\mu = \eta\otimes\nu$, $\mu_1(dt) = \eta(dt) = \mu_1(dt, x)$, $\mu_2(dx) = \nu(dx) = \mu_2(t, dx)$ and so

$$E_X^\lambda(\mu) = \iint \mathrm{Re}([\lambda + \Psi(z) - it]^{-1})|\mathscr{F}_t\eta(t)|^2|\mathscr{F}_x\nu(z)|^2\,dtdz\,.$$

Proof of i) *of the part* (II). First note that $E_X^\lambda(\nu) < \infty$ follows from $E_Y^\lambda(\mu) < \infty$ by i) of (I). If ν charges a semipolar set, then charges a compact set K such that $K\subset\{x;\ E^x(\exp(-\lambda T_K)) < \delta\}$ for some $\delta < 1$. Let ν_K be the restriction of ν to the set K. Then $E_X^\lambda(\nu_K) \le E_X^\lambda(\nu) < \infty$ by Corollary of Lemma 2.1, and therefore K must be non-polar for X by Lemma 2.3. So $C^\lambda(K)\uparrow C$ as $\lambda\uparrow\infty$ for some positive finite constant C by Lemma 2.4. Then it follows from Lemma 2.2 that

$$\lim_{\lambda\uparrow\infty} E_X^\lambda(\nu) \ge \lim_{\lambda\uparrow\infty} E_X^\lambda(\nu_K) \ge (2\pi)^d\nu(K)^2/2C\,.$$

Thus we have

$$\liminf_{\lambda\uparrow\infty}\int \mathrm{Re}([\lambda + \Psi(z) - it]^{-1})|\mathscr{F}\nu(z)|^2dz \ge (2\pi)^d\nu(K)^2/2C$$

for every fixed t. Hence

$$\lim_{\lambda\uparrow\infty} E_Y^\lambda(\mu) \ge \int |\mathscr{F}_t\eta(t)|^2dt(2\pi)^d\nu(K)^2/2C\,.$$

So $\mathscr{F}_t\eta$ belongs to $L^2(R^1)$, which implies that η is absolutely continuous and that the density belongs to $L^2(R^1)$. The proof of i) of the part (II) is finished.

Proof of ii) *of the part* (II). Let ν be a bounded measure with finite λ-energy for X. Assume that the measure ν does not charge any semipolar set. Then, by Lemma 2.6,

$$E_X^\lambda(\nu) \downarrow 0 \quad \text{as} \quad \lambda \uparrow \infty . \tag{4.1}$$

Set

$$g_\lambda(t, x) = \int W_S^\lambda(t, y - x)\nu * \tilde{\nu}(dy) .$$

Then

$$\int_{-\infty}^{\infty} g_\lambda(t, x)dt = U_S^\lambda * \nu * \tilde{\nu}(x) .$$

Since $U_S^\lambda * \nu * \tilde{\nu}$ is bounded by Lemma 2.1, $g_\lambda(t, 0)$ is L^1 in t. So it follows from Lemma 2.7 that there exists a bounded singular measure η on R^1 (we may suppose its support is compact) such that $g_\lambda(\cdot, 0) *_{(t)} \eta$ equals a continuous function on R^1, a.e., and therefore $g_\lambda(\cdot, 0) *_{(t)} \eta$ is locally bounded because of its lower semicontinuity. Hence $g_\lambda(\cdot, 0) *_{(t)} \eta *_{(t)} \tilde{\eta}$ is locally bounded in t. Clearly it belongs to $L^1(R^1)$. Further, for every t,

$$\mathscr{F}_t(g_\lambda(\cdot, 0))(t) = [\mathscr{F}_t(W_S^\lambda(\cdot, x))(t) *_{(x)} \nu *_{(x)} \tilde{\nu}](0)$$

by Fubuni's theorem. (In the above we denote by $*_{(t)}$ and $*_{(x)}$ the convolution operation in t and x respectively.) On the other hand, since

$$\mathscr{F}(W_S^\lambda)(t, z) = \mathscr{F}_x[\mathscr{F}_t(W_S^\lambda(\cdot, x))(t)](z) = \mathrm{Re}([\lambda + \Psi(z) - it]^{-1}) ,$$

we have, for each fixed t,

$$\mathscr{F}_x[\mathscr{F}_t(W_S^\lambda(\cdot, x))(t) *_{(x)} \nu *_{(x)} \tilde{\nu}](z) = \mathrm{Re}([\lambda + \Psi(z) - it]^{-1})|\mathscr{F}_x\nu(z)|^2 \geq 0 .$$

Hence it follows from Bochner's theorem that, for each fixed t,

$$\begin{aligned}(\mathscr{F}_t(W_S^\lambda(\cdot, \cdot\cdot))(t) *_{(x)}\nu *_{(x)}\tilde{\nu})(x) \\ = \mathscr{F}_x^{-1}[\mathrm{Re}([\lambda + \Psi(\cdot) - it]^{-1})|\mathscr{F}_x\nu(\cdot)|^2](x)\end{aligned} \tag{4.2}$$

for almost all x. In general the equality does not hold for all x. In the following we shall show the equality holds for $x = 0$ (hence it holds everywhere) by the use of (4.1). Since $\mathscr{F}_t(g_\lambda(\cdot, 0))(t) = (\mathscr{F}_t(W_S^\lambda(\cdot, \cdot\cdot))(t) *_{(x)} \nu *_{(x)} \tilde{\nu})(0)$, we must show

$$(4.3)\qquad \mathscr{F}_t(g_\lambda(\cdot,0))(t)=(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)-it]^{-1})|\mathscr{F}_x\nu(z)|^2dz\,.$$

Define

$$V_t^\lambda(x)=\int \exp(itu)W^\lambda(u,x)du/2\,,\quad \hat{V}_t^\lambda(x)=\int \exp(itu)W^\lambda(-u,-x)du/2\,.$$

Then it is easily proved that

$$V_t^\lambda(z)-V_t^{\lambda'}(z)=2(\lambda'-\lambda)\int V_t^\lambda(y)V_t^{\lambda'}(z-y)dy\,.$$

The same equality is also valid for $\hat{V}_t^\lambda$. Setting $H^\lambda(t,z)=((V_t^\lambda+\hat{V}_t^\lambda)*\nu*\tilde{\nu})(z)$, we have

$$H^\lambda(t,z)-H^{\lambda'}(t,z)=2(\lambda'-\lambda)\int V_t^\lambda(x+z)\Big[\int V_t^{\lambda'}(y-x)\nu*\tilde{\nu}(dy)\Big]dx$$
$$+2(\lambda'-\lambda)\int \hat{V}_t^\lambda(x+z)\Big[\int \hat{V}_t^{\lambda'}(y-x)\nu*\tilde{\nu}(dy)\Big]dx\,.$$

Since $\int V_t^\lambda(y-x)\nu*\tilde{\nu}(dy)$ and $\int \hat{V}_t^{\lambda'}(y-x)\nu*\tilde{\nu}(dy)$ are bounded measurable, each term of the right side is a continuous function of z, and so $H^\lambda(t,z)-H^{\lambda'}(t,z)$ is continuous. Since $H^\lambda(t,z)=(\mathscr{F}_t(W_S^\lambda(\cdot,x))(t)*_{(x)}\nu*_{(x)}\tilde{\nu})(z)$, it follows from (4.2) that

$$\begin{aligned}(\mathscr{F}_t(W_S^\lambda(\cdot,x))(t)*_{(x)}\nu*_{(x)}\tilde{\nu})(z)&-(\mathscr{F}_t(W_S^{\lambda'}(\cdot,x))(t)*_{(x)}\nu*_{(x)}\tilde{\nu})(z)\\ =\mathscr{F}_x^{-1}[\mathrm{Re}([\lambda+\Psi(\cdot)-it]^{-1})&|\mathscr{F}_x\nu(\cdot)|^2](z)\\ &-\mathscr{F}_x^{-1}[\mathrm{Re}([\lambda'+\Psi(\cdot)-it]^{-1})|\mathscr{F}_x\nu(\cdot)|^2](z)\end{aligned}$$

for every z. In particular, putting $z=0$ and letting $\lambda'\uparrow\infty$, we have

$$\mathscr{F}_t(g_\lambda(\cdot,0))(t)=(2\pi)^{-d}\int \mathrm{Re}[(\lambda+\Psi(z)-it]^{-1})|\mathscr{F}_x\nu(z)|^2dz$$
$$-\lim_{\lambda'\uparrow\infty}(2\pi)^{-d}\int \mathrm{Re}([\lambda'+\Psi(z)-it]^{-1})|\mathscr{F}_x\nu(z)|^2dz\,.$$

But it follows from (4.1) that the last term in the above equality is zero. Thus the equality (4.3) is proved. Finally we shall prove that the λ-energy of $\mu=\eta\otimes\nu$ for Y is finite. Since

$$\begin{aligned}\mathscr{F}_t(g(\cdot,0)*_{(t)}\eta*_{(t)}\tilde{\eta})(t)&=\mathscr{F}_t(g_\lambda(\cdot,0))(t)|\mathscr{F}_t\eta(t)|^2\\ &=(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)-it]^{-1})|\mathscr{F}_x\nu(z)|^2dz|\mathscr{F}_t\eta(t)|^2\geq 0\end{aligned}$$

by (4.3), Bochner's theorem ensures that

$$\int \mathrm{Re}([\lambda + \Psi(z) - it]^{-1})|\mathscr{F}_x\nu(z)|^2 dz|\mathscr{F}_t\eta(t)|^2$$

belongs to $L^1(R^1)$ as a function of t, which implies $E_Y^\lambda(\eta\otimes\nu) < \infty$. The proof of ii) of the part (II) is now finished.

§ 5. Proof of Corollary

First we shall prove the "only if" part. Assume that the set B is semipolar for X. If B is polar, the assertion is trivial. So we assume that B is non-polar. If there exists a set A in $]0, \infty[$ of Lebesgue measure zero such that $P^x(X_t \in B$ for some $t \in A) > 0$ for some x. Then the product set $A \times B$ in $R^1 \times R^d$ is non-polar for the space-time process Y over X. So there exists a bounded measure μ whose support is compact and in $A \times B$ with finite λ-energy for Y by Lemma 2.3. Then the R^1-marginal μ_1 of μ is carried by A and the R^d-marginal μ_2 of μ is carried by B. This contradicts the statement ii) of the part (I) in Theorem.

Before proving the "if" part, we prepare

LEMMA 5.1. *Let B be a non-semipolar closed set. Then there exists a non-trivial bounded measure ν on B of compact support with finite λ-energy for X that charges no semipolar set. Indeed we can choose the restriction of the regular part (explained below) of the λ-capacitary measure of B for X to some compact subset of B as the measure ν.*

Proof. We can decompose any bounded measure μ as $\mu = \mu_1 + \mu_2 + \mu_3$ where μ_1 is carried by a polar Borel set, μ_2 is carried by a semipolar Borel set but charges no polar set and μ_3 charges no semipolar set. See Blumenthal and Getoor [1], p. 283. We say that μ_3 is the regular part of μ. We show that the regular part of the λ-capacitary measure π_B of B for X is non-trivial (i.e. $(\pi_B)_3 \neq 0$). Suppose, on the contrary, that the regular part is trivial. Since π_B charges no polar set, we have then $\pi_B = (\pi_B)_2$. Let E be a semipolar Borel subset of B for X such that $\pi_B(B - E) = 0$. Then E is a countable union of thin sets for X by definition. Let H be any compact subset of one of such thin sets satisfying $\pi_B(H) > 0$. Let μ and ν be the restrictions of π_B to B and $B - H$, respectively. Then $U^\lambda\mu$ is discontinuous at μ-almost all points by Pop-Stojanovic [10]. But $E^x(\exp(-\lambda T_B)) = \tilde{U}^\lambda * \pi_B(x) = \tilde{U}^\lambda * \mu(x) + \tilde{U} * \nu(x)$, and so $E^x(\exp(-\lambda T_B))$ is continuous at x if and only if both $\tilde{U}^\lambda * \mu$ and $\tilde{U}^\lambda * \nu$ are continuous at x, because the both are lower-semicontinuous. Since $E^x(\exp(-\lambda T_B))$

is continuous at every point of $B^r(=\{x; E^x(\exp(-\lambda T_B))=1\})$, we see $\mu(B^r)=0$. Therefore $\pi_B(B^r)=0$, because $\pi_B(B^r\cap H)=\mu(B^r)=0$ for every H and so $0=\pi_B(B^r\cap E)=\pi_B(B^r\cap B)=\pi_B(B^r)$. For the last equality we used the closedness of B. Setting $D=B-B^r$, we have then $\pi_B|_D$ (= the restriction of π_B to D) $\leq \pi_D$, where π_D is the λ-capacitary measure of D for X, because

$$\pi_B(S)=\lambda\int \hat{E}^x(\exp(-\lambda\hat{T}_B), \hat{X}_{\hat{T}_B}\in S)dx \leq \lambda\int \hat{E}^x(\exp(-\lambda\hat{T}_D), \hat{X}_{\hat{T}_D}\in S)dx$$
$$=\pi_D(S)$$

for $S\subset D$. So $E^x(\exp(-\lambda T_B))=\tilde{U}^\lambda * \pi_B|_D(x)\leq \tilde{U}^\lambda * \pi_D(x)=E^x(\exp(-\lambda T_D))$. Since $T_D\geq T_B$ almost surely, we have $P^x(T_B=T_D)=1$ for every x. But the set D is semipolar so that almost surely $X_t\in D$ for only countable many values of t. See Blumenthal and Getoor [1], p. 80. Then it follows from $D=B-B^r$ and $T_B=T_D$ almost surely that $X_t\in B$ for only countably many values of t almost surely. Hence the set B must be semipolar. See Sharpe [13], p. 281. This contradicts the assumption that B is non-semipolar.

Now we prove the "if" part of Corollary. Assume that B is non-semipolar for X. Then there exists a bounded measure ν on B of compact support with finite λ-energy for X which charges no semipolar set. For the measure ν, by ii) of the part (II) in Theorem, we can find a singular measure η on R^1 such that $\eta\otimes\nu$ has finite λ-energy for Y. Then the product set $E\times B$ is non-polar for Y by Lemma 2.3, where E is a set of Lebesgue measure zero such that $\eta(R^1-E)=0$. This implies $P^x(X_t\in B$ for some $t\in A)>0$ for some x and for some set $A\subset]0,\infty[$ of Lebesgue measure zero (which is indeed a translation of E). The proof of Corollary is finished.

Remark. If the process X satisfies Hunt's condition (H), that is, every semipolar set for X is polar for X, then *a set B is polar if and only if $P^x(X_t\in B$ for some $t\in A)=0$ for every x and every set $A\subset]0,\infty[$ of Lebesgue measure zero.*

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REFERENCES

[1] R. G. Blumenthal and R. K. Getoor, Markov Processes and Potential Theory, Academic Press, New York, 1968.

[2] S. Bochner, Harmonic Analysis and Theory of Probability, Berkeley: Univ. of California Press 1955.

[3] P. J. Fitzsimmons, Remarks on a paper of Kanda, (private communication), (1989).

[4] J. Hawkes, Potential theory of Lévy processes, Proc. London Math. Soc., (3) **38** (1979), 335–352.

[5] M. Kanda, Two theorems on capacity for Markov processes with stationary independent increments, Z. Wahrsch. Verw. Gebiete, **35** (1976), 159–165.

[6] ——, Characterizations of semipolar sets for processes with stationary independent increments, Z. Wahrsch. Verw. Gebiete, **42** (1978), 141–154.

[7] ——, Characterization of semipolar sets for Lévy processes by Fourier transform of measures, Surikaiseki Kenkyusho Kokyuroku, **502** (1983), 161–170.

[8] ——, A note on capacitary measures of semipolar sets, The Annals of Probability, (1) **17** (1989), 379–384.

[9] ——, A note on a characterization of semipolar sets, Abstracts, Singapore Prob. Conference (1989).

[10] Z. R. Pop-Stojanovic, Energy in Markov processes, Functional analysis, (II) (Dubrovnik 1985), Lecture Notes in Math., **1242** (1987), 374–395.

[11] M. Rao, On polar sets for Lévy processes, J. London Math. Soc., (2) **35** (1987), 569–576.

[12] ——, Hunt's hypothesis for Lévy processes, Proc. Amer. Math. Soc., (2) **104** (1988), 621–624.

[13] M. Sharpe, General Theory of Markov Processes, Academic Press, San Diego, 1988.

[14] J. Zabczyk, A note on semipolar sets for processes with independent increments, Winter School on Probability, Karpacz, Poland, Lecture Notes in Math., **472** (1975) 277–283.

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