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WHITE NOISE ANALYSIS AND TANAKA FORMULA FOR INTERSECTIONS OF PLANAR BROWNIAN MOTION

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§ 1. Introduction

In this paper, we shall use Hida's [5, 7, 9] theory of generalized Brownian functionals (or named white noise analysis) to establish a stochastic integral formula concerning the multiple intersection local times of planar Brownian motion $B(t)$. First, we review some basic facts. It is well-known that, for each $r \geqslant 2$, a.s. $B(t)$ has points of multiplicity r, see Dvoretzky-Erdös-Kakutani [1] and Geman-Horowitz-Rosen [4]. A formal measure on such r-fold self-intersections is

$$\int \cdots \int_D \delta(B(t_2) - B(t_1)) \cdots \delta(B(t_r) - B(t_{r-1}))dt_1 \cdots dt_r, \tag{1.1}$$

where $D \subset \{(t_1, \cdots, t_r) \mid 0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_r < \infty\}$. Rosen [15] proved that (a.s.) there exists measurable $a_r(x, D)$, $x \in R^{2(r-1)}$, called the r-fold intersection local time, such that for all bounded measurable $g(x)$ on $R^{2(r-1)}$

$$\int \cdots \int_D g(B(t_2) - B(t_1), \cdots, B(t_r) - B(t_{r-1}))dt_1 \cdots dt_r$$
$$= \int_{R^{2(r-1)}} \cdots \int g(x)\alpha_r(x, D)dx .$$

In case $D = \prod_{i=1}^r [a_i, b_i]$, $0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_r$ (the off-diagonal case), Rosen [15, 17] proved that (a version of) $\alpha_r(x, D)$ can be chosen so that it is (a.s.) jointly continuous in x, a_i, b_i. Thus, (1.1) is represented as $\alpha_r(0, D)$ in this case. However, in case $D = D_T = \{(t_1, \cdots, t_r) \mid 0 \leqslant t_1 \leqslant \cdots t_r \leqslant T\}$ (the diagonal case), then $\alpha_r(x, T) \equiv \alpha_r(x, D_T)$ can only be continuous on $(R^2 \backslash \{0\})^{r-1}$. To study the asymptotics of $\alpha_r(x, T)$ as $x \to 0$, it is intended to find that after substracting off certain explicit "infinite part" from $\alpha_r(x, T)$ the remainder $\tilde{\alpha}_r(x, T)$, called the renormalized intersection local time, admits a jointly continuous extension to all $(x, T) \in R^{2(r-1)} \times R_+$.

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The research was initiated by Varadhan [22] and has been studied intensively by Dynkin, LeGall, Rosen and Yor, see Dynkin [2]. In the investigations Tanaka type formulae for α_r and $\tilde{\alpha}_r$, analogous to the classical Tanaka's formula for one-dimensional Brownian local time, play an important role. In the off-diagonal case, let $\alpha_r(x, a_1, \cdots, a_r, dt_r)$, $t_r \in [a_r, b_r]$, denote the Borel measure induced by $t_r \to \alpha_r(\cdots, t_r)$ and $\alpha_1(dt_1) \equiv dt_1$, the Lebesgue measure on R^1. The Tanaka formulae relating α_{r+1} and α_r were studied by Rosen [16, 17] and Yor [23]. Then, the formulae were used by them [18, 20, 23] to find the explicit forms of $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$.

In this paper, we shall prove the following formula, which is an extension of Yor [23] from $r = 1$ to general $r \geqslant 1$. It corresponds also to Rosen [17], who used the potential kernel for the killed Brownian motion instead of $\log|x|$.

THEOREM 1.1. *Assume that* $0 < a_i < b_i < a_{i+1} < b_{i+1}$, $i = 1, \cdots, r$. *For each* $x = (x_1, \cdots, x_{r-1}) \in R^{2(r-1)}$ *and* $x_r \in R^2$ *the following equality holds almost surely.*

$$
\begin{aligned}
(1.2) \qquad & \int_{a_r}^{b_r} \log|B(b_{r+1}) - B(t_r) - x_r|\alpha_r(x, a_1, \cdots, a_r, dt_r) \\
& \qquad - \int_{a_r}^{b_r} \log|B(a_{r+1}) - B(t_r) - x_r|\alpha_r(x, a_1, \cdots, a_r, dt_r) \\
& = \int_{a_{r+1}}^{b_{r+1}} \left(\int_{a_r}^{b_r} \frac{B(t_{r+1}) - B(t_r) - x_r}{|B(t_{r+1}) - B(t_r) - x_r|^2} \alpha_r(x, a_1, \cdots, a_r, dt_r) \right) \cdot dB(t_{r+1}) \\
& \qquad + \pi\alpha_{r+1}((x, x_r), a_1, \cdots, a_{r+1}, b_{r+1});
\end{aligned}
$$

in the above, the integrand for the (Ito's) stochastic integral will be proved to be the $L^2(dt_{r+1} \times dP)$ *limit of certain approximation sequence.*

Our formula will suggest an explicit form for $\tilde{\alpha}_r$, which is an extension of Yor [23, $r = 2$] and Rosen-Yor [20, $r = 3$] to general r. See § 5.

We explain our viewpoint on using white noise analysis to derive (1.2). Kubo [11] established an Ito formula for generalized Brownian functionals and indicated briefly to view the classical Tanaka's formula as a case of his result [11, Remark 4.1]. Thus, we are motivated to derive (1.2) from Kubo's formula. The crucial point is that Kubo's formula holds in Hida's space $(L^2)^-$, while our (1.2) is an a.s. equality. Therefore, in order to achieve our formula we must "realize" a Brownian functional from generalized one to ordinary one. Our procedure in based on or-

thogonal series expansions of generalized Brownian functionals. The expansions are known to hold in Hida's space $(L^2)^-$, and we carry out precise computations to show that the expansions are in fact convergent in L^2 sense. In a previous work Kuo-Sheih [13], Yor's result (i.e. (1.2) with $r = 1$) was reproved by using realization techniques. However, the setting in [13, § 4] is improper, because of the doubtful definition $f(B(t, \omega), \omega)$ given there.

