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EXPLICIT DESCRIPTIONS OF TRACE RINGS OF GENERIC 2 BY 2 MATRICES

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§ 1. Introduction

Let K be a field of characteristic zero and let

$$X_1 = (x_{ij}(1)), \cdots, X_m = (x_{ij}(m)), \qquad m \geq 2,$$

be m generic n by n matrices over K. That is, $x_{ij}(k)$ are independent commuting indeterminates over K. The K-subalgebra generated by $X_1, \cdots, X_m$ is called a ring of n by n generic matrices and is denoted by $R(n, m)$. Let $M_n(K[x_{ij}(k)])$ denote the n by n matrix algebra over the polynomial ring $K[x_{ij}(k)]$. The ring $R(n, m)$ is a K-subalgebra of $M_n(K[x_{ij}(k)])$. Let $C(n, m)$ be the subring of the polynomial ring $K[x_{ij}(k)]$ generated by all traces $\mathrm{Tr}(X_{i_1} \cdots X_{i_a})$, where $X_{i_1} \cdots X_{i_a}$ is a monomial in the generic matrices $X_1, \cdots, X_m$. The trace ring $T(n, m)$ of m generic n by n matrices is the K-subalgebra of $M_n(K[x_{ij}(k)])$ generated by $R(n, m)$ and $C(n, m)$. Here we identify elements of $C(n, m)$ with scalar matrices.

In this paper we will be concerned with the trace ring $T(2, m)$ of generic 2 by 2 matrices. L. Le Bruyn [1. Chap. 3, Theorem 5.1] proved that $T(2, m)$ is a Cohen-Macaulay module over $C(n, m)$. Apart from this general result, very little is known about explicit structure on $T(2, m)$. Explicit descriptions of $T(2, m)$ are known only for $m \leq 4$ (cf. [2], [3], [4]) and except these cases nothing is known on an explicit description of $T(2, m)$. In this paper we will give explicit descriptions of $T(2, m)$ for all m.

A Young tableau on numbers $1, 2, \cdots, m$

$$Y = \begin{bmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{bmatrix}$$

is called standard if the entries strictly increase down columns and non-decrease across rows. Let $X_1, \cdots, X_m$ be m generic 2 by 2 matrices. We

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denote by $\mathrm{Tr}(Y)$ the element $\mathrm{Tr}(X_{i_1}X_{j_1})\,\mathrm{Tr}(X_{i_2}X_{j_2})\cdots\mathrm{Tr}(X_{i_r}X_{j_r})$ of $C(2, m)$. A standard monomial of $T(2, m)$ is an element of the form

$$\mathrm{Tr}(Y)\,\mathrm{Tr}(X_1)^{\alpha_1}\cdots\mathrm{Tr}(X_m)^{\alpha_m}X_1^{\beta_1}\cdots X_m^{\beta_m}\,,$$

where α_i, β_i are non-negative integers and Y is a standard tableau. We include the case that the shape of Y is the empty Young diagram, and in that case we set $\mathrm{Tr}(Y) = 1$. An S-standard monomial of $T(2, m)$ is an element of the form

$$\mathrm{Tr}(Y)X_{i_1}X_{i_2}\cdots X_{i_k}\,,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, $k \geq 0$, and Y is an S-standard Young tableau. Here if $k = 0$, we set $X_{i_1}X_{i_2}\cdots X_{i_k} = 1$. The definition of an S-standard Young tableau is given in the next section. Let $p_3, \cdots, p_{2m-1}$ be the elements of $C(2, m)$ defined by

$$(1.1)\qquad p_k = \textstyle\sum_{i+j=k}\mathrm{Tr}(X_iX_j)\,, \qquad 3 \leq k \leq 2m-1\,,$$

and denote by $B(2, m)$ the subring of $C(2, m)$ generated by

$$\mathrm{Tr}(X_i)\,,\quad \mathrm{Tr}(X_i^2)\,,\quad 1 \leq i \leq m\,,\quad \text{and}\quad p_k\,,\quad 3 \leq k \leq 2m-1\,.$$

Then it can be easily verified that the elements above are algebraically independent over K and hence $B(2, m)$ is a polynomial ring.

