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ANALYTIC CAPACITY FOR TWO SEGMENTS

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§ 1. Introduction

The analytic capacity $\gamma(E)$ of a compact set E in the complex plane $\mathbf{C}$ is defined by $\gamma(E) = \sup|f'(\infty)|$, where $-f'(\infty)$ is the $1/z$-coefficient of $f(\zeta)$ at infinity and the supremum is taken over all bounded analytic functions $f(\zeta)$ outside E with supremum norm less than or equal to 1. Analytic capacity $\gamma(\cdot)$ plays various important roles in the theory of bounded analytic functions.

It is known that $\gamma(E) \leq |E|$, where $|\cdot|$ is the (generalized) length (i.e., the 1-dimension Hausdorff measure [3, CHAP. III]) and that the inverse relation does not exist, in general. In fact, Vitushkin [14] constructs an example of a set with positive length but zero analytic capacity, and Garnett [3, p. 87] also points out that the planar Cantor set with ratio 1/4

$$E(1/4) = \bigcap_{n=0}^{\infty} E_n$$

satisfies the same property. Here E_0 is the unit square $[0, 1] \times [0, 1]$ and E_n is inductively defined from E_{n-1} with each square Q of E_{n-1} replaced by four squares with sides 4^{-n} in the four corners of Q. The set E_n is a union of 4^n squares with sides 4^{-n}, and the projections of these 4^n squares to the line $\mathscr{L}$: $y = x/2$ do not mutually overlap. Hence if we choose $\mathscr{L}$ as a new axis, then E_n seems like a discontinuous graph. From this point of view, the author [8, CHAP. III] defined cranks and studied their analytic capacities: Cranks are nothing but deformations of sets of Vitushkin-Garnett type, however, these discontinuous graphs simplify the computation of analytic capacity and enable us to construct various examples [8, Theorem F], [9]. Hence clarifying the geometric meaning of cranks is important and would be applicable to study analytic capacities of general sets. (Cranks are closely related to fractals (Mandelbrot [6]).)

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Here are simple cranks of degree 1:

$$\Gamma(1+iy) = [-1/2, 1/2] \cup (1 + iy + [-1/2, 1/2]) \qquad (y > 0).$$

This is a subclass of

$$\Gamma(z) = [-1/2, 1/2] \cup (z + [-1/2, 1/2]) \qquad (z \in \mathbf{C}),$$

where, in general, $(z + wE) = \{z + w\zeta;\ \zeta \in E\}$ $(z, w \in \mathbf{C};\ E \subset \mathbf{C})$. The purpose of this note is to study $\gamma(z) = \gamma(\Gamma(z))$ $(z \in \mathbf{C})$ and show a role of cranks $\Gamma(1+iy)$ $(y > 0)$ in an extremum problem.

In fluid dynamics, $\Gamma(z)$ is a model of biplane wing sections, and the study of flows obstructed by $\Gamma(z)$ is classical (Ferrari [1], Garrick [3]). As is well known, there exists uniquely an analytic function $f_z(\zeta)$ outside $\Gamma(z)$ such that

(1) $f_z(\zeta)$ is integrable on $\partial\Gamma(z)$[†] (with respect to the length element $|d\zeta|$), $f_z(\zeta)$ is real-valued continuous on $\partial\Gamma(z)$ and $f_z(\infty) = -i$,

(2) $|f_z(p)|$ exists at the right endpoint p of each component of $\Gamma(z)$ (Joukowksi's hypothesis).

Here $\partial\Gamma(z)$ is the subboundary of $\Gamma(z)^c$ which corresponds to $\Gamma(z)$-{endpoints of $\Gamma(z)$} topologically; $\partial\Gamma(z)$ has two sides. Condition (1) means that $f_z(\zeta)$ is a velocity field obstructed by $\Gamma(z)$ with velocity i at infinity, and (2) means that vortexes at endpoints of $\Gamma(z)$ are negligible. We define the lift coefficient for $\Gamma(z)$ by

$$\mathscr{L}(z) = \frac{1}{4}\left|\frac{1}{2\pi}\int_{\partial\Gamma(z)} f_z(\zeta)^2 d\zeta\right| \left(= \frac{1}{2}|f_z'(\infty)|\right).$$

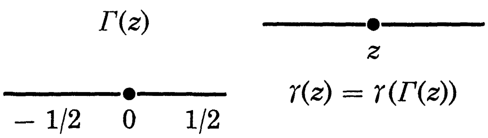
Using Blasius' theorem [7, p. 173], Kutta-Joukowski shows that $4\pi\mathscr{L}(z)\sin\alpha$ gives the lift for $\Gamma(z)$ with respect to the velocity field with density 1 and velocity $e^{i\alpha}$ at infinity $(0 \le \alpha \le 2\pi)$ (cf. [7, CHAP. VII], [3]). In the section 2, we shall give a formula for $\gamma(z)$ in terms of $\mathscr{L}(z)$ and shall show that $\mathscr{L}(z) \le \gamma(z)$ (Theorems 1 and 2). To compute $\gamma(z)$ practically, it is necessary to study the so-called modulus-invariant arcs. In the section 2, we shall show two lemmas (with respect to modulus-invariant arcs) which will be used later. Using our formula along modulus-invariant arcs, we shall show, in the section 4, that the behaviour of $\gamma(z)$ near 1 is critical (Theorem 8). In the section 5, we shall show that

$$\sigma_0 = \min_{y \ge 0} \gamma(1+iy)/\gamma(1),$$

where σ_0 is defined by the infimum of $\gamma(x+iy)/\gamma(x)$ over all real numbers

[†] The condition "$\lim_{\varepsilon\downarrow 0}\int_{|\zeta-p|=\varepsilon}|f_z||d\zeta|=0$ $(p=\pm 1/2,\ z\pm 1/2)$" is required.

x and y (Theorem 13). Since $\gamma(z) = 1/2$, $2\sigma_0$ equals the minimum of analytic capacities of cranks $\Gamma(1+iy)$ $(y>0)$. This shows that the computation of $\gamma(1+iy)$ $(y>0)$ is essential in this extremum problem. We shall also show a practical method to estimate σ_0. Theorem 13 suggests that $E(1/4)$ is an extreme in a sense. Our method works for unions of two segments with different length, however, this is not applicable to unions of three segments.



§ 2. A formula for $\gamma(z)$

In this section, we give a formula for $\gamma(z)$ $(z \in \mathbf{C})$. Without loss of generality, we may assume that z is contained in $P = \{\zeta \in \mathbf{C};\ \operatorname{Re}\zeta \geq 0, \operatorname{Im}\zeta \geq 0\}$, where $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ are the real part and the imaginary part of ζ, respectively. A domain $\Gamma(z)^c$ is univalently mapped onto a ring $\{\zeta \in \mathbf{C};\ r < |\zeta| < r'\}$. The modulus of $\Gamma(z)^c$ is defined by $\operatorname{mod}(\Gamma(z)^c) = r'/r$ [12, p. 199]. An arc λ in P is called modulus-invariant, if $\operatorname{mod}(\Gamma(z)^c)$ is a constant on λ. For $z \in P$, $\operatorname{Im} z > 0$, $\lambda(z)$ denotes the modulus-invariant arc in P with endpoints z and a real number; this real number is uniquely determined by z and larger than 1. In this section, we show the following two theorems.

THEOREM 1. *For* $z \in P$, $\operatorname{Im} z > 0$,

$$\gamma(z) = \frac{1}{2} + \frac{\operatorname{Im} z}{2} \int_{\lambda(z)} \left\{ \frac{\gamma(\zeta)}{\mathscr{L}(\zeta)} - 1 \right\} \frac{d(\operatorname{Im}\zeta)}{(\operatorname{Im}\zeta)^2}, \tag{3}$$

where z is chosen as the initial point of this curvilinear integral.

THEOREM 2. $\mathscr{L}(z) \leq \gamma(z)$ $(z \in P)$. *Equality holds if and only if z is real.*

Since z is the initial point of the integral in (3), Theorems 1 and 2 show that $\gamma(z) < 1/2$ $(z \in P, \operatorname{Im} z > 0)$. Here are some lemmas necessary for the proof. The following lemma is a version of biplane theory to analytic capacity (Ferrari [1], Garrick [3], Sasaki [13, pp. 208–213]).