The remaining part of this paper will be devided into four sections. In § 2, we shall summarize briefly Hida's theory. In § 3, we shall prove two realization results of generalized Brownian functionals. Then, we shall prove a Tanaka type formula in § 4 and derive Theorem 1.1. In the final § 5, we shall discuss how to apply (1.2) to obtain an explicit form of the renormalized $\tilde{\alpha}_r$.

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§ 2. Summary on generalized Brownian functionals

Let $E \subset L^2(R) \subset E^*$ be a Gel'fand triplet and μ be the standard Gaussian measure on E^*, i.e. its characteristic functional is given by $C(\xi) = \exp[-\|\xi\|^2/2)]$, $\|\cdot\|$ being the $L^2(R)$ norm. Let K_n, $K_n^{(n)}$, $K_n^{(-n)}$ consist respectively of the multiple Wiener integrals of symmetric functions in Hilbert space $L^2(R^n)$, Sobolev space $H^{(n+1)/2}(R^n)$ and negative Sobolev space $H^{-(n+1)/2}(R^n)$; see Hida [6]. By Wiener-Ito decomposition theorem, we have

$$(L^2) \equiv L^2(E^*, \mu) = \sum_{n=0}^{\infty} \oplus K_n .$$

Hida [5, 7, 9] introduced the following two spaces

$$(L^2)^+ \equiv \sum_{n=0}^{\infty} \oplus K_n^{(n)} ,$$

$$(L^2)^- \equiv \sum_{n=0}^{\infty} \oplus K_n^{(-n)} ,$$

and established the continuous inclusions $(L^2)^+ \hookrightarrow (L^2) \hookrightarrow (L^2)^-$. Members in these three spaces are called respectively test, ordinary and generalized Brownian functionals. A calculus has been developed on $(L^2)^-$ which is usually referred to white noise analysis or Hida's calculus. To

visualize an element $\phi \in (L^2)^-$, we consider its $\mathscr{S}$-transform (or say $\mathscr{U}$-functional):

$$(\mathscr{S}\phi)(\xi) = \exp\left(-\frac{\|\xi\|^2}{2}\right)\langle\langle e^{(\cdot,\xi)}, \phi(\cdot)\rangle\rangle\,, \qquad \xi \in E\,;$$

$\langle\langle\cdot,\cdot\rangle\rangle$ being the pairing between $(L^2)^+$ and $(L^2)^-$.

Note that $B(t, \omega) = (\langle\omega, 1_{[0,t]}\rangle, \langle\omega, 1_{[-t,0]}\rangle)$, $t \geqslant 0$ and $\omega \in E^*$, is a planar Brownian motion. For $0 < t_1 < t_2 < \cdots < t_r$ and $u_1, \cdots, u_{r-1} \in \mathscr{S}^*(R^2)$, the space of tempered distributions on R^2, the formal product

$$\prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i)) \tag{2.1}$$

can be defined as a member of $(L^2)^-$; its $\mathscr{S}$-transform is given by

$$\begin{aligned}
&\mathscr{S}\left(\prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i))\right)(\xi) \\
&\quad = \prod_{i=1}^{r-1} \mathscr{S}(u_i(B(t_{i+1}) - B(t_i))(\xi) \\
&\quad = \prod_{i=1}^{r-1} (u_i * g_{t_{i+1}-t_i})\left(\int_{t_i}^{t_{i+1}} \xi(s)ds, \int_{-t_{i+1}}^{-t_i} \xi(s)ds\right),
\end{aligned}$$

see Kubo [11], Kuo [12] and Kallianpur-Kuo [10]. Moreover, the orthogonal series expansion of (2.1) in $(L^2)^-$ is given by

$$\begin{aligned}
&(2\pi)^{-r+1} \prod_{i=1}^{r-1} (t_{i+1} - t_i)^{-1} \sum_{\substack{(m_i=n_i)=(0,0)\\ i=1,\cdots,r-1}} \prod_{i=1}^{r-1} \left\{\frac{1}{m_i!\, n_i!\, 2^{m_i} 2^{n_i}}\right. \\
&\quad \times \left\langle u_i(x_1, x_2), H_{m_i}\left(\frac{x_1}{\sqrt{2(t_{i+1} - t_i)}}\right) H_{n_i}\left(\frac{x_2}{\sqrt{2(t_{i+1} - t_i)}}\right)\right. \\
&\quad \cdot \exp\left(-\frac{(x_1^2 + x_2^2)}{2(t_{i+1} - t_i)}\right)\Bigg\rangle_{(x_1,x_2)} \\
&\quad \left. \times H_{m_i}\left(\frac{\langle\omega, 1_{[t_i,t_{i+1}]}\rangle}{\sqrt{2(t_{i+1} - t_i)}}\right) H_{n_i}\left(\frac{\langle\omega, 1_{[-t_{i+1},-t_i]}\rangle}{\sqrt{2(t_{i+1} - t_i)}}\right)\right\}.
\end{aligned} \tag{2.2}$$

In the above, $\langle u(x_1, x_2), \psi(x_1, x_2)\rangle_{(x_1,x_2)}$ denotes the pairing between $u \in \mathscr{S}^*(R^2)$ and $\psi \in \mathscr{S}(R^2)$, and $H_n(x)$, $x \in R$, is the Hermite polynomial of degree n.

For $0 < t_1 < t_2$ and $u \in \mathscr{S}^*(R^2)$, Kubo [11] established the following formula in $(L^2)^-$:

$$u(B(t_2)) - u(B(t_1)) = \int_{t_1}^{t_2} \partial_t^*(\nabla u)(B(t))dt + \frac{1}{2}\int_{t_1}^{t_2} (\Delta u)(B(t))dt\,. \tag{2.3}$$

In the above $\partial_t^* = (\partial_{(t,1)}^*, \partial_{(t,2)}^*)$ is the adjoint of $\dot{B}(t)$-differentiation operator $(\partial_{(t,1)}, \partial_{(t,2)})$, see Kubo [11] and Hida [8].

§ 3. Realization of generalized Brownian functionals

Assuming $0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{r-1} < a_r < b_r$, we shall prove

PROPOSITION 3.1. *Let u_i, $i = 1, \cdots, r-1$, be positive tempered distributions on R^2 (or equivalently, they are positive tempered Radon measures on R^2). Then*

$$\int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i))dt_1 \cdots dt_r \tag{3.1}$$

is in (L^2).