C. Procesi [5] founded a K-basis for $T(2, m)$. The following theorem gives a natural K-basis for $T(2, m)$:

THEOREM 1. *The set of standard monomials of $T(2, m)$ is a K-basis of $T(2, m)$ over the polynomial ring $B(2, m)$.*

The main result of this paper is the following

THEOREM 2. *The set of S-standard monomials of $T(2, m)$ is a basis of $T(2, m)$ over the polynomial ring $B(2, m)$.*

§ 2. S-standard Young tableaux

Consider the finite subset Λ_m of N^2:

$$\Lambda_m = \{(i, j) \in N^2 \mid 1 \leq i < j \leq m\}\,.$$

The set Λ_m is a partially ordered set by defining

$$(i, j) \leq (k, l) \Leftrightarrow i \leq k \quad \text{and} \quad j \leq l\,.$$

We denote the Hasse diagram associated with the pratially ordered set

Λ_m also by Λ_m, and assign to every edge in Λ_m a natural number according to the following rule:

$$\mu((i,j),\ (i,j+1)) = 2(j-1)$$

and

$$\mu((i,j),\ (i+1,j)) = 2i+1\,.$$

Moreover we assign to each maximal chain in Λ_m

$$(1,2) = (i_0,j_0) < (i_1,j_1) < \cdots < (i_{2m-4},j_{2m-4}) = (m-1,m)$$

a standard Young tableau

$$\begin{bmatrix} \cdots & i_\alpha & \cdots \\ \cdots & j_\alpha & \cdots \end{bmatrix} \qquad \alpha \in S\,,$$

where S is the subset of indices $\alpha \in \{0, 1, \cdots, 2m-4\}$ such that

$$\mu((i_{\alpha-1}, j_{\alpha-1}),\ (i_\alpha, j_\alpha)) > \mu((i_\alpha, j_\alpha),\ (i_{\alpha+1}, j_{\alpha+1}))\,.$$

We call a standard tableau, obtained as above, an S-standard tableau.

§ 3. Grassmannian Gr (2, *m*) and Procesi's identity

Let Gr $(2, m)$ be the Grassmannian of the 2-dimensional K-vector spaces of an m-dimensional fixed K-vector space. The homogeneous coordinate ring $K[\mathrm{Gr}\,(2, m)]$ of Gr $(2, m)$ is generated by the Prücker coordinates p_{ij}, $1 \leq i < j \leq m$. A monomial in the Prücker coordinates

$$p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_r j_r}$$

is called a standard monomial if the associated Young tableau

$$\begin{bmatrix} i_1 i_2 \cdots i_r \\ j_1 j_2 \cdots j_r \end{bmatrix}$$

is standard. Let

$$\theta_k = \textstyle\sum_{i+j=k} p_{ij}\,, \qquad \text{for } k = 3, 4, \cdots, 2m-1\,.$$

Then θ_3, θ_4, $\cdots$, θ_{2m-1} are algebraically independent over K. We now recall the following basic results on the homogeneous coordinate ring of the Grassmannian Gr $(2, m)$.

PROPOSITION 1 (cf. [6]). *The set of standard monomials is a K-basis of the homogeneous coordinate ring* $K[\mathrm{Gr}\,(2, m)]$.

PROPOSITION 2 (cf. [7]). *The homogeneous coordinate ring* $K[\mathrm{Gr}(2, m)]$ *is a free module of finite rank over the polynomial ring* $K[\theta_3, \theta_4, \cdots, \theta_{2m-1}]$ *and the set of standard monomials associated with S-standard tableaux is a basis of* $K[\mathrm{Gr}(2, m)]$ *over* $K[\theta_3, \theta_4, \cdots, \theta_{2m-1}]$.

We make $K[\mathrm{Gr}(2, m)]$ into a graded ring by giving each p_{ij} degree 2. Denoting by $K[\mathrm{Gr}(2, m)]_d$ the K-vector space of degree d-part, we consider the Poincare series associated with $K[\mathrm{Gr}(2, m)]$:

$$P(K[\mathrm{Gr}(2, m)], t) = \sum_{d\geq 0} \dim K\{\mathrm{Gr}(2, m)]_d t^d .$$

By Proposition 1, we have

$$P(K[\mathrm{Gr}(2, m)], t) = \sum_{d\geq 0} \# \begin{Bmatrix} \text{standard monomials of} \\ K[\mathrm{Gr}(2, m)] \text{ with degree } d \end{Bmatrix} t^d . \tag{3.1}$$