LEMMA 3. *For* $0 < k < 1$ *and* $t \geq 0$, *we define*

$$(4)\qquad \xi_k(t) = \left[\frac{2m_k^2 + (1+k^2)t^2 - \sqrt{\{2m_k^2 + (1+k^2)t^2\}^2 - 4(1+k^2t^2)(m_k^4 + t^2)}}{2(1+k^2t^2)}\right]^{1/2},$$

$$(5)\qquad \eta_k(t) = \left[\frac{2m_k^2 + (1+k^2)t^2 + \sqrt{\{2m_k^2 + (1+k^2)t^2\}^2 - 4(1+k^2t^2)(m_k^4 + t^2)}}{2(1+k^2t^2)}\right]^{1/2},$$

$$l_k(t) = \tau_k + \int_0^t \{\eta_k(s) - \xi_k(s)\}ds,$$

where

$$m_k = \frac{1}{k}\sqrt{\frac{E(k')}{K(k')}}, \qquad \tau_k = 2\int_1^{m_k} \frac{m_k^2 - s^2}{\sqrt{s^2-1}\sqrt{1-k^2s^2}}ds,$$

$$E(k') = \int_0^1 \sqrt{\frac{1-k'^2s^2}{1-s^2}}ds, \quad K(k') = \int_0^1 \frac{ds}{\sqrt{1-s^2}\sqrt{1-k'^2s^2}}, \quad k' = \sqrt{1-k^2}.$$

Let

$$\begin{aligned} z_k(t) &= x_k(t) + iy_k(t) \\ &= 1 + \left\{-\tau_k + 2\int_0^t \xi_k(s)ds + \frac{i\pi}{k^2K(k')}\right\}/l_k(t). \end{aligned}$$

Then

$$(6)\qquad \gamma(z_k(t)) = \left\{\frac{1-k}{2k}\sqrt{t^2 + k^{-2}}\right\}/l_k(t).$$

Proof. Since this lemma plays an important role in the proof of Theorems 1 and 2, we give the proof of this lemma, for the sake of completeness. For $0 < k < 1$ and $t \geq 0$, we write $\xi = \xi_k(t)$ and $\eta = \eta_k(t)$. Take a Schwarz-Christoffel transformation

$$f(\zeta) = \int_0^\zeta \frac{s^2 - m_k^2}{\sqrt{s-1}\sqrt{s+1}\sqrt{ks-1}\sqrt{ks+1}}ds - it\zeta,$$

where we choose a branch of the square root so that the upper half plane is mapped to the positive orthant. Since

$$m_k^2 = \int_1^{1/k} \frac{s^2ds}{\sqrt{s^2-1}\sqrt{1-k^2s^2}} \Big/ \int_1^{1/k} \frac{ds}{\sqrt{s^2-1}\sqrt{1-k^2s^2}}.$$

$f(\zeta)$ univalently maps $\{[-1/k, -1] \cup [1, 1/k]\}^c$ onto $\{(-a + i[\alpha_-, \beta_-]) \cup (a + i[\alpha_+, \beta_+])\}^c$ for some $a > 0$, $\alpha_\pm < \beta_\pm$. (See [13, pp. 208–213].) Pommerenke [11] shows that $\gamma(E) = |E|/4$ if E is a compact set on the real line. Since

$$\lim_{\zeta\to\infty} f(\zeta)/\zeta = (1/k) - \mathrm{it}\,,$$

the conformal invariance of $\gamma(\cdot)$ and Pommerenke's theorem show that

$$\begin{aligned}&\gamma((-a+i[\alpha_-,\beta_-])\cup(a+i[\alpha_+,\beta_+]))\\&\quad=\left|\frac{1}{k}-it\right|\gamma([-1/k,-1]\cup[1,1/k])=\frac{1-k}{2k}\sqrt{t^2+k^{-2}}\,.\end{aligned}$$

Legendre's formula

$$E(k)K(k')+E(k')K(k)-K(k)K(k')=\pi/2 \qquad [4, \text{p. } 291]$$

shows that

$$2a = 2\,\mathrm{Re}\, f(1) = 2\int_0^1 \frac{m_k^2 - s^2}{\sqrt{1-s^2}\,\sqrt{1-k^2s^2}}\,ds = \frac{\pi}{k^2K(k')}\,.$$

Let

$$\psi_k(x) = \int_1^x \frac{m_k^2 - s^2}{\sqrt{s^2-1}\,\sqrt{1-k^2s^2}}\,ds \qquad (1\le x\le 1/k)\,.$$

Then (4) and (5) show that

$$1<\xi<m_k\,,\quad \psi_k'(\xi)=t\,;\quad m_k<\eta<1/k\,,\quad \psi_k'(\eta)=-t\,.$$

These inequalities yield that

$$\beta_+ = \psi_k(\xi) - t\xi\,,\quad \alpha_+ = -\psi_k(\eta) - t\eta\,,\quad \alpha_- = -\beta_+\,,$$

and hence

$$\begin{aligned}\beta_+ - \alpha_+ &= \psi_k(\eta) + \psi_k(\xi) + t(\eta-\xi)\,,\\ \alpha_- - \beta_+ &= 2t\xi - 2\psi_k(\xi)\,.\end{aligned}$$

Rotating, translating and normalizing $(-a+i[\alpha_-,\beta_-])\cup(a+i[\alpha_+,\beta_+])$, we obtain

$$\begin{aligned}\gamma(z_k^*(t)) &= \frac{1-k}{2k}\sqrt{t^2+k^{-2}}\,\frac{1}{\psi_k(\eta)+\psi_k(\xi)+t(\eta-\xi)}\,,\\ z_k^*(t) &= 1 + \frac{2t\xi - 2\psi_k(\xi) + i\pi/\{k^2K(k')\}}{\psi_k(\eta)+\psi_k(\xi)+t(\eta-\xi)}\,.\end{aligned}$$

Since

$$\frac{d}{dt}\{\psi_k(\xi_k(t)) - t\xi_k(t)\} = -\xi_k(t)\,,\qquad \psi_k(\xi_k(0)) = \tau_k/2\,,$$

we have

$$(7)\qquad \psi_k(\xi_k(t)) - t\xi_k(t) = \frac{\tau_k}{2} - \int_0^t \xi_k(s)ds\,.$$

In the same manner,

$$(8)\qquad \psi_k(\eta_k(t)) + t\eta_k(t) = \frac{\tau_k}{2} + \int_0^t \eta_k(s)ds\,.$$

Thus

$$(9)\qquad \psi_k(\eta_k(t)) + \psi_k(\xi_k(t)) + t\{\eta_k(t) - \xi_k(t)\} = l_k(t)\,, \qquad z_k^*(t) = z_k(t)\,,$$

which yields (6).

LEMMA 4 (the lift formula). *The function $\mathscr{L}(z)$ is continuous on P and*

$$(10)\qquad \mathscr{L}(z_k(t)) = \left\{kt + \frac{1}{kt}\right\}\frac{\eta_k(t) - \xi_k(t)}{2kl_k(t)} \qquad (0 < k < 1,\ t > 0)\,.$$

This lemma is known in fluid dynamics ([1], [3], [13, p. 213]). The outline of the proof is as follows. For $0 < k < 1$ and $t > 0$, let $f(\zeta)$ be the Schwarz-Christoffel transformation used in the proof of Lemma 3. Then $if(\zeta)$ univalently maps $\{[-1/k, -1]\cup[1, 1/k]\}^c$ onto a domain similar to $\Gamma(z_k(t))^c$, say R. For real numbers U, V, ρ, n, we take

$$\Omega(\zeta) = U\zeta - iV\int_0^\zeta \frac{s^2 - m_k^2}{\sqrt{s^2 - 1}\sqrt{k^2s^2 - 1}}ds - i\rho\int_0^\zeta \frac{s - n}{\sqrt{s^2 - 1}\sqrt{k^2s^2 - 1}}ds\,.$$

Then $\dfrac{d}{dw}\Omega(h(w))$ is an analytic function in R, where $h(w)$ is the inverse function of $if(\zeta)$. Using Joukowski's hypothesis and (the argument of $\dfrac{d}{dw}\Omega(h(\infty))) = -\pi/2$, we determine U, V, ρ, n. Translating and normalizing R, we obtain $f_{z_k(t)}(\zeta)$. Computing $f'_{z_k(t)}(\infty)$, we obtain (10).

LEMMA 5. $\dfrac{\tau_k}{2} = \displaystyle\int_0^\infty \left\{\frac{1}{k} - \eta_k(s)\right\}ds = \int_0^\infty \{\xi_k(s) - 1\}ds \qquad (0 < k < 1)\,.$

Proof. Since

$$\frac{1}{k} - \eta_k(t) = O(t^{-2})\,, \qquad \xi_k(s) - 1 = O(t^{-2}) \qquad (t \longrightarrow \infty)\,,$$

two integrals in the required equalities converge. Equality (8) shows that

$$\int_0^t \left\{\frac{1}{k} - \eta_k(s)\right\} ds = \frac{\tau_k}{2} - \psi_k(\eta_k(t)) + t\left\{\frac{1}{k} - \eta_k(t)\right\}.$$

Letting t tend to infinity, we obtain

$$\int_0^\infty \left\{\frac{1}{k} - \eta_k(s)\right\} ds = \frac{\tau_k}{2} - \psi_k(1/k) = \frac{\tau_k}{2}.$$

Thus the first equality holds. Analogously, (7) yields the second equality.

In order to prove Theorems 1 and 2, it is necessary to use the following property:

(11) To $z \in P$, $\operatorname{Im} z > 0$, there corresponds uniquely a pair (k, t) so that $z_k(t) = z$ and $\lambda(z) = \{z_k(s);\ s \geq t\} \cup \{(1+k)/(1-k)\}$.

This property will be shown in the next section. Here we give the proof of Theorems 1 and 2, assuming (11). First we give the proof of Theorem 1. For $z \in P$, $\operatorname{Im} z > 0$, let (k, t) be the pair in (11). Equality (10) shows that

$$\begin{aligned} y_k'(s) &= -\frac{\pi}{k^2 K(k')} \frac{l_k'(s)}{l_k(s)^2} = -\frac{\eta_k(s) - \xi_k(s)}{l_k(s)} y_k(s) \\ &= \frac{2k^2 s}{1 + k^2 s^2} \mathscr{L}(z_k(s)) y_k(s) \qquad (s > 0). \end{aligned}$$

Thus we have, by Lemmas 4, 5, (6) and (10),

$$\begin{aligned} \frac{\gamma(z) - 1/2}{\operatorname{Im} z} &= \frac{\gamma(z_k(t)) - 1/2}{y_k(t)} = \frac{k^2 K(k') l_k(t)}{2\pi}\{2\gamma(z_k(t)) - 1\} \\ &= \frac{k^2 K(k')}{2\pi}\left\{\frac{1-k}{k}\sqrt{t^2 + k^{-2}} - \tau_k - \int_0^t (\eta_k(s) - \xi_k(s)) ds\right\} \\ &= \frac{k^2 K(k')}{2\pi}\left[\frac{1-k}{k}\left\{\sqrt{t^2 + k^{-2}} - t\right\} - \tau_k + \int_0^t \left\{\frac{1}{k} - 1 - \eta_k(s) + \xi_k(s)\right\} ds\right] \\ &= \frac{k^2 K(k')}{2\pi}\left[\int_t^\infty \left\{\frac{1}{k} - 1 - \frac{(1-k)s}{\sqrt{1 + k^2 s^2}}\right\} ds - \int_t^\infty \left\{\frac{1}{k} - 1 - \eta_k(s) + \xi_k(s)\right\} ds\right] \\ &= -\frac{k^2 K(k')}{2\pi}\int_t^\infty \frac{2k^2 s}{1 + k^2 s^2}\left\{\frac{1-k}{2k}\sqrt{s^2 + k^{-2}} - \frac{1 + k^2 s^2}{2k^2 s}(\eta_k(s) - \xi_k(s))\right\} ds \\ &= -\frac{1}{2}\int_t^\infty \frac{2k^2 s}{1 + k^2 s^2}\{\gamma(z_k(s)) - \mathscr{L}(z_k(s))\}\frac{1}{y_k(s)} ds \\ &= \frac{1}{2}\int_t^\infty \frac{\gamma(z_k(s)) - \mathscr{L}(z_k(s))}{\mathscr{L}(z_k(s))}\frac{y_k'(s)}{y_k(s)^2} ds = \frac{1}{2}\int_{\lambda(z)} \left\{\frac{\gamma(\zeta)}{\mathscr{L}(\zeta)} - 1\right\}\frac{d(\operatorname{Im}\zeta)}{(\operatorname{Im}\zeta)^2}. \end{aligned}$$

This completes the proof of Theorem 1. Next we give the proof of Theorem 2. For $z \in P$, $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, let (k, t) be the pair in (11). We write $\xi = \xi_k(t)$ and $\eta = \eta_k(t)$. Equalities (4) and (5) show that

$$(\eta - \xi)^2 = \eta^2 + \xi^2 - 2\eta\xi$$
$$= \frac{1}{1 + k^2t^2}\{2m_k^2 + (1 + k^2)t^2 - 2\sqrt{(1 + k^2t^2)(m_k^4 + t^2)}\}.$$

Thus we have, by Lemmas 3 and 4,

$$(12) \quad \gamma(z) - \mathscr{L}(z) = \frac{\gamma(z)^2 - \mathscr{L}(z)^2}{\gamma(z) + \mathscr{L}(z)}$$
$$= \frac{1}{\{\gamma(z) + \mathscr{L}(z)\}l_k(t)^2}\left\{\frac{(1-k)^2}{4k^4}(1 + k^2t^2) - \frac{(1 + k^2t^2)^2}{4k^4t^2}(\eta - \xi)^2\right\}$$
$$= \frac{1 + k^2t^2}{4\{\gamma(z) + \mathscr{L}(z)\}l_k(t)^2k^4t^2}\{(1-k)^2t^2 - (1 + k^2t^2)(\eta - \xi)^2\}$$
$$= \frac{\gamma(z)^2}{\{\gamma(z) + \mathscr{L}(z)\}(1-k)^2t^2}$$
$$\times [(1-k)^2t^2 - \{2m_k^2 + (1 + k^2)t^2 - 2\sqrt{(1 + k^2t^2)(m_k^4 + t^2)}\}]$$
$$= \frac{2\gamma(z)^2}{\{\gamma(z) + \mathscr{L}(z)\}(1-k)^2t^2}\{\sqrt{(kt^2 + m_k^2)^2 + (km_k^2 - 1)^2t^2} - (kt^2 + m_k^2)\}.$$

A simple calculation shows that $km_k^2 > 1$. Thus $\mathscr{L}(z) < \gamma(z)$ ($z \in P$, $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$). If $\operatorname{Re} z = 0$ and $\operatorname{Im} z > 0$, then we have

$$(13) \qquad \gamma(z) - \mathscr{L}(z) = \frac{\gamma(z)^2(km_k^2 - 1)^2}{\{\gamma(z) + \mathscr{L}(z)\}(1-k)^2m_k^2} > 0,$$

by (12) and the continuity of $\gamma(z)$ and $\mathscr{L}(z)$. We now show that

$$(14) \qquad \gamma(z) \leq \mathscr{L}(z) + \frac{C}{\log(1/\operatorname{Im} z)} \qquad (z \in P,\ 0 < \operatorname{Im} z < 1/2)$$

for some absolute constant C. By (12), we have, with two absolute constants C_1 and C_2,

$$\gamma(z) - \mathscr{L}(z) \leq \frac{\gamma(z)^2(km_k^2 - 1)^2}{\{\gamma(z) + \mathscr{L}(z)\}(1-k)^2(kt^2 + m_k^2)} \leq \frac{(km_k^2 - 1)^2}{(1-k)^2(kt^2 + m_k^2)}$$
$$\leq \frac{k^2m_k^4}{(1-k)^2m_k^2} = \frac{E(k')}{(1-k)^2K(k')} \leq \frac{C_1}{(1-k)^2\log(1 + (1/k))}$$

and

$$\begin{aligned}\gamma(z)-\mathscr{L}(z)&\leq\frac{(km_k^2-1)^2}{(1-k)^2(kt^2+m_k^2)}=\frac{k(km_k^2-1)^2}{(1-k)^2(k^2t^2+km_k^2)}\\&\leq\frac{k^3m_k^4}{(1-k)^2(1+k^2t^2)}=\frac{m_k^4}{4\gamma(z)^2kl_k(t)^2}=\frac{k^3m_k^4K(k')^2y_k(t)^2}{4\pi^2\gamma(z)^2}\\&=\frac{E(k')^2(\operatorname{Im}z)^2}{4\pi^2\gamma(z)^2k}\leq C_2(\operatorname{Im}z)^2/k\,,\end{aligned}$$

where (k, t) is the pair associated with z. Thus

$$\gamma(z)-\mathscr{L}(z)\leq\min\left\{\frac{C_1}{(1-k)^2\log(1+(1/k))},\ C_2(\operatorname{Im}z)^2/k\right\}.$$

If $\operatorname{Im} z \leq k$, then $\gamma(z)-\mathscr{L}(z)\leq C_2 \operatorname{Im} z$. If $\operatorname{Im} z > k$, then

$$\gamma(z)-\mathscr{L}(z)\leq\frac{C_1}{(1-k)^2\log(1+(1/k))}\leq\frac{C_3}{\log(1/\operatorname{Im}z)}$$

for some absolute constant C_3, because of $0<\operatorname{Im} z<1/2$. Thus

$$\gamma(z)-\mathscr{L}(z)\leq\max\left\{\frac{C_3}{\log(1/\operatorname{Im}z)},\ C_2\operatorname{Im}z\right\},$$

which gives (14). Since $\gamma(z)$ and $\mathscr{L}(z)$ are continuous on P, (14) shows that the equality holds for real numbers z. This completes the proof of Theorem 2.

Inequality (13) yields that

$$\gamma(iy)-\mathscr{L}(iy)\geq C_4y\qquad(0<y<1/2)$$

for some absolute constant C_4. We do not know whether the order $\dfrac{1}{\log(1/\operatorname{Im}z)}$ in (14) is best possible or not.

§ 3. Modulus-invariant arcs

To compute $\gamma(z)$ practically, it is necessary to study modulus-invariant arcs. To use later, we prepare, in this section, the following two lemmas; (15) and (16) in Lemma 6 give (11) which was used in the proof of Theorems 1 and 2.

LEMMA 6.

(15) *$z_k(t)$ is a continuous homeomorphism from $Q=\{(k,t);\ 0<k<1,\ t\geq 0\}$ to $P-[0,\infty)$.*

(16) *For* $(k, t) \in Q$, $\lambda(z_k(t)) = \{z_k(s);\ s \geq t\} \cup \{(1+k)/(1-k)\}$.

(17) *For* $0 < k < 1$, $x_k(t)$ *is strictly increasing, and* $y_k(t)$ *is strictly decreasing with respect to* t.

LEMMA 7. *Let* $a \geq 0$. *Then, for any* k *satisfying* $k_a < k < 1$ ($k_a = \max\{(a-1)/(a+1), 0\}$), *there exists uniquely* $t_{a,k} > 0$ *such that* $x_k(t_{a,k}) = a$. *We have*

(18) $y_k(t_{a,k})$ *is continuous and strictly increasing with respect to* k.

(19) $\lim_{k \to k_a} y_k(t_{a,k}) = 0$.

(20) $a\tau_k = \int_0^{t_{a,k}} \{(1-a)\eta_k(s) + (1+a)\xi_k(s)\} ds$.

Proof of Lemma 6. For $0 < k < 1$, we have

$$\text{(21)} \qquad \begin{cases} x_k(0) = 0\,, & \lim_{t \to \infty} x_k(t) = \dfrac{1+k}{1-k}\,, \\ y_k(0) = \dfrac{\pi}{k^2 K(k')\tau_k}\,, & \lim_{t \to \infty} y_k(t) = 0\,. \end{cases}$$

In fact, (4) and (5) show that

$$\lim_{t \to \infty} \eta_k(t) = 1/k\,, \qquad \lim_{t \to \infty} \xi_k(t) = 1\,,$$

and hence

$$\begin{aligned} \lim_{t \to \infty} x_k(t) &= 1 + 2 \lim_{t \to \infty} \int_0^t \xi_k(s) ds \Big/ \int_0^t \{\eta_k(s) - \xi_k(s)\} ds \\ &= 1 + \frac{2}{(1/k) - 1} = \frac{1+k}{1-k}\,. \end{aligned}$$

The other three equalities in (21) are easily seen. We have

$$\text{(22)} \qquad \lim_{k \to 0} y_k(0) = 0\,, \qquad \lim_{k \to 1} x_k(1/k') = \lim_{k \to 1} y_k(1/k') = \infty\,.$$

In fact, we have

$$\begin{aligned} \lim_{k \to 0} k^2 \tau_k &= 2 \lim_{k \to 0} k^2 \int_1^{m_k} \frac{m_k^2 - s^2}{\sqrt{s^2 - 1}\sqrt{1 - k^2 s^2}} ds \\ &= 2 \lim_{k \to 0} k^2 m_k^2 \log m_k = \lim_{k \to 0} \frac{E(k')}{K(k')} \log \left\{ \frac{E(k')}{k^2 K(k')} \right\} = 2\,, \end{aligned}$$

which gives

$$\lim_{k\to 0} y_k(0) = \lim_{k\to 0} \frac{\pi}{k^2 K(k')\tau_k} = \frac{\pi}{2} \lim_{k\to 0} \frac{1}{K(k')} = 0\,.$$

Since $\lim_{k\to 1} m_k = 1$, we have, with $n_k = \sqrt{1 - k^2 m_k^2}/k'$,

$$\lim_{k\to 1} \tau_k = 2 \lim_{k\to 1} \left\{ m_k^2 \int_{n_k}^{1} \frac{ds}{\sqrt{1-s^2}\sqrt{1-k'^2 s^2}} - k^{-2} \int_{n_k}^{1} \sqrt{\frac{1-k'^2 s^2}{1-s^2}}\, ds \right\}$$
$$= 2 \lim_{k\to 1} (m_k^2 - k^{-2}) \int_{n_k}^{1} \frac{ds}{\sqrt{1-s^2}} = 0\,.$$

Recall that $\xi_k(s) > 1$, $0 < \eta_k(s) - \xi_k(s) < (1/k) - 1$. We have

$$\liminf_{k\to 1} x_k(1/k') = 1 + \liminf_{k\to 1} 2 \int_0^{1/k'} \xi_k(s) ds \Big/ \int_0^{1/k'} \{\eta_k(s) - \xi_k(s)\} ds$$
$$\geq 1 + \liminf_{k\to 1} \frac{2}{(1/k) - 1} = \infty$$

and

$$\liminf_{k\to 1} y_k(1/k') = \liminf_{k\to 1} \pi \Big/ \left\{ k^2 K(k') \int_0^{1/k'} (\eta_k(s) - \xi_k(s)) ds \right\}$$
$$= \liminf_{k\to 1} \frac{2k'}{(1/k) - 1} = \infty\,.$$

Thus (22) holds.

Since

$$l'_k(t) = \eta_k(t) - \xi_k(t) > 0\,,$$

$l_k(t)$ is strictly increasing, and hence $y_k(t)$ is strictly decreasing. Recall (7) and (9). Since

$$x_k(t) = 1 + \frac{2}{l_k(t)} \{ - \psi_k(\xi_k(t)) + t\xi_k(t) \}\,,$$

we have, with $\xi = \xi_k(t)$ and $\eta = \eta_k(t)$,

$$x'_k(t) = \frac{2}{l_k(t)^2} \{ \xi l_k(t) - (-\psi_k(\xi) + t\xi)(\eta - \xi) \}$$
$$= \frac{2}{l_k(t)^2} \{ \xi \psi_k(\eta) + \eta \psi_k(\xi) \}\,.$$

Since $\psi'_k(t) > 0$ $(1 < t < m_k)$, we have $\psi_k(\xi) > 0$. Since $\psi'_k(t) < 0$ $(m_k < t < 1/k)$, we have $\psi_k(\eta) > \psi_k(1/k) = 0$. Consequently, $x'_k(t) > 0$. Thus (17) holds. Inequalities (21) show that $\lim_{t\to\infty} z_k(t) = (1 + k)/(1 - k)$. Thus (17) yields (16). Let W_k be the compact set bounded by the x, y axes and $\lambda(iy_k(0))$. Then (16) and (17) show that

$$W_k \subset \left\{x + iy;\ 0 \leq x \leq \frac{1+k}{1-k},\ 0 \leq y \leq y_k(0)\right\},$$
$$W_k \supset \{x + iy;\ 0 \leq x \leq x_k(1/k'),\ 0 \leq y \leq y_k(1/k')\},$$

and hence, by (22),

$$\bigcap_{0<k<1} W_k = [0, 1], \qquad \bigcup_{0<k<1} W_k = P.$$

This shows that $z_k(t)$ is an onto mapping from Q to $P - [0, \infty)$. Recall that $\lambda(iy_k(0))$ is a modulus-invariant arc with modulus $\mathrm{mod}(\{[-1/k, -1] \cup [1, 1/k]\}^c)$. The domain $\{[-1/k, -1] \cup [1, 1/k]\}^c$ is univalently mapped onto a Grötzsch's domain $G_{p_k} = \{z \in \mathbf{C};\ |z| > 1\} - [p_k, \infty)$ with

$$p_k = 1 + \frac{8k}{(1-k)^2}\left\{1 + \frac{1+k}{2\sqrt{k}}\right\}.$$

Since $\mathrm{mod}(G_p)$ is strictly increasing with respect to p [5, p. 72] and p_k, $(1+k)/(1-k)$ $(= \lim_{t\to\infty} z_k(t))$ are strictly increasing with respect to k, we have

$$W_k \subset W_{k'}, \quad W_k \cap \lambda(iy_{k'}(0)) = \varnothing \quad (k < k'). \tag{23}$$

Notice that $z_k(t)$ is continuous on Q (with respect to (k, t)). Since $(1+k)/(1-k)$ $(= \lim_{t\to\infty} z_k(t))$ is continuous with respect to k, we have $\bigcap_{k<\mu<1} W_\mu = W_k$. Thus (15) holds. This completes the proof of Lemma 6.

Proof of Lemma 7. Let $\mu(a) = \{\zeta \in \mathbf{C};\ \mathrm{Re}\,\zeta = a\}$ $(a \geq 0)$. Then Lemma 6 shows that

$$\mu(a) \cap \lambda(iy_k(0)) = \varnothing \qquad (0 < k < k_a),$$
$$\mu(a) \cap \lambda(iy_k(0)) \text{ is a singleton} \qquad (k_a < k < 1).$$

Hence, if $k > k_a$, then, by (17), there exists uniquely $t_{a,k} \geq 0$ such that $z_k(t_{a,k})$ is the unique element of $\mu(a) \cap \lambda(iy_k(0))$. Evidently, $x_k(t_{a,k}) = a$. By (15) and (23), $y_k(t_{a,k})$ is continuous and strictly increasing with respect to k. If $a > 1$, then $k_a = (a-1)/(a+1)$, and hence (16) gives (19). If $0 \leq a \leq 1$, then $k_a = 0$, and hence

$$\limsup_{k\to k_a} y_k(t_{a,k}) \leq \lim_{k\to 0} y_k(0) = 0.$$

Since

$$a = x_k(t_{a,k}) = 1 + \left\{- \tau_k + 2\int_0^{t_{a,k}} \xi_k(s)ds\right\}/l_k(t)\,,$$

we have (20). This completes the proof of Lemma 7.

§ 4. Asymptotic behaviour of $\gamma(z)$

In this section, we show

THEOREM 8.

(24) $\gamma_y^+(0) = +\infty\,,$

(25) $\gamma_y^+(a) = \dfrac{1}{4\pi}\log\dfrac{1}{a}\ (>0)\quad (0 < a < 1)\,,$

(26) $\gamma_y^+(1) < 0\,,$

(27) $\gamma_y(a) = 0\,,$

$$\gamma_{yy}(a) = -\frac{1}{8\pi^2}\frac{a+1}{a-1}\left\{E\left(\frac{2\sqrt{a}}{a+1}\right) - \frac{a-1}{a+1}K\left(\frac{2\sqrt{a}}{a+1}\right)\right\}^2$$
$$(<0)\quad (a>1)\,,$$

where $\gamma_y^+(a) = \lim_{y\downarrow 0}\{\gamma(a+iy) - \gamma(a)\}/y$, $\gamma_y = \partial\gamma/\partial y$ *and* $\gamma_{yy} = \partial^2\gamma/\partial y^2$.

Equalities (25)–(27) show that $\gamma_y^+(a)$ is discontinuous at $a = 1$. We see that $\gamma_y(1) = 1/\{2\pi\sqrt{c^2-1}\} = 0.662\cdots/2\pi$, where c is the number satisfying $c/\sqrt{c^2-1} = \log(c + \sqrt{c^2-1})$ (cf. Lemma 10). Since

$$\gamma(1) = 1/2\,,\qquad \lim_{y\to\infty}\gamma(1+iy) = 1/2\,,$$

(26) shows that $\gamma(1+iy)$ has the minimum in $(0, \infty)$. If $0 < a_0 < 1$ is sufficiently near to 1, the behaviour of $\gamma(a_0+iy)$ $(y>0)$ is more complicated. Let $y_0 > 0$ be a point such that $\gamma(1+iy_0) = \min_{y\geq 0}\gamma(1+iy)$. Since $\gamma(1+iy_0) < 1/2$, we can choose $0 < a_1 < 1$ so that $\max_{a_1\leq a\leq 1}\gamma(a+iy_0)$ $(=\gamma_0$, say) is less than 1/2. If we choose a_0 so that $\max\{a_1, 1-2(1-2\gamma_0)\} < a_0 < 1$, then $\gamma(a_0+iy_0) < \gamma(a_0)$, and hence (25) shows that $\gamma(a_0+iy)$ has a local maximum in $(0, y_0)$. Since $\gamma(a_0+iy_0) < \gamma(a_0)$ and $\lim_{y\to\infty}\gamma(a_0+iy) = 1/2$, $\gamma(a_0+iy)$ has the minimum in $(0,\infty)$. Thus $\gamma(a_0+iy)$ has at least two extrema. A calculation shows that $\lim_{a\downarrow 1}\gamma_{yy}(a) = -\infty$ and

$$\gamma_{yy}^+(1) = 2\lim_{y\downarrow 0}\{\gamma(1+iy) - \gamma(1) - y\gamma_y^+(y)\}/y^2 = +\infty\,.$$

Thus $\gamma_{yy}^+(a)$ $(a\geq 1)$ is also discontinuous at $a = 1$.

Here are some lemmas necessary for the proof.

LEMMA 9. $\lim_{k\to 0} kt_{a,k} = \dfrac{2\sqrt{a}}{1-a} \qquad (0 < a < 1)$.

Proof. Equalities (4) and (5) show that, with $\xi_{a,k} = \xi_k(t_{a,k})$ and $\eta_{a,k} = \eta_k(t_{a,k})$,

$$\frac{1 - \xi_{a,k}^2 m_k^{-2}}{\sqrt{\xi_{a,k}^2 - 1}\sqrt{1 - k^2\xi_{a,k}^2}} = t_{a,k} m_k^{-2}, \tag{28}$$

$$\frac{1 - m_k^2 \eta_{a,k}^{-2}}{\sqrt{1 - \eta_{a,k}^{-2}}\sqrt{1 - k^2\eta_{a,k}^2}} = t_{a,k}\eta_{a,k}^{-1}. \tag{29}$$

Equality (20) shows that

$$\begin{aligned} 0 &= -a\tau_k + \int_0^{t_{a,k}} \{(1-a)\eta_k(s) + (1+a)\xi_k(s)\} ds \\ &= (1-a)\left\{\frac{\tau_k}{2} + \int_0^{t_{a,k}} \eta_k(s) ds\right\} + (1+a)\left\{-\frac{\tau_k}{2} + \int_0^{t_{a,k}} \xi_k(s) ds\right\} \\ &= (1-a)\{\psi_k(\eta_{a,k}) + t_{a,k}\eta_{a,k}\} + (1+a)\{-\psi_k(\xi_{a,k}) + t_{a,k}\xi_{a,k}\}, \end{aligned}$$

and hence

$$\begin{aligned} \eta_{a,k}^{-2}\Big\{(1+a)\int_1^{\xi_{a,k}} &\frac{m_k^2 - s^2}{\sqrt{s^2-1}\sqrt{1-k^2s^2}} ds \\ &- (1-a)\int_{\eta_{a,k}}^{1/k} \frac{s^2 - m_k^2}{\sqrt{s^2-1}\sqrt{1-k^2s^2}} ds\Big\} \\ &= \eta_{a,k}^{-2}\{(1+a)\psi_k(\xi_{a,k}) - (1-a)\psi_k(\eta_{a,k})\} \\ &= t_{a,k}\eta_{a,k}^{-1}\{(1-a) + (1+a)\xi_{a,k}\eta_{a,k}^{-1}\}. \end{aligned} \tag{30}$$

Let $(k_j)_{j=1}^{\infty}$ be a sequence tending to 0 such that $\lim_{j\to\infty} k_j\eta_{a,k_j}$ ($= d$, say) exists. Evidently, $0 \le d \le 1$. If $0 < d < 1$, then (29) shows that

$$\lim_{j\to\infty} t_{a,k_j}\eta_{a,k_j}^{-1} = \frac{1}{\sqrt{1-d^2}},$$

and hence

$$\lim_{j\to\infty} k_j t_{a,k_j} = \frac{d}{\sqrt{1-d^2}}.$$

By (28), we have

$$\lim_{j\to\infty} \xi_{a,k_j} k_j \log(1/k_j) = \frac{\sqrt{1-d^2}}{d}.$$

By (30), we have

$$\frac{1}{d^2}\{(1+a)-(1-a)\sqrt{1-d^2}\}=\frac{1-a}{\sqrt{1-d^2}},$$

which gives $d=2\sqrt{a}/(1+a)$. We show that $d\neq 0, 1$. Let $u(k)$ and $v(k)$ be the first quantity and the last quantity in (30), respectively. It holds that $u(k)=v(k)$ $(0<k<1)$. If $d=1$, then (29) shows that $\lim_{j\to\infty} v(k_j)=\infty$. We have

$$\limsup_{j\to\infty} u(k_j)\leq \limsup_{j\to\infty}(1+a)\eta_{a,k_j}^{-2}m_{k_j}^2K(k_j')=(1+a),$$

which contradicts (30). If $d=0$, then (29) shows that $\limsup_{j\to\infty} v(k_j)<\infty$. By (28) and (29), we have

$$\lim_{j\to\infty}\xi_{a,k_j}k_j\log(1/k_j)=1.$$

Hence

$$\begin{aligned}\lim_{j\to\infty} u(k_j)&=\lim_{j\to\infty}\eta_{a,k_j}^{-2}\{(1+a)m_{k_j}^2\log\xi_{a,k_j}-(1-a)k_j^{-2}\}\\&=2a\lim_{j\to\infty}\eta_{a,k_j}^{-2}k_j^{-2}=\infty,\end{aligned}$$

which contradicts (30). Thus $d\neq 0, 1$. Since $(k_j)_{j=1}^{\infty}$ is arbitrary as long as $(k_j\eta_{a,k_j})_{j=1}^{\infty}$ converges, we obtain $\lim_{k\to 0}k\eta_{a,k}=d=2\sqrt{a}/(1+a)$. Thus

$$\lim_{k\to 0}kt_{a,k}=\frac{2\sqrt{a}/(1+a)}{\sqrt{1-\{4a/(1+a)^2\}}}=\frac{2\sqrt{a}}{1-a}.$$

LEMMA 10. *We have*

$$\lim_{k\to 0}t_{1,k}m_k^{-2}=\frac{1}{\sqrt{c^2-1}},$$

where $c>0$ *is the number satisfying*

$$c/\sqrt{c^2-1}=\log(c+\sqrt{c^2-1}).$$

Proof. Equalities (4) and (20) show that, with $\xi_{1,k}=\xi_k(t_{1,k})$,

$$(31)\qquad \frac{1-\xi_{1,k}^2m_k^{-2}}{\sqrt{\xi_{1,k}^2-1}\sqrt{1-k^2\xi_{1,k}^2}}=t_{1,k}m_k^{-2},$$

$$\begin{aligned}&\int_1^{\xi_{1,k}}\frac{1-s^2m_k^{-2}}{\sqrt{s^2-1}\sqrt{1-k^2s^2}}ds-t_{1,k}m_k^{-2}\xi_{1,k}\\&\quad=m_k^{-2}\{\psi_k(\xi_{1,k})-t_{1,k}\xi_{1,k}\}=m_k^{-2}\left\{\frac{\tau_k}{2}-\int_0^{t_{1,k}}\xi_k(s)ds\right\}=0,\end{aligned}$$

and hence

$$\frac{\{1-\xi_{1,k}^2 m_k^{-2}\}\xi_{1,k}}{\sqrt{\xi_{1,k}^2-1}\sqrt{1-k^2\xi_{1,k}^2}} = \int_1^{\xi_{1,k}} \frac{1-s^2m_k^{-2}}{\sqrt{s^2-1}\sqrt{1-k^2s^2}}ds\,.$$

This shows that $\lim_{k\to 0}\xi_{1,k}=c$. Thus (31) yields the required equality.

LEMMA 11. *Let*

$$\Delta\gamma(z_k(t)) = \frac{\gamma(z_k(t))-(1+x_k(t))/4}{y_k(t)} \qquad (0<k<1,\ t\geq 0)\,.$$

Then

$$\Delta\gamma(z_k(t)) = \frac{k^2K(k')}{2\pi}\int_t^\infty \left\{\eta_k(s)-\frac{s}{\sqrt{1+k^2s^2}}\right\}ds - \frac{kK(k')}{2\pi}\sqrt{1+k^2t^2}\,.$$

Proof. We have

$$\begin{aligned}
\Delta\gamma(z_k(t)) &= \frac{2\gamma(z_k(t))-1+(1-x_k(t))/2}{2y_k(t)} \\
&= \frac{1}{2y_k(t)l_k(t)}\left\{\frac{1-k}{k^2}\sqrt{1+k^2t^2}-l_k(t)+\frac{\tau_k}{2}-\int_0^t \xi_k(s)ds\right\} \\
&= \frac{k^2K(k')}{2\pi}\left\{\frac{1-k}{k^2}\sqrt{1+k^2t^2}-\frac{\tau_k}{2}-\int_0^t \eta_k(s)ds\right\} \\
&= \frac{k^2K(k')}{2\pi}\left\{\frac{1}{k^2}\sqrt{1+k^2t^2}-\frac{t}{k}-\frac{\tau_k}{2}+\int_0^t\left(\frac{1}{k}-\eta_k(s)\right)ds\right\} \\
&\quad - \frac{kK(k')}{2\pi}\sqrt{1+k^2t^2} \\
&= \frac{k^2K(k')}{2\pi}\left\{\frac{1}{k^2}\sqrt{1+k^2t^2}-\frac{t}{k}-\int_t^\infty\left(\frac{1}{k}-\eta_k(s)\right)ds\right\} \\
&\quad - \frac{kK(k')}{2\pi}\sqrt{1+k^2t^2} \\
&= \frac{k^2K(k')}{2\pi}\left\{\int_t^\infty\left(\frac{1}{k}-\frac{s}{\sqrt{1+k^2s^2}}\right)ds-\int_t^\infty\left(\frac{1}{k}-\eta_k(s)\right)ds\right\} \\
&\quad - \frac{kK(k')}{2\pi}\sqrt{1+k^2t^2} \\
&= \frac{k^2K(k')}{2\pi}\int_t^\infty\left\{\eta_k(s)-\frac{s}{\sqrt{1+k^2s^2}}\right\}ds - \frac{kK(k')}{2\pi}\sqrt{1+k^2t^2}\,.
\end{aligned}$$

LEMMA 12. $\gamma(z) = \dfrac{1}{2} + c_{k_z}\operatorname{Im} z\displaystyle\int_{\lambda(z)} \frac{\gamma(\zeta)^2}{\mathscr{L}(\zeta)\{\gamma(\zeta)+\mathscr{L}(\zeta)\}}h_{k_z}(\zeta)d(\operatorname{Im}\zeta)$ $\quad(z\in P)$, *where k_z is the first number in the pair associated with z in* (11),

$$c_k = \frac{1}{4\pi^2 k}\{E(k') - kK(k')\}^2 ,$$

$$h_k(\zeta) = \{\gamma(\zeta)\sqrt{\gamma(\zeta)^2 + c'_k(\operatorname{Im}\zeta)^2} + \gamma(\zeta)^2 + c''_k(\operatorname{Im}\zeta)^2\}^{-1} ,$$

$$c'_k = \frac{1}{4\pi^2}(1-k)^2 K(k')^2\{(km_k^2 - 1)^2 + 2(km_k^2 - 1)\} ,$$

$$c''_k = \frac{1}{4\pi^2}(1-k)^2 K(k')^2 (km_k^2 - 1) .$$

Proof. Let $\zeta \in \lambda(z)$. Then $k_\zeta = k_z$ $(= k$, say$)$. By (12), we have

$$\begin{aligned}
&\frac{\gamma(\zeta)}{\mathscr{L}(\zeta)} - 1 = \frac{\gamma(\zeta) - \mathscr{L}(\zeta)}{\mathscr{L}(\zeta)} \\
&= \frac{2\gamma(\zeta)^2}{\mathscr{L}(\zeta)\{\gamma(\zeta) + \mathscr{L}(\zeta)\}(1-k)^2 t^2}\{\sqrt{(kt^2 + m_k^2)^2 + (km_k^2 - 1)^2 t^2} - (kt^2 + m_k^2)\} \\
&= \frac{2\gamma(\zeta)^2 (km_k^2 - 1)^2}{\mathscr{L}(\zeta)\{\gamma(\zeta) + \mathscr{L}(\zeta)\}(1-k)^2}\,\frac{1}{\sqrt{(kt^2 + m_k^2)^2 + (km_k^2 - 1)^2 t^2} + (kt^2 + m_k^2)} .
\end{aligned}$$

Since

$$\begin{aligned}
&\sqrt{(kt^2 + m_k^2)^2 + (km_k^2 - 1)^2 t^2} + (kt^2 + m_k^2) \\
&= \frac{1}{k}[\sqrt{\{(1 + k^2t^2) + (km_k^2 - 1)\}^2 + (km_k^2 - 1)^2(1 + k^2t^2) - (km_k^2 - 1)^2} \\
&\quad + (1 + k^2t^2) + (km_k^2 - 1)] \\
&= \frac{1}{k}[\sqrt{1 + k^2t^2}\sqrt{(1 + k^2t^2) + (km_k^2 - 1)^2 + 2(km_k^2 - 1)} \\
&\quad + (1 + k^2t^2) + (km_k^2 - 1)] \\
&= \frac{4k^4 l_k(t)^2}{k(1-k)^2}\Bigg[\frac{(1-k)\sqrt{1 + k^2t^2}}{2k^2 l_k(t)} \\
&\quad \times \sqrt{\frac{(1-k)^2(1 + k^2t^2)}{4k^4 l_k(t)^2} + \frac{(1-k)^2\{(km_k^2 - 1)^2 + 2(km_k^2 - 1)\}}{4k^4 l_k(t)^2}} \\
&\quad + \frac{(1-k)^2(1 + k^2t^2)}{4k^4 l_k(t)^2} + \frac{(1-k)^2(km_k^2 - 1)}{4k^4 l_k(t)^2}\Bigg] \\
&= \frac{4\pi^2}{k(1-k)^2 K(k')^2 (\operatorname{Im}\zeta)^2}\{\gamma(\zeta)\sqrt{\gamma(\zeta)^2 + c'_k(\operatorname{Im}\zeta)^2} + \gamma(\zeta)^2 + c''_k(\operatorname{Im}\zeta)^2\} \\
&= \frac{4\pi^2}{k(1-k)^2 K(k')^2 (\operatorname{Im}\zeta)^2}\, h_k(\zeta)^{-1} ,
\end{aligned}$$

we have

$$\frac{1}{2}\int_{\lambda(z)}\left\{\frac{\gamma(\zeta)}{\mathscr{L}(\zeta)}-1\right\}\frac{d(\operatorname{Im}\zeta)}{(\operatorname{Im}\zeta)^2}$$
$$=\frac{kK(k')^2(km_k^2-1)^2}{4\pi^2}\int_{\lambda(z)}\frac{\gamma(\zeta)^2}{\mathscr{L}(\zeta)\{\gamma(\zeta)+\mathscr{L}(\zeta)\}}h_k(\zeta)d(\operatorname{Im}\zeta)$$
$$=c_k\int_{\lambda(z)}\frac{\gamma(\zeta)^2}{\mathscr{L}(\zeta)\{\gamma(\zeta)+\mathscr{L}(\zeta)\}}h_k(\zeta)d(\operatorname{Im}\zeta)\,,$$

which gives the required equality.

We now give the proof of Theorem 8. Since

$$\Delta\gamma(z_k(0))=\frac{\gamma(z_k(0))-1/4}{y_k(0)}=\frac{k^2K(k')}{\pi}\left\{\frac{1-k}{k^2}-\frac{\tau_k}{4}\right\}$$
$$=\frac{K(k')}{\pi}\left\{1-k-\frac{k^2\tau_k}{4}\right\},$$

we have (24). Let $0<a<1$. Then

$$\frac{k^2K(k')}{2\pi}\int_{t_{a,k}}^{\infty}\left\{\eta_k(s)-\frac{s}{\sqrt{1+k^2s^2}}\right\}ds$$
$$=\frac{K(k')}{2\pi}\int_{kt_{a,k}}^{\infty}\left\{\eta_k^*(u)-\frac{u}{\sqrt{1+u^2}}\right\}du\,,$$

where

$$\eta_k^*(u)=\frac{1}{\sqrt{2(1+u^2)}}[2k^2m_k^2+(1+k^2)u^2$$
$$+\sqrt{\{2k^2m_k^2+(1+k^2)u^2\}^2-4(k^4m_k^4+k^2u^2)(1+u^2)}]^{1/2}$$

Let $d_k=k^2m_k^2+k^2(m_k^2-1)(1-k^2m_k^2)(1-k^2)^{-1}$. Then we can write

$$\eta_k^*(u)=\frac{1}{\sqrt{2(1+u^2)}}[2k^2m_k^2+(1+k^2)u^2$$
$$+\sqrt{(1-k^2)^2u^4+4\{k^2m_k^2(1+k^2)-(k^2+k^4m_k^4)\}u^2}]^{1/2}$$
$$=\frac{u}{\sqrt{1+u^2}}\left[k^2m_k^2u^{-2}+\frac{1+k^2}{2}\right.$$
$$\left.+\frac{1-k^2}{2}\sqrt{1+4k^2(m_k^2-1)(1-k^2m_k^2)(1-k^2)^{-2}u^{-2}}\right]^{1/2}$$
$$=\frac{u}{\sqrt{1+u^2}}[1+d_ku^{-2}\{1+d_k\omega_1(k,u)\}]^{1/2}$$
$$=\frac{u}{\sqrt{1+u^2}}+\frac{d_k}{2u\sqrt{1+u^2}}\{1+d_k\omega_2(k,u)\}$$

with two functions $\omega_j(k, u)$ $(j = 1, 2)$ satisfying $\sup|\omega_j(k, u)| < \infty$, where the supremum is taken over all pairs (k, u) such that $0 < k \leq 1/2$ and $u \geq \sqrt{a}/(1-a)$. Notice that $\lim_{k\to 0} d_k = 0$ and $\lim_{k\to 0} d_k K(k') = 2$. Thus Lemmas 9 and 11 show that

$$\begin{aligned}
\gamma_y^+(a) &= \lim_{k\to 0} \Delta\gamma(z_k(t_{a,k})) \\
&= \lim_{k\to 0}\left[\frac{k^2K(k')}{2\pi}\int_{t_{a,k}}^{\infty}\left\{\eta_k(s) - \frac{s}{\sqrt{1+k^2s^2}}\right\}ds - \frac{kK(k')}{2\pi}\sqrt{1+k^2t_{a,k}^2}\right] \\
&= \lim_{k\to 0}\frac{K(k')}{2\pi}\int_{kt_{a,k}}^{\infty}\left\{\eta_k^*(u) - \frac{u}{\sqrt{1+u^2}}\right\}du \\
&= \lim_{k\to 0}\frac{d_kK(k')}{4\pi}\int_{kt_{a,k}}^{\infty}\frac{1}{u\sqrt{1+u^2}}\{1 + d_k\omega_2(k, u)\}du \\
&= \lim_{k\to 0}\frac{1}{2\pi}\int_{2\sqrt{a}/(1-a)}^{\infty}\frac{du}{u\sqrt{1+u^2}} = \frac{1}{4\pi}\log\frac{1}{a}\,.
\end{aligned}$$

Thus (25) holds. Lemma 10 shows that $\lim_{k\to 0} kt_{1,k} = \infty$, and hence

$$\begin{aligned}
&\lim_{k\to 0}\frac{k^2K(k')}{2\pi}\int_{t_{1,k}}^{\infty}\left\{\eta_k(s) - \frac{s}{\sqrt{1+k^2s^2}}\right\}ds \\
&\quad = \lim_{k\to 0}\frac{d_kK(k')}{4\pi}\int_{kt_{1,k}}^{\infty}\frac{1}{u\sqrt{1+u^2}}\{1 + d_k\omega_2(k, u)\}du = 0\,.
\end{aligned}$$

By Lemmas 10 and 11, it follows that

$$\begin{aligned}
\gamma_y^+(1) &= \lim_{k\to 0}\Delta\gamma(z_k(t_{1,k})) = -\lim_{k\to 0}\frac{kK(k')}{2\pi}\sqrt{1+k^2t_{1,k}^2} \\
&= -\frac{1}{2\pi}\lim_{k\to 0} t_{1,k}m_k^{-2} = -\frac{1}{2\pi\sqrt{c^2-1}} < 0\,.
\end{aligned}$$

Thus (26) holds. Let $a > 1$. Theorem 2 shows that

$$\lim_{\zeta\to k_a,\,\zeta\in P}\frac{\gamma(\zeta)^2}{\mathscr{L}(\zeta)\{\gamma(\zeta) + \mathscr{L}(\zeta)\}}h_{k_a}(\zeta) = \frac{1}{4}\,.$$

Thus Lemmas 7 and 12 yield that

$$\gamma_y^+(a + iy) = \lim_{y\downarrow 0}\frac{\gamma(a+iy) - 1/2}{y} = -\frac{1}{4}c_{k_a}\lim_{y\downarrow 0}\int_0^y ds = 0$$

and

$$\gamma_{yy}^+(a) = 2\lim_{y\downarrow 0}\frac{\gamma(a+iy) - 1/2}{y^2} = -\frac{1}{2}c_{k_a}\lim_{y\downarrow 0}\frac{1}{y}\int_0^y ds$$

$$= -\frac{1}{2}c_{k_a} = -\frac{1}{8\pi^2 k_a}\{E(k_a') - k_a K(k_a')\}^2$$
$$= -\frac{1}{8\pi^2}\frac{a+1}{a-1}\left\{E\left(\frac{2\sqrt{a}}{a+1}\right) - \frac{a-1}{a+1}K\left(\frac{2\sqrt{a}}{a+1}\right)\right\}^2,$$

which shows (27). This completes the proof of Theorem 8.

§ 5. The constant σ_0

In this section, we study the following extremum problem: $\sigma_0 = \inf \gamma(x+iy)/\gamma(x)$, where the infimum is taken over all real numbers x and y. We show

THEOREM 13. *Let* $\rho(a) = \min_{y\geq 0}\gamma(a+iy)/\gamma(a)$ $(a \geq 0)$. *Then* $\sigma_0 = \rho(1)$ *and* $\sigma_0 < \rho(a)$ $(a \neq 1)$.

Here is a lemma necessary for the proof.

LEMMA 14. *For each* $0 < k < 1$,

(32) $\gamma(z_k(t))$ *is strictly increasing,*

(33) $4\gamma(z_k(t))/(1 + x_k(t))$ *is strictly decreasing.*

Proof. Theorem 1 shows that

$$\gamma(z_k(t)) = \frac{1}{2} + \frac{y_k(t)}{2}\int_t^\infty \left\{\frac{\gamma(z_k(s))}{\mathscr{L}(z_k(s))} - 1\right\}\frac{y_k'(s)}{y_k(s)^2}ds,$$

and hence

$$\frac{d}{dt}\gamma(z_k(t)) = \frac{y_k'(t)}{2}\int_t^\infty \left\{\frac{\gamma(z_k(s))}{\mathscr{L}(z_k(s))} - 1\right\}\frac{y_k'(s)}{y_k(s)^2}ds$$
$$- \frac{y_k(t)}{2}\left\{\frac{\gamma(z_k(t))}{\mathscr{L}(z_k(t))} - 1\right\}\frac{y_k'(t)}{y_k(t)^2}.$$

Thus Theorem 2 and (17) yield (32). Since

$$\frac{1 + x_k(t)}{4} = \frac{1}{2l_k(t)}\left\{l_k(t) - \frac{\tau_k}{2} + \int_0^t \xi_k(s)ds\right\}$$
$$= \frac{1}{2l_k(t)}\left\{\frac{\tau_k}{2} + \int_0^t \eta_k(s)ds\right\} = \frac{1}{2l_k(t)}\{\psi_k(\eta_k(t)) + t\eta_k(t)\},$$

we have, by (6),

$$(34)\quad \frac{d}{dt}\frac{4\gamma(z_k(t))}{1+x_k(t)} = \frac{1-k}{k^2}\frac{d}{dt}\frac{\sqrt{1+k^2t^2}}{\psi_k(\eta_k(t))+t\eta_k(t)\}}$$
$$= \frac{1-k}{k^2\{\psi_k(\eta_k(t))+t\eta_k(t)\}^2}$$
$$\times\left[\frac{k^2t}{\sqrt{1+k^2t^2}}\{\psi_k(\eta_k(t))+t\eta_k(t)\}-\sqrt{1+k^2t^2}\,\eta_k(t)\right]$$
$$= \frac{1-k}{k^2\sqrt{1+k^2t^2}\,\{\psi_k(\eta_k(t))+t\eta_k(t)\}^2}\{k^2t\psi_k(\eta_k(t))-\eta_k(t)\}\,.$$

Since $m_k > 1$, we have, with $\eta = \eta_k(t)$,

$$k^2t\psi_k(\eta) = k^2t\{\psi_k(\eta)-\psi_k(1/k)\}$$
$$= \frac{k^2(\eta^2-m_k^2)}{\sqrt{\eta^2-1}\,\sqrt{1-k^2\eta^2}}\int_\eta^{1/k}\frac{s^2-m_k^2}{\sqrt{s^2-1}\,\sqrt{1-k^2s^2}}\,ds$$
$$< \frac{k^2\eta}{\sqrt{1-k^2\eta^2}}\int_\eta^{1/k}\frac{s}{\sqrt{1-k^2s^2}}\,ds = \eta\,.$$

Hence the first quantity in (34) is negative, which gives (33).

We now give the proof of Theorem 13. Let $a > 1$. Since $\lim_{y\to\infty}\gamma(a+iy)/\gamma(a) = 1$, there exists $y_a \geq 0$ such that

$$\rho(a) = \gamma(a+iy_a)/\gamma(a) = 2\gamma(a+iy_a)\,.$$

By (27), we have $y_a > 0$. Hence there exists a pair (k^0, t^0) such that $a + iy_a = z_{k^0}(t^0)$. Let $t^1 > 0$ be the number such that $x_{k^0}(t^1) = 1$. Then $t^1 < t^0$. Hence, by (32), it follows that

$$\rho(1) \leq \gamma(z_{k^0}(t^1))/\gamma(1) = 2\gamma(z_{k^0}(t^1)) < 2\gamma(z_{k^0}(t^0)) = \rho(a)\,.$$

Inequality (26) shows that $\rho(1) < 1$. Let $0 \leq a < 1$. Then there exists $y_a \geq 0$ such that

$$\rho(a) = \frac{\gamma(a+iy_a)}{\gamma(a)} = \frac{4\gamma(a+iy_a)}{1+a}\,.$$

If $y_a = 0$, then $\rho(1) < 1 = \rho(a)$. If $y_a > 0$, then there exists a pair (k^0, t^0) such that $a + iy_a = z_{k^0}(t^0)$. Let $t^1 > 0$ be the number such that $x_{k^0}(t^1) = 1$. Then $t^1 > t^0$. Hence, by (33), it follows that

$$\rho(1) \leq 4\gamma(z_{k^0}(t^1))/(1+x_{k^0}(t^1))$$
$$< 4\gamma(z_{k^0}(t^0))/(1+x_{k^0}(t^0)) = \rho(a)\,.$$

Thus

$$\rho(1) = \min_{a \geq 0} \rho(a) , \qquad \rho(1) < \rho(a) \quad (a \neq 1) .$$

which gives the required inequalities in Theorem 13. This completes the proof of Theorem 13.

From the point of view of Vitushkin-Garnett's example, it is interesting to estimate σ_0. A rough estimate is given as follows. The Garabedian function [2, p. 19] of an interval $[-1/2, 1/2]$ is given by

$$\psi(\zeta) = \frac{1}{2}\left\{1 + \frac{\zeta}{\sqrt{\zeta^2 - (1/4)}}\right\} ;$$

in fact,

$$\frac{1}{2\pi} \int_{\partial[-1/2,1/2]} |\psi(\zeta)||d\zeta| = \frac{1}{4\pi} \int_{-1/2}^{1/2} \frac{ds}{\sqrt{(1/4) - s^2}} = \frac{1}{4} .$$

Since $\psi(\zeta)\psi(\zeta + 1 + iy)$ is analytic outside $\Gamma(1 + iy)$ and equal to 1 at infinity, we have

$$\gamma(1 + iy) \leq \frac{1}{2\pi} \int_{\partial \Gamma(1+iy)} |\psi(\zeta)\psi(\zeta + 1 + iy)||d\zeta| \qquad (\text{cf. } [2, \text{ p. } 19]) .$$

Thus Theorem 13 shows that

$$\sigma_0 \leq \inf_{y \geq 0} \frac{1}{\pi} \int_{\partial \Gamma(1+iy)} |\psi(\zeta)\psi(\zeta + 1 + iy)||d\zeta| . \tag{35}$$

We can easily compute the right-hand side of (35). The estimate by this method is rough, however, this method gives a new approach to the construction of sets of Vitushkin-Garnett type (cf. [8, p. 81]). In order to get a better estimate, it is necessary to study, in detail, incomplete elliptic integrals. Recall that

$$\sigma_0 = \min_{0<k<1} 2\gamma(z_k(t_{1,k})) ,$$

$$\gamma(z_k(t)) = \left\{\frac{1 - k}{2k}\sqrt{t^2 + k^{-2}}\right\}/l_k(t) ,$$

$$l_k(t) = \psi_k(\eta_k(t)) + \psi_k(\xi_k(t)) + t\{\eta_k(t) - \xi_k(t)\} ,$$

$$\psi_k(x) = \int_1^x \frac{m_k^2 - s^2}{\sqrt{s^2 - 1}\sqrt{1 - k^2 s^2}} ds \qquad (1 \leq x \leq 1/k) .$$

Since

$$\psi_k(x) = -\psi_k(1/k) + \psi_k(x) = \int_x^{1/k} \frac{s^2 - m_k^2}{\sqrt{s^2 - 1}\sqrt{1 - k^2 s^2}} ds ,$$

we have, by making the substitution $1 - k^2s^2 = k'^2u^2$,

$$\psi_k(x) = k^{-2}\int_0^{\nu(x)} \sqrt{\frac{1-k'^2u^2}{1-u^2}}\,du - m_k^2\int_0^{\nu(x)} \frac{du}{\sqrt{1-u^2}\sqrt{1-k'^2u^2}}$$
$$= k^{-2}E(\arcsin \nu(x), k') - m_k^2 F(\arcsin \nu(x), k')\,,$$

where $\nu(x) = \sqrt{1-k^2x^2}/k'$. Thus $\psi_k(x)$ can be computed with the aid of Landen's transformation [4, p. 250] or Jacobian theta functions [4, p. 292]. (As is well known, Landen's transformation yields that

$$F(\varphi, k') = \frac{1}{1+k}F\left(\psi, \frac{1-k}{1+k}\right),$$
$$E(\varphi, k') = -\frac{k(1+k)}{2}F(\varphi, k') + \frac{1+k}{2}E\left(\psi, \frac{1-k}{1+k}\right) + \frac{1-k}{2}\sin\psi\,,$$

where ψ is defined by $\tan(\psi - \varphi) = k\tan\varphi$. Since $(1-k)/(1+k) < k'$, we can compute $E(\varphi, k')$ and $F(\varphi, k')$ by repeating this formula.) Equality (20) for $a = 1$ can be rewritten as

$$0 = \frac{\tau_k}{2} - \int_0^{t_{1,k}} \xi_k(s)ds = \psi_k(\xi_k(t_{1,k})) - t_{1,k}\xi_k(t_{1,k})\,,$$

and hence

$$m_k t_{1,k} = t_{1,k}\{m_k - \xi_k(t_{1,k})\} + \psi_k(\xi_k(t_{1,k}))\,.$$

We now inductively define a sequence $(t_{1,k}^{(n)})_{n=0}^{\infty}$ by $t_{1,k}^{(0)} = 0$,

$$m_k t_{1,k}^{(n)} = t_{1,k}^{(n-1)}\{m_k - \xi_k(t_{1,k}^{(n-1)})\} + \psi_k(\xi_k(t_{1,k}^{(n-1)})) \qquad (n \geq 1)\,.$$

Since

$$t\{m_k - \xi_k(t)\} + \psi_k(\xi_k(t)) = \frac{\tau_k}{2} + \int_0^t \{m_k - \xi_k(s)\}ds\,,$$

we have

$$m_k|t_{1,k}^{(n)} - t_{1,k}^{(n-1)}| = \left|\int_{t_{1,k}^{(n-2)}}^{t_{1,k}^{(n-1)}} \{m_k - \xi_k(s)\}ds\right|$$
$$\leq (m_k - 1)|t_{1,k}^{(n-1)} - t_{1,k}^{(n-2)}| \qquad (n \geq 2)\,,$$

and hence

$$|t_{1,k} - t_{1,k}^{(n)}| \leq \sum_{l=n}^{\infty}(1 - m_k^{-1})^l|t_{1,k}^{(1)}| = \frac{m_k\tau_k}{2}(1 - m_k^{-1})^n \qquad (n \geq 0)\,.$$

This shows that $(t_{1,k}^{(n)})_{n=0}^{\infty}$ converges to $t_{1,k}$. (In the case where k is small, the speed of the convergence of $(t_{1,k}^{(n)})_{n=0}^{\infty}$ is slow. Hence, by using $(t_{1,k}^{(n)})_{n=0}^{\infty}$, we choose first $\tilde{t}_{1,k}$ sufficiently near to $t_{1,k}$ and define next $(\tilde{t}_{1,k}^{(n)})_{n=0}^{\infty}$ by $\tilde{t}_{1,k}^{(0)} = \tilde{t}_{1,k}$,

$$\tilde{t}_{1,k}^{(n)} = \tilde{t}_{1,k}^{(n-1)}\{1 - \varepsilon_k \xi_k(\tilde{t}_{1,k}^{(n-1)})\} + \varepsilon_k \psi_k(\xi_k(\tilde{t}_{1,k}^{(n-1)})) \qquad (n \geq 1),$$

where $\varepsilon_k > 0$ is chosen so that the convergence of $(\tilde{t}_{1,k}^{(n)})_{n=0}^{\infty}$ is rapid. Notice that $t_{1,k} = \lim_{n\to\infty} \tilde{t}_{1,k}^{(n)}$.) Thus we can compute $2\gamma(z_k(t_{1,k}))$ $(0 < k < 1)$. The author expresses his thanks to Prof. Yonezawa and Mr. Sakurai who practiced our program. Prof. Yonezawa shows that $0.95 \leq \sigma_0 \leq 0.97$. ($\sigma_0$ is attained when k is near to 0.1.)

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