Remark. From the arguments in Kallianpur-Kuo [10], it is not difficult to see that

$$(t_1, \cdots, t_r) \in \prod_{i=1}^{r} [a_i, b_i] \to \prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i)) \in (L^2)^-$$

is continuous. Thus, (3.1) exists as a generalized Brownian functional. Proposition 3.1 asserts that it can be "realized" as an ordinary one.

To prove Proposition 3.1, we write the summund in (2.2) as

$$\prod_{i=1}^{r-1} \phi_i(m_i, n_i, t_i, t_{i+1}), \quad m_i, n_i \geqslant 0 \quad \text{and} \quad t_i \in [a_i, b_i]\,.$$

LEMMA 3.1. (i) *If $(m_i, n_i) \neq (m'_i, n'_i)$ for some i, then*

$$E \prod_{i=1}^{r-1} \phi_i(m_i, n_i, t_i, t_{i+1})\phi_i(m'_i, n'_i, t'_i, t'_{i+1}) = 0\,.$$

(ii)
$$\begin{aligned} E \prod_{i=1}^{r-1} \phi_i(m_i, n_i, t_i, t_{i+1})\phi_i(m_i, n_i, t'_i, t'_{i+1}) \\ \leqslant \text{const.} \left(\frac{\prod_{i=1}^{r-1} m_i!n_i!}{(\sum_{i=1}^{r-1} m_i)!(\sum_{i=1}^{r-1} n_i)!(\prod_{i=1}^{r-1} m_i n_i)^{1/6}} \right) \\ \times \prod_{i=1}^{r-1} \left(\frac{t_{i+1} \wedge t'_{i+1} - t_i \vee t'_i}{\sqrt{(t_{i+1} - t_i)(t'_{i+1} - t'_i)}} \right)^{m_i+n_i}, \end{aligned}$$

where $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

Proof. We prove (ii) first. Since u_i is positive, $|\langle u_i, fg\rangle \leqslant \langle u_i, f\rangle \cdot \|g\|_\infty$ for all $f, g \in \mathscr{S}(R^2)$ and $f \geqslant 0$. Thus

$$\begin{aligned} \left| \left\langle u_i(x_1, x_2), H_{m_i}\left(\frac{x_1}{\sqrt{2(t_{i+1} - t_i)}}\right) H_{n_i}\left(\frac{x_2}{\sqrt{2(t_{1+1} - t_i)}}\right) \exp\left(-\frac{x_1^2 + x_2^2}{2(t_{i+1} - t_i)}\right) \right\rangle \right| \\ \leqslant \left(\left\langle u_i(x), \exp\left(-\frac{|x|^2}{4(t_{i+1} - t_i)}\right) \right\rangle_{x=(x_1,x_2)} \right) \end{aligned}$$

$$\times \sup_{\substack{t>0\\ x_1,x_2\in R}} \left|\exp\left(-\frac{x_1^2+x_2^2}{4t}\right)H_{m_i}\left(\frac{x_1}{\sqrt{2t}}\right)H_{n_i}\left(\frac{x_2}{\sqrt{2t}}\right)\right| .$$

The first term in the above product is less than a constant. This is because $t_{i+1} - t_i \geqslant a_{i+1} - b_i \geqslant \theta > 0$ and the mapping $t \in [T_0, T_1] \to \langle u_i(\cdot), \exp(-|\cdot|^2/4t)\rangle$ is continuous, hence is bounded, on each $[T_0, T_1]$, $0 < T_0 < T_1 < \infty$. The second term is $O((m_i n_i 2^{m_i} 2^{n_i})^{1/2}(m_i n_i)^{-1/12})$, according to Szegö [21, (8.91.10)]. Thus, the expectation in (ii) is not greater than a constant multiple of

(3.2)
$$\left(\prod_{i=1}^{r-1} \frac{1}{m_i!\, n_i!\, 2^{m_i} 2^{n_i} \sqrt[6]{m_i n_i}}\right) E\left(\prod_{i=1}^{r-1} H_{m_i}\left(\frac{X_i}{\sqrt{2}}\right) H_{m_i}\left(\frac{X_i'}{\sqrt{2}}\right) H_{n_i}\left(\frac{Y_i}{\sqrt{2}}\right) H_{n_i}\left(\frac{Y_i'}{\sqrt{2}}\right)\right),$$

where

$$X_i(\omega) = \frac{\langle \omega, 1_{[t_i, t_{i+1}]}\rangle}{\sqrt{t_{i+1} - t_i}}, \qquad X_i'(\omega) = \frac{\langle \omega, 1_{[t_i', t_{i+1}']}\rangle}{\sqrt{t_{i+1}' - t_1'}}$$

$$Y_i(\omega) = \frac{\langle \omega, 1_{[-t_{i+1}, -t_i]}\rangle}{\sqrt{(t_{i+1} - t_i)}}, \qquad Y_i'(\omega) = \frac{\langle \omega, 1_{[-t_{i+1}', -t_i']}\rangle}{\sqrt{(t_{1+1}' - t_i')}} .$$

Since X_i and X_i' are independent of Y_i and Y_i', the expectation in (3.2) is equal to the product of (3.3) and (3.4) below, and (ii) will follow from these two displays once they have been proved.

(3.3)
$$E\prod_{i=1}^{r-1} H_{m_i}\left(\frac{X_i}{\sqrt{2}}\right) H_{m_i}\left(\frac{X_i'}{\sqrt{2}}\right) = \left(\frac{\prod_{i=1}^{r-1}(m_i!)^2}{(\sum_{i=1}^{r-1} m_i)!} 2^{\sum_{i=1}^{r-1} m_i}\right) \prod_{i=1}^{r-1}\left(\frac{t_{i+1} \wedge t_{i+1}' - t_i \vee t_i'}{\sqrt{(t_{i+1} - t_i)(t_{i+1}' - t_i')}}\right)^{m_i},$$

(3.4)
$$E\prod_{i=1}^{r-1} H_{n_i}\left(\frac{Y_i}{\sqrt{2}}\right) H_{n_i}\left(\frac{Y_i'}{\sqrt{2}}\right) = \left(\frac{\prod_{i=1}^{r-1}(n_i!)^2}{(\sum_{i=1}^{r-1} n_i)!} 2^{\sum_{i=1}^{r-1} n_i}\right) \prod_{i=1}^{r-1}\left(\frac{t_{i+1} \wedge t_{i+1}' - t_i \vee t_i'}{\sqrt{(t_{i+1} - t_i)(t_{i+1}' - t_i')}}\right)^{n_i}.$$

We prove (3.3), and (3.4) is proved in the same way. Each X_i and X_i' is $N(0, 1)$ distributed. In case $r = 2$, we apply the ready formula in Hida [7, p. 139] and the covariance

$$E(X_i X_i') = \frac{t_{i+1} \wedge t_{i+1}' - t_i \vee t_i'}{\sqrt{(t_{i+1} - t_i)(t_{i+1}' - i_i')}} .$$

In case $r > 2$, we appeal to the diagram formula listed in (for example) Major [14, p. 51]. Note that, in our case, the summation in the diagram

formula is reduced to only one term, namely the right-handed side of (3.3). This is due to the covariances $E(X_i X'_i)$ and

$$E(X_i X'_{i+1}) = \frac{(t_{i+1} - t'_{i+1})^+}{\sqrt{(t_{i+1} - t_i)(t'_{i+2} - t'_{i+1})}}, \qquad a^+ = \max(a, 0),$$

$$E(X_{i+1} X'_i) = \frac{(t'_{i+1} - t_{i+1})^+}{\sqrt{(t_{i+2} - t_{i+1})(t'_{i+1} - t'_i)}},$$

and that all other X_i and X'_i have zero correlations. In the above, we rely on the relation $t_{i+1}, t'_{i+1} > t_i, t'_i$.

The diagram formula also asserts the validity of (i), thus, the proof of Lemma 3.1 is completed. □

Now, we prove Proposition 3.1. By (2.2), the series expansion of (3.1) is

$$\frac{1}{(2\pi)^{r-1}} \sum_{\substack{(m_i, n_i) \\ i=1,\cdots,r-1}} \int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \frac{\prod_{i=1}^{r-1} \phi_i(m_i, n_i, t_i, t_{i+1})}{\prod_{i=1}^{r-1} (t_{i+1} - t_i)} dt_1 \cdots dt_r .$$

Let the integral in the above summation be denoted by $\psi(m_1, n_1, \cdots, m_{r-1}, n_{r-1})$. By Lemma 3.1 (i), $\psi(m_i, n_i)$'s are orthogonal in (L^2). By Lemma 3.1 (ii), the square of the (L^2) norm of $\psi(m_j, n_j)$ is not greater than a constant multiple of

$$\left(\frac{\prod_{i=1}^{r-1} m_i! n_i!}{(\sum_{i=1}^{r-1} m_i)!(\sum_{i=1}^{r-1} n_i)!(\prod_{i-1}^{r-1} m_i n_i)^{1/6}}\right) \int_{t_r, t'_r = a_r}^{b_r} \cdots \int_{t_1, t'_r = a_1}^{b_1} \tag{3.5}$$
$$\times \prod_{i=1}^{r-1} \frac{(t_{i+1} \wedge t'_{i+1} - t_i \vee t'_i)^{m_i + n_i}}{((t_{i+1} - t_i)(t'_{i+1} - t'_i))^{(m_i + n_i)/2 + 1}} dt_1 \cdots dt_t dt'_1 \cdots dt'_r .$$

(i) $r = 2$. That (3.5) is reduced to

$$\frac{1}{\sqrt[6]{mn}} \int_{t_2, t'_2} \int_{t_1, t'_1} \frac{(t_2 \wedge t'_2 - t_1 \vee t'_1)^{m+n}}{((t_2 - t_1)(t'_2 - t'_1))^{(m+n)/2+1}} dt_1 dt_2 dt'_1 dt'_2 . \tag{3.6}$$

The integration domain of t_1, t'_1, t_2, t'_2 is split into four parts. On $t_1 \leqslant t'_1 < t_2 \leqslant t'_2$, we integrate in the order of t'_2, t'_1, t_2, t_1 and find that the resultant is $\leqslant$ const. $/(m + n)^2$. On $t_1 \leqslant t'_1 < t'_2 \leqslant t_2$, we integrate in the order of t'_1, t'_2, t_2, t_1 and find that the same dominating factor $1/(m + n)^2$ appears. The other two parts are treated similarly. We conclude that (3.6) is dominated by a constant multiple of

$$\frac{1}{(m + n)^2 \sqrt[6]{mn}} \leqslant \frac{1}{4(mn)^{7/6}} .$$

Therefore, the series expansion of (3.7), for $r = 2$, converges in (L^2), and thus (3.1), for $r = 2$, is in (L^2).

(ii) $r > 2$. The integration domain of $t_1, t'_1, \cdots, t_r, t'_r$ is split into 2^r parts. On $t_1 \leqslant t'_1 < t_2 \leqslant t'_2 < \cdots < t_r \leqslant t'_r$, we integrate in the order of t'_r, $t'_{r-1}, \cdots, t'_1$ and in the meantime drop the terms involving $(t_i - t'_{i-1})/(t_i - t_{i-1})$, the resultant $\leqslant$ const. $/(m_{r-1} + n_{r-1}) \cdots (m_1 + n_1)$ (Compare with (i); we have no $(m_i + n_i)^2$ here). We treat all other parts similarly, and conclude that (3.5) is dominated by a constant multiple of

$$\frac{\prod_{i=1}^{r-1} m_i!\, n_i!}{(\sum_{i=1}^{r-1} m_i)!\,(\sum_{i=1}^{r-1} n_i)! \prod_{i=1}^{r-1}(m_i + n_i)\sqrt[6]{m_i n_i}}\,.$$

Using $m_i + n_i \geqslant 2\sqrt{m_i n_i}$ and Stirling formula, we see that the above is not greater than a constant multiple of

$$\left(\frac{\prod_{i=1}^{r-1} m_i^{m_i+1} n_i^{n_i+1}}{(\sum_{i=1}^{r-1} m_i)^{1/2+\Sigma m_i}(\sum_{i=1}^{r-1} n_i)^{1/2+\Sigma n_i}}\right)\frac{1}{(\prod_{i=1}^{r-1} m_i n_i)^{7/6}}\,.$$

Since the quantity in the parenthesis above is $\leqslant 1$, we have shown again that the series expansion of (3.1) converges in (L^2). This completes the proof. □

Remark. The above proof depends on the off-diagonal assumption $a_i > b_{i-1}$. In the diagonal case $\{t_1 \leqslant t_2 \leqslant \cdots \leqslant T\}$, we even cannot assert that (3.1) is in $(L^2)^-$, since the limiting case $\lim_{t \downarrow s} u(B(t) - B(s))$ is in general unknown. Thus, it seems challenging to "renormalize" (3.1) in this case.

Let $f(x)$ be a measurable function on R^2, which is locally square-integrable and is slowly increasing as $|x| \to \infty$. That $\log|x|$ is such a function. Then, for $0 \leqslant s \leqslant t$,

$$Ef^2(B(t) - B(s)) = \frac{1}{2\pi(t-s)}\int_{R^2} f^2(x)\exp\left(-\frac{x^2}{2(t-s)}\right)dx < \infty\,.$$

PROPOSITION 3.2. *Assume that $a_1 < b_1 < a_2 < b_2 \cdots$, u_i are given in Proposition* 3.1 *and $f(x)$ is the function mentioned above. Then, for each $t \in [a_{r+1}, b_{r+1}]$,*

$$\int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} f(B(t) - B(t_r)) \prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i))dt_1 \cdots dt_r \tag{3.8}$$

is in (L^2).

Proof. As in the proof of Proposition 3.1, the series expansion of (3.8) is

$$\frac{1}{(2\pi)^r} \sum_{\substack{(m_i,n_i)\\ i=1,\cdots,r}} \int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \frac{\prod_{i=1}^r \phi_i(m_i, n_i, t_i, t_{i+1})}{\prod_{i=1}^r (t_{i+1} - t_i)} dt_1 \cdots dt_r , \tag{3.9}$$

where $t_{r+1} = t$ and $\phi_r(m_r, n_r, t_r, t_{r+1})$ is the term corresponding to $f(B(t_{r+1} - B(t_r))$. To estimate the squares of (L^2) norms of the above summunds and their sum, we note that

$$\frac{1}{2\pi(t_{r+1} - t_r)} \sum_{m_r,n_r} \phi_r(m_r, n_r, t_r, t_{r+1})$$

is the series expansion of $f(B(t_{r+1}) - B(t_r))$ in (L^2). Thus,

$$\begin{aligned} &\sum_{m_r,n_r} E \prod_{i=1}^r \frac{\phi_i(m_i, n_i, t_i, t_{i+1})\phi_i(m_i, n_i, t'_i, t'_{i+1})}{(t_{i+1} - t_i)(t'_{i+1} - t_i)}, \quad t_{r+1} = t'_{r+1} = t, \\ &= \left[E \prod_{i=1}^{r-1} \cdots\right] \cdot (4\pi)^2 E[f(B(t) - B(t_r))f(B(t) - B(t'_r))] \\ &\leqslant \left[E \prod_{i=1}^{r-1} \cdots\right]\left[\frac{2\pi}{(t - b_r)} \int_{R^2} f^2(x) \exp\left(-\frac{x^2}{(t - a_r)}\right)dx\right]. \end{aligned} \tag{3.10}$$

Therefore, the arguments in the proof of Proposition 3.1 show that (3.9) converges in (L^2). □

§ 4. A Tanaka type formula and the proof of Theorem 1.1

Let $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{r+1} < b_{r+1}$ and $u_1, \cdots, u_{r-1} \in \mathscr{S}^*(R^2)$ be positive, we shall prove

THEOREM 4.1. *Let $f(x)$, $x = (x_1, x_2) \in R^2$, be a measurable function such that $f(x)$ itself and its distributional derivatives $\partial f/\partial x_1$, $\partial f/\partial x_2$ are all locally square-integrable and slowly increasing as $|x| \to \infty$. Let Δf be the distributional Liplacian of f. Assume also that $\Delta f \in \mathscr{S}^*(R^2)$ is positive. Then the following equality holds in (L^2).*

$$\begin{aligned} &\int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \left\{f(B(b_{r+1}) - B(t_r)) \prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i))\right\} dt_1 \cdots dt_r \\ &\quad - \int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \left\{f(B(a_{r+1}) - B(t_r)) \prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i))\right\} dt_1 \cdots dt_r \\ &= \sum_{j=1}^{2} \int_{a_{r+1}}^{b_{r+1}} \left(\int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \left\{\frac{\partial f}{\partial x_j}(B(t_{r+1}) - B(t_r)) \right.\right. \\ &\qquad \left.\left. \times \prod_{i=1}^{r-1} u_i(B(t_{i+1}) - B(t_i))\right\} dt_1 \cdots dt_r\right) dB_j(t_{r+1}) \end{aligned} \tag{4.1}$$

$$+\frac{1}{2}\int_{a_{r+1}}^{b_{r+1}}\cdots\int_{a_1}^{b_1}\Big\{(\Delta f)(B(t_{r+1})-B(t_r))$$
$$\times\prod_{i=1}^{r-1}u_i(B(t_{r+1})-B(t_i))\Big\}dt_1\cdots dt_{r+1}.$$

In the above, the first term on the right-handed side is an Ito's integral.

Proof. For each $t_1,\cdots,t_r$, fixed, we have the following Kubo's formula in $(L^2)^-$

$$\frac{d}{dt_{r+1}}\Big\{f(B(t_{r+1})-B(t_r))\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}$$
$$=\partial^*_{t_{r+1}}\Big\{(\nabla f)(B(t_{r+1})-B(t_r))\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}$$
$$+\frac{1}{2}\Big\{(\Delta f)(B(t_{r+1})-B(t_r))\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}.$$

The above equality can be proved by calculating the $\mathscr{S}$-transform of each term, as Kubo did. Now, integrating in $t_1,\cdots,t_{r+1}$, we have the following equality in $(L^2)^-$:

$$\text{(4.2)}\quad \int_{a_r}^{b_r}\cdots\int_{a_1}^{b_1}\Big\{f(B(b_{r+1})-B(t_r))\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}dt_1\cdots dt_r$$
$$-\int_{a_r}^{b_r}\cdots\int_{a_1}^{b_1}\Big\{f(B(a_{r+1})-B(t_r))\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}dt_1\cdots dt_r$$
$$=\int_{a_{r+1}}^{b_{r+1}}\cdots\int_{a_1}^{b_1}\Big\{\partial^*_{t_{r+1}}(\nabla f)(B(t_{r+1})-B(t_r))$$
$$\times\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}dt_1\cdots dt_r dt_{r+1}$$
$$+\frac{1}{2}\int_{a_{r+1}}^{b_{r+1}}\cdots\int_{a_1}^{b_1}\Big\{(\Delta f)(B(t_{r+1})-B(t_r))$$
$$\times\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}dt_1\cdots dt_r dt_{r+1}.$$

Propositions 3.1 and 3.2 assert respectively that the last term and the left-handed side of (4.2) are in (L^2). Therefore, the first term on the right-handed side is also in (L^2) and (4.2) becomes an a.s. eauality. By Proposition 3.2,

$$t_{r+1}\in[a_{r+1},b_{r+1}]$$
$$\to\int_{a_r}^{b_r}\cdots\int_{a_1}^{b_1}\Big\{\frac{\partial f}{\partial x_j}(B(t_{r+1})-B(t_r))\prod_{i=1}^{r-1}u_i(B(t_{i+1})-B(t_i))\Big\}dt_1\cdots dt_r$$

is an nonanticipating (L^2) process. Thus, according to Kubo [11, Theorem

2.1] or Hida [8, the concluding statement], the two first terms on the right-handed sides of (4.1) and (4.2) must be equal a.s., and (4.1) has been proved. □

To derive Theorem 1.1, we need the following lemma.

LEMMA 4.2. *The following equalities hold in* (L^2).

$$\text{(i)} \quad \int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{r-1} \delta_{x_i}(B(t_{i+1}) - B(t_i)) dt_1 \cdots dt_r$$
$$= \alpha_r(x, a_1, b_1, \cdots, a_r, b_r), \qquad x = (x_1, \cdots, x_{r-1}) \in R^{2(r-1)}.$$

$$\text{(ii)} \quad \int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} f(B(t) - B(t_r)) \prod_{i=1}^{r-1} \delta_{x_i}(B(t_{i+1}) - B(t_i)) dt_1 \cdots dt_r$$
$$= \int_{a_r}^{b_r} f(B(t) - B(t_r)) \alpha_r(x, a_1, b_1, \cdots, a_r, dt_r),$$

where $t \in [a_{r+1}, b_{r+1}]$ *and* $f(y)$ *is the function in Proposition* 3.2.

Proof. By Propositions 3.1 and 3.2, the left-handed sides of (i) and (ii) are in (L^2). Set

$$\alpha_r^\varepsilon(x, a_1, \cdots, b_r) = \int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{r-1} p_\varepsilon(B(t_{i+1}) - B(t_i) - x_i) dt_1 \cdots dt_r,$$

$$\beta_r^\varepsilon(x, a_1, \cdots, b_r)$$
$$= \int_{a_r}^{b_r} \cdots \int_{a_1}^{b_1} f(B(t) - B(t_r)) \prod_{i=1}^{r-1} p_\varepsilon(B(t_{i+1}) - B(t_i) - x_i) dt_1 \cdots dt_r,$$

where $p_\varepsilon(y)$ is the standard planar $N(0, \varepsilon)$ density function. On one hand, by computing the $\mathscr{S}$-transforms we see that α_r^ε and β_r^ε converge respectively in $(L^2)^-$ to the left-handed sides of (i) and (ii). On the other hand, by Rosen [17, Lemma 1 and p. 135] we see that α_r^ε converges in (L^2) to the right-handed side of (i) and β_r^ε converges pathwise to the right-handed side of (ii). This proves the assertion. □

We cannot derive (1.2) directly from Theorem 4.1, since $(x_1, x_2) \to x_i/(x_1^2 + x_2^2)$ is merely in $L^1_{\text{loc}}(R^2)$ not in $L^2_{\text{loc}}(R^2)$. However, we can prove (1.2) as follows. Let $f_\varepsilon(y) = (1/2\pi) \log (|y|^2 + \varepsilon^2)^{1/2}$, $0 < \varepsilon \leqslant 1$ and $y \in R^2$. For each $x = (x_1, \cdots, x_{r+1}) \in R^{2(r-1)}$ and $x_r \in R^2$, by Theorem 4.1 and Lemma 4.2 the following equality holds in (L^2).

$$\text{(4.3)} \quad \int_{a_r}^{b_r} f_\varepsilon(B(b_{r+1}) - B(t_r) - x_r) \alpha_r(x, a_1, \cdots, a_r, dt_r)$$
$$- \int_{a_r}^{b_r} f_\varepsilon(B(a_{r+1}) - B(t_r) - x_r) \alpha_r(x, a_1, \cdots, a_r, dt_r)$$

$$= \int_{a_{r+1}}^{b_{r+1}} \left(\int_{a_r}^{b_r} (\nabla f_\varepsilon)(B(t_{r+1}) - B(t_r) - x_r)\alpha_r(x, a_1, \cdots, a_r, dt_r) \right) \cdot dB(t_{r+1})$$
$$+ \frac{1}{2} \int_{y \in R^2} (\Delta f_\varepsilon)(y - x_r)\alpha_{r+1}((x, y), a_1, \cdots, b_{r+1})dy,$$

where the last term is due to the definition of α_{r+1} and the fact that Δf_ε is a positive continuous function. Since

$$\int_{y \in R^2} \left(f_\varepsilon(y - x_r) - \frac{1}{2\pi} \log|y - x_r| \right)^2 \exp\left(-\frac{|y|^2}{2s} \right) dy \to 0 \qquad \text{for all } s > 0,$$

by Lemma 4.2 and (3.10) in the proof of Proposition 3.2 the left-handed side of (4.3) converges in (L^2) to that of (1.2) as $\varepsilon \downarrow 0$. Since $(\Delta f_\varepsilon)(y) = (\varepsilon^2/\pi)(1/(|x|^2 + \varepsilon^2)$, $0 < \varepsilon \leqslant 1$, is an approximation to the identity on R^2, by Rosen [17, Lemma 1 and its proof], the last term of (4.3) converges in (L^2) to $\alpha_{r+1}((x, x_r), a_1, \cdots, b_{r+1})$ as $\varepsilon \downarrow 0$. Therefore, the stochastic integral term of (4.3) must also converge in (L^2). By the isometry property of stochastic integrals, the limit must be of the form

$$\int_{a_{r+1}}^{b_{r+1}} \phi_r(t_{r+1}, \omega) \cdot dB(t_{r+1}, \omega)$$

where $\phi_r(t_{r+1}, \omega)$ is the $L^2(dt_{r+1} \times dP)$ limit of

$$\int_{a_r}^{b_r} \frac{B(t_{r+1}) - B(t_r) - x_r}{(2\pi)(|B(t_{r+1}) - B(t_r) - x_r|^2 + \varepsilon^2)} \alpha_r(x, a_1, \cdots, a_r, dt_r)$$

since $(\nabla f_\varepsilon)(y) = y/(2\pi)(|y|^2 + \varepsilon^2)$. Note that $1/(|y|^2 + \varepsilon^2) \uparrow 1/|y|^2$. Thus, by the monotone convergence, for a.e. t_{r+1}

$$\int_{a_r}^{b_r} \frac{B(t_{r+1}, \omega) - B(t_r, \omega) - x_r}{(2\pi)|B(t_{r+1}, \omega) - B(t_r, \omega) - x_r|^2} \alpha_r(x, a_1, \cdots, a_r, dt_r, \omega)$$
$$= \phi_r(t_{r+1}, \omega), \text{ which is of finite value for a.s. } \omega.$$

Hence, we have proved that the integrand of the stochastic integral in (1.2) is well-defined and that (1.2) holds in (L^2). □

§ 5. An explicit form of $\tilde{\alpha}_r(x, T)$

We have mentioned in § 1 that, in the diagonal case, we intend to substract off certain "infinite part" from $\alpha_r(x, T)$ so that the remainder $\tilde{\alpha}_r(x, T)$ will have a jointly continuous extension to all $(x, T) \in R^{2(r-1)} \times R_+$. In this section, we shall discuss how (1.2) suggests an explicit form of such a "cut-off". We shall show that the explicit forms in Yor [23, $r = 2$]

and Rosen-Yor [20, $r = 3$] can be extended to general r. Recall that

$$\tilde{\alpha}_2(x, T) = \alpha_2(x, T) - \frac{T}{\pi} \log \frac{1}{|x|}, \qquad x \in R^2 \text{ and } x \neq 0.$$

Renormalization of α_r. For each $r \geqslant 3$, suppose that $\tilde{\alpha}_2, \cdots, \tilde{\alpha}_{r-1}$ have been constructed. Set

$$N(x) = \frac{1}{\pi} \log \frac{1}{|x|}, \qquad x \neq R^2 \text{ and } x \neq 0.$$

Then, we can construct $\tilde{\alpha}_r$ as follows.

$$\begin{aligned}
(5.2) \qquad \tilde{\alpha}_r(x_1, \cdots, x_{r-1}, T) &= \alpha_r(x_1, \cdots, x_{r-1}, T) \\
&- \sum_{i=1}^{r} N(x_i)\tilde{\alpha}_{r-1}(\overbrace{x_1, \cdots, x_{r-1}}^{i}, T) \\
&- \sum_{i,j=1}^{r} N(x_i)N(x_j)\tilde{\alpha}_{r-2}(\overbrace{x_1, \cdots, x_{r-1}}^{i,j}, T) \\
&- \sum_{i,j,k=1}^{r} N(x_i)N(x_j)N(x_k)\tilde{\alpha}_{r-3}(\overbrace{x_1, \cdots, x_{r-1}}^{i,j,k}, T) \\
&\quad \vdots \\
&- N(x_1) \cdots N(x_{r-1})T;
\end{aligned}$$

in the above, $\overbrace{x_1, \cdots, x_{r-1}}^{i}$ indicates that the i-th one has been deleted. etc.

Thus, in case $r = 3$, we have

$$\begin{aligned}
(5.3) \qquad \tilde{\alpha}_3(x_1, x_2, T) &= \alpha_3(x_1, x_2, T) - \frac{1}{\pi} \log \frac{1}{|x_1|} \tilde{\alpha}_2(x_2, T) \\
&- \frac{1}{\pi} \log \frac{1}{|x_2|} \tilde{\alpha}_2(x_1, T) - \frac{T}{\pi^2} \log \frac{1}{|x_1|} \log \frac{1}{|x_2|},
\end{aligned}$$

which is consistant with the one obtained by Rosen-Yor [20]. We show in the below how to obtain $\tilde{\alpha}_4$ from $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ by using Theorem 1.1 (it is in the same way to obtain $\tilde{\alpha}_3$ from $\tilde{\alpha}_2$). First, we may rewrite (1.2) as follows. For each $T > 0$, each $x_1, x_2, x_3 \in R^2$ and $x_i \neq 0$, with probability one

$$\begin{aligned}
(5.4) \qquad \int_0^T \log |B(T) - B(t) - x_3| \alpha_3(x_1, x_2, dt) \\
- \int_0^T \log |B(t) - B(t) - x_3| \alpha_3(x_1, x_2, dt) \\
= \int_0^T \int_0^u \frac{B(u) - B(t) - x_3}{|B(u) - B(t) - x_3|^2} \alpha_3(x_1, x_2, dt) \cdot dB(u) + \pi\alpha_4(x_1, x_2, x_3, T).
\end{aligned}$$

The proof of the above equality is parallel to that of (1.2), with the simple observation that $\lim_{t\downarrow 0}(1/t)\exp(-a^2/2t)=0$ for all $a\neq 0$. Now, we write $\tilde{\alpha}_3(x_1, x_2, dt)=\alpha_3(x_1, x_2, dt)-N_3(x_1, x_2, dt)$, where N_3 denotes the "infinite part". Thus, we have, by (5.4),

$$
\begin{aligned}
(5.5)\quad & \int_0^T \log|B(T)-B(t)-x_3|\tilde{\alpha}_3(x_1, x_2, dt) \\
& \quad -\int_0^T\int_0^u \frac{B(u)-B(t)-x_3}{|B(u)-B(t)-x_3|^2}\tilde{\alpha}_3(x_1, x_2, dt)\cdot dB(u) \\
& = \pi\alpha_4(x_1, x_2, x_3, T)-\log\frac{1}{|x_3|}\tilde{\alpha}_3(x_1, x_2, T)-\log\frac{1}{|x_3|}N_3(x_1, x_2, T) \\
& \quad -\Big\{\int_0^T \log|B(T)-B(t)-x_3|N_3(x_1, x_2, dt) \\
& \qquad -\int_0^T\int_0^u \frac{B(u)-B(t)-x_3}{|B(u)-B(t)-x_3|^2}N_3(x_1, x_2, dt)\cdot dB(u)\Big\}\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
N_3(x_1, x_2, dt) &= \frac{1}{\pi}\log\frac{1}{|x_1|}\tilde{\alpha}_2(x_2, dt) \\
&\quad +\frac{1}{\pi}\log\frac{1}{|x_2|}\tilde{\alpha}_2(x_1, dt) \\
&\quad +\frac{1}{\pi}\log\frac{1}{|x_1|}\log\frac{1}{|x_2|}dt\,,
\end{aligned}
$$

and

$$
\tilde{\alpha}_2=\alpha_2-N_2, \qquad N_2(x, dt)=\frac{1}{\pi}\log\frac{1}{|x|}dt\,,
$$

we see that the last term of (5.5) is equal to

$$
\begin{aligned}
& \frac{1}{\pi}\log\frac{1}{|x_1|}\Big\{\int_0^T \log|B(T)-B(t)-x_3|\alpha_2(x_2, dt) \\
& \qquad -\int_0^T\int_0^u \frac{B(u)-B(t)-x_3}{|B(u)-B(t)-x_3|^2}\alpha_2(x_2, dt)\cdot dB(u)\Big\} \\
& +\frac{1}{\pi}\log\frac{1}{|x_2|}\Big\{\int_0^T \log|B(T)-B(t)-x_3|\alpha_2(x_1, dt) \\
& \qquad -\int_0^T\int_0^u \frac{B(u)-B(t)-x_3}{|B(u)-B(t)-x_3|^2}\alpha_2(x_1, dt)\cdot dB(u)\Big\} \\
& +\frac{1}{\pi^2}\log\frac{1}{|x_1|}\log\frac{1}{|x_2|}\Big\{\int_0^T \log|B(T)-B(t)-x_3|dt \\
& \qquad -\int_0^T\int_0^u \frac{B(u)-B(t)-x_3}{|B(u)-B(t)-x_3|^2}dt\cdot dB(u)\Big\}
\end{aligned}
$$

$$= \frac{1}{\pi} \log \frac{1}{|x_1|} \left\{ \pi\alpha_3(x_2, x_3, T) - \log \frac{1}{|x_3|} \alpha_2(x_2, T) \right\}$$
$$+ \frac{1}{\pi} \log \frac{1}{|x_2|} \left\{ \pi\alpha_3(x_1, x_3, T) - \log \frac{1}{|x_3|} \alpha_2(x_1, T) \right\}$$
$$+ \frac{1}{\pi^2} \log \frac{1}{|x_1|} \log \frac{1}{|x_2|} \left\{ \pi\alpha_2(x_3, T) - \log \frac{1}{|x_3|} T \right\},$$

by (5.4) with α_3 and α_4 being replaced by α_2 and α_3.

Substituting $\alpha_3(x_2, x_3, T) = \tilde{\alpha}_3(x_2, x_3, T) + N_3(x_2, x_3, T)$ etc. into the above display, we see that the right-handed side of (5.5) is equal to

$$\pi\alpha_4(x_1, x_2, x_3, T) - \log \frac{1}{|x_1|} \tilde{\alpha}_3(x_2, x_3, T) - \log \frac{1}{|x_2|} \tilde{\alpha}_3(x_1, x_3, T)$$
$$- \log \frac{1}{|x_3|} \tilde{\alpha}_3(x_1, x_2, T) - \frac{1}{\pi} \log \frac{1}{|x_1|} \log \frac{1}{|x_2|} \tilde{\alpha}_2(x_3, T)$$
$$- \frac{1}{\pi} \log \frac{1}{|x_1|} \log \frac{1}{|x_3|} \tilde{\alpha}_2(x_2, T) - \frac{1}{\pi} \log \frac{1}{|x_2|} \log \frac{1}{|x_3|} \tilde{\alpha}_2(x_1, T)$$
$$- \frac{1}{\pi^2} \log \frac{1}{|x_1|} \log \frac{1}{|x_2|} \log \frac{1}{|x_3|} T,$$

which we shall denote by $\pi\tilde{\alpha}_4(x_1, x_2, x_3, T)$. This is just the form $\tilde{\alpha}_4$ which we intend to find.

Now, we consider the left-handed side of (5.5). The following two multiparameter functionals are well-defined for $T > 0$, $0 < \varepsilon \leqslant 1$ and $x_i \in R^2$

$$\Phi_1(T, \varepsilon, x_1, x_2, x_3) = \int_0^T \log (|B(T) - B(t) - x_3|^2 + \varepsilon^2)^{1/2} \tilde{\alpha}_3(x_1, x_2, dt)$$
$$\Phi_2(T, \varepsilon, x_1, x_2, x_3) = \int_0^T \int_0^u \frac{B(u) - B(t) - x_3}{|B(u) - B(t) - x_3|^2 + \varepsilon^2} \tilde{\alpha}_3(x_1, x_2, dt) \cdot dB(u).$$

According to Rosen [17], to show that the left-handed side of (5.5) is jointly continuous in T and x_1, x_2, x_3 it suffices to show that for any even integer m

$$E|\Phi_i(T, \varepsilon, x_1, x_2, x_3) - \Phi_i(T', \varepsilon', x_1', x_2', x_3')|^m$$
$$\leqslant C_{m,\alpha} |(T, \varepsilon, x_1, x_2, x_3) - (T', \varepsilon', x_1', x_2', x_3')|^{m\alpha},$$

where α is some positive real number independent on m. However, since the above display involves the higher moments of random variables, it seems that we cannot establish it by merely using white noises analysis,

although the latter does have helped us to prove (1.2).

Finally, we mention that the renormalizations of $\alpha_r(x, T)$ have also been investigated by Dynkin [3] and Rosen [19]. We do not know whether these seemingly different renormalizations are in fact mutually inducible.

Add in the proof. It has been found that the arguments for proving Proposition 3.1, especially Lemma 3.1, are not perfectly right. The corrections together with some improvements will appear in "A W.N.C. Viewpoint on Intersection Local Times" (Proceedings on Gaussian Random Fields Conference, World Scientific 1991).

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