The trace ring $T(2, m)$ of m generic 2 by 2 matrices is also a graded ring by giving each $x_{ij}(k)$ degree 1. Denoting by $T(2, m)_d$ the K-vector space of $T(2, m)$ spanned by all homogeneous elements of degree d, we consider the Poincare series of $T(2, m)$:

$$P(T(2, m), t) = \sum_{d\geq 0} \dim T(2, m)_d t^d .$$

C. Procesi discovered the following identity between $P(T(2, m), t)$ and $P(K[\mathrm{Gr}(2, m), t)$:

PROPOSITION 3 (C. Procesi).

$$P(T(2, m), t) = (1 - t)^{-2m} P(K[\mathrm{Gr}(2, m)], t) . \tag{3.2}$$

For the proof we refer the reader to [1. Chap. 5] or [4, Proposition 8.1]. Procesi used a sort of Pieri's formula. A direct proof is given in [4].

§ 4. The Streightening formula

In this section we will prove Theorem 1. Let $X_1, \cdots, X_m$ be m generic 2 by 2 matrices. The matrices $X_1^0, \cdots, X_m^0$ defined by

$$X_i^0 = X_i - \frac{1}{2} \mathrm{Tr}(X_i) , \qquad \text{for } i = 1, \cdots, m ,$$

are called 2 by 2 generic trace zero matrices. The K-subalgebra of $T(2, m)$ generated by $X_1^0, \cdots, X_m^0$ and all traces of the monomials in X_i^0, $1 \leq i \leq m$, is called the ring of m generic 2 by 2 trace zero matrices, and will be denoted by $T^0(2, m)$. The trace ring $T(2, m)$ is clearly a polynomial ring over $T^0(2, m)$:

$$(4.1) \qquad T(2, m) = T^0(2, m)[\mathrm{Tr}\,(X_1), \cdots, \mathrm{Tr}\,(X_m)]\,.$$

By using the Cayley-Hamilton formula for 2 by 2 matrices, it can be easily shown that $T^0(2, m)$ is generated by $X_1^0, \cdots, X_m^0$ and they satisfy the following relation:

$$(4.2) \qquad X_i^0X_j^0 + X_j^0X_i^0 = \mathrm{Tr}\,(X_i^0X_j^0)\,, \qquad \text{for all } i, j\,.$$

Using the relation (4.2), we see that any element of $T^0(2, m)$ is a K-linear combination of monomials of the form

$$(4.3) \qquad \begin{gathered} \mathrm{Tr}\,(X_{i_1}^0X_{j_1}^0) \cdots \mathrm{Tr}\,(X_{i_r}^0X_{j_r}^0)X_{k_1}^0 \cdots X_{k_t}^0\,, \\ 1 \le i_\alpha \le j_\alpha \le m\,, \quad 1 \le k_\beta \le m\,, \quad \text{and} \quad r, t \ge 0\,. \end{gathered}$$

We call $X_{k_1}^0 \cdots X_{k_t}^0$ the matrix part and $\mathrm{Tr}\,(X_{i_1}^0X_{j_1}^0) \cdots \mathrm{Tr}\,(X_{i_r}^0X_{j_r}^0)$ the trace part. If $t = 0$ (resp. $r = 0$), we set

$$X_{k_1}^0 \cdots X_{k_t}^0 \qquad (\text{resp. } \mathrm{Tr}\,(X_{i_1}^0X_{j_1}^0) \cdots \mathrm{Tr}\,(X_{i_r}^0X_{j_r}^0) = 1)\,.$$

Using the relation (4.2) again, we can normalize the matrix part of (4.3) into regular order. Therefore any element of $T^0(2, m)$ is a K-linear combination of monomials of the form

$$(4.4) \qquad \mathrm{Tr}\,(X_{i_1}^0X_{j_1}^0)\,\mathrm{Tr}\,(X_{i_2}^0X_{j_2}^0) \cdots \mathrm{Tr}\,(X_{i_r}^0X_{j_r}^0)(X_1^0)^{\alpha_1} \cdots (X_m^0)^{\alpha_m}\,,$$

with $\alpha_i \in \mathbf{N}$, $i_\alpha < j_\alpha$, for all α, and $1 \le i_1 \le i_2 \le \cdots \le i_r$. Such a monomial is called a semi-standard monomial, and a semi-standard monomial is called a standard monomial if the Young tableau associated with its trace part is a standard tableau.

Proof of Theorem 1. First, we prove that any semi-standard monomial of $T^0(2, m)$ is a K-linear combination of standard monomials. Take a semi-standard monomial (4.4) with degree d and let

$$\underline{a} = (\underbrace{1 \cdots 1}_{\alpha_1}, \underbrace{2 \cdots 2}_{\alpha_2}, \cdots, \underbrace{m \cdots m}_{\alpha_m})\,.$$

We insert the numbers $i_1, j_1, i_2, j_2, \cdots, i_r, j_r$ into the sequence $\underline{a}$ as follows: if $i_1 = \cdots = i_k < i_{k+1}$ for some k, insert the numbers $i_1, j_1, \cdots, i_k, j_k$ into $\underline{a}$ by means the rule below and we get a sequence $\underline{a}[i_1j_1 \cdots i_kj_k]$ of numbers;

$$\underline{a}[i_1 j_1 \cdots i_k j_k] = (\cdots, \underbrace{i_1 \cdots i_1}_{\alpha_{i_1}}, i_1 j_1 \cdots i_k j_k, \underbrace{i_1 + 1 \cdots i_1 + 1}_{\alpha_{i_1+1}}, \cdots).$$

Repeating this procedure successibly, we obtain a sequence of numbers $\underline{a}[i_1 j_1 \cdots i_r j_r] \in N^d$ and call it the content of f (denoted by $c(f)$). For example, if $f = \operatorname{Tr}(X_1^0 X_2^0) X_1^0 X_2^0$, we have $c(f) = (1, 1, 2, 2)$.

The following identity on 2 by 2 trace zero matrices $X_1, \cdots, X_4$ is a consequence of the Cayley-Hamilton theorem for 2 by 2 matrices.

$$\begin{aligned} (4.5) \quad & \operatorname{Tr}(X_1X_2)\operatorname{Tr}(X_3X_4) \\ &= \operatorname{Tr}(X_1X_3)\operatorname{Tr}(X_2X_4) - \operatorname{Tr}(X_1X_4)\operatorname{Tr}(X_2X_3) - 4X_1X_2X_3X_4 \\ &\quad + 2\{\operatorname{Tr}(X_1X_2)X_3X_4 + \operatorname{Tr}(X_3X_4)X_1X_2 - \operatorname{Tr}(X_1X_3)X_2X_4 \\ &\quad - \operatorname{Tr}(X_2X_4)X_1X_3 + \operatorname{Tr}(X_1X_4)X_2X_3 + \operatorname{Tr}(X_2X_3)X_1X_4\}. \end{aligned}$$

Suppose now that a semi-standard monomial (4.4) is not a standard monomial. Then there exists a number k such that

$$i_k < i_{k+1} < j_{k+1} < j_k .$$

Then applying the identity (4.5) to $\operatorname{Tr}(X_{i_k}X_{j_k})\operatorname{Tr}(X_{i_{k+1}}X_{j_{k+1}})$, we obtain:

$$\begin{aligned} (4.6) \quad & \operatorname{Tr}(X_{i_k}^0 X_{j_k}^0)\operatorname{Tr}(X_{i_{k+1}}^0 X_{j_{k+1}}^0)(X_1^0)^{\alpha_1} \cdots (X_m^0)^{\alpha_m} \\ &= \operatorname{Tr}(X_{i_k}^0 X_{i_{k+1}}^0)\operatorname{Tr}(X_{j_{k+1}}^0 X_{j_k}^0)(X_1^0)^{\alpha_1} \cdots (X_m^0)^{\alpha_m} \\ &\quad - \operatorname{Tr}(X_{i_k}^0 X_{j_{k+1}}^0)\operatorname{Tr}(X_{i_{k+1}}^0 X_{j_k}^0)(X_1^0)^{\alpha_1} \cdots (X_m^0)^{\alpha_m} \\ &\quad - 4(X_1^0)^{\alpha_1} \cdots (X_t^0)^{\alpha_t}(X_{i_k}^0 X_{j_k}^0 X_{i_{k+1}}^0 X_{j_{k+1}}^0)(X_{t+1}^0)^{\alpha_{t+1}} \cdots (X_m^0)^{\alpha_m} \\ &\quad + 2\{\operatorname{Tr}(X_{i_k}^0 X_{j_k}^0)(X_1^0)^{\alpha_1} \cdots (X_t^0)^{\alpha_t}(X_{i_{k+1}}^0 X_{j_{k+1}}^0)(X_{t+1}^0)^{\alpha_{t+1}} \cdots (X_m^0)^{\alpha_m} \\ &\quad + \operatorname{Tr}(X_{i_{k+1}}^0 X_{j_{k+1}}^0)(X_1^0)^{\alpha_1} \cdots (X_t^0)^{\alpha_t}(X_{i_k}^0 X_{j_k}^0)(X_{t+1}^0)^{\alpha_{t+1}} \cdots (X_m^0)^{\alpha_m} \\ &\quad - \operatorname{Tr}(X_{i_k}^0 X_{i_{k+1}}^0)(X_1^0)^{\alpha_1} \cdots (X_t^0)^{\alpha_t}(X_{j_k}^0 X_{j_{k+1}}^0)(X_{t+1}^0)^{\alpha_{t+1}} \cdots (X_m^0)^{\alpha_m} \\ &\quad - \operatorname{Tr}(X_{j_{k+1}}^0 X_{j_k}^0)(X_1^0)^{\alpha_1} \cdots (X_t^0)^{\alpha_t}(X_{i_k}^0 X_{i_{k+1}}^0)(X_{t+1}^0)^{\alpha_{t+1}} \cdots (X_1^0)^{\alpha_m} \\ &\quad + \operatorname{Tr}(X_{i_k}^0 X_{j_{k+1}}^0)(X_1^0)^{\alpha_1} \cdots (X_0^t)^{\alpha_t}(X_{j_k}^0 X_{i_{k+1}}^0)(X_{t+1}^0)^{\alpha_{t+1}} \cdots (X_m^0)^{\alpha_m} \\ &\quad + \operatorname{Tr}(X_{i_{k+1}}^0 X_{j_k}^0)(X_1^0)^{\alpha_1} \cdots (X_t^0)^{\alpha_t}(X_{i_k}^0 X_{j_{k+1}}^0)(X_{t+1}^0)^{\alpha_{t+1}} \cdots (X_m^0)^{\alpha_m}, \end{aligned}$$

where $t = j_{k+1} - 1$.

Substitute the relation (4.6) into (4.4). Then applying the relation (4.2), we see that the semi-standard monomial f is a linear combination of monomials of the following types:

(1) semi-standard monomials with lexicographically smaller contents than that of f, and (2) the monomial

$$\begin{aligned} (4.7) \qquad & \operatorname{Tr}(X_{i_1}^0 X_{j_1}^0) \cdots \operatorname{Tr}(X_{i_k}^0 X_{j_k}^0)\operatorname{Tr}(X_{i_{k+2}}^0 X_{j_{k+2}}^0) \cdots \operatorname{Tr}(X_i^0 X_j^0) \\ & \times (X_1)^{\alpha_1} \cdots (X_t)^{\alpha_t} X_{i_{k+1}} X_{j_{k+1}} (X_{t+1})^{\alpha_{t+1}} \cdots (X_m)^{\alpha_m}. \end{aligned}$$

Using again (4.2), we make the monomial (4.7) into a semi-standard monomial g. Then the content of g is equal to $c(f)$ or lexicographically smaller than $c(f)$. If $c(g) = c(f)$, then one sees immediately that the degree of the trace part of g is smaller than that of f. We repeat this process. Then the process terminates within finitely many steps. Therefore any semi-standard monomial is a linear combination of standard monomials. To finish the proof, we have to show that the standard monomials are linearly independent. To do so, we employ the following convention: for given formal power series,

$$f(t) \leq g(t) \quad \text{means that} \quad a_i \leq b_i \quad \text{for all } i .$$

Because any element of $T^0(2, m)$ is a linear combination of standard monomials, we have

$$P(T^0(2, m), t) \leq \textstyle\sum_{d \geq 0} \#\{\text{standard monomials of degree } d\} t^d .$$

Then by (3.1), we obtain

$$P(T^0(2, m), t) \leq \frac{(1 + t)^m}{(1 - t^2)^m} P(K[\mathrm{Gr}\,(2, m)], t) .$$

Here the equality holds if and only if the standard monomials are linearly independent. On the other hand, by Proposition 3 and (4.1),

$$P(T^0(2, m), t) = \frac{1}{(1 - t)^m} P(K[\mathrm{Gr}\,(2, m]), t) ,$$

and hence the set of standard monomials of $T^0(2, m)$ constitutes a K-basis of $T^0(2, m)$. Then again by (4.1), this completes the proof.

§ 5. Proof of Theorem 2

Let $B^0(2, m)$ be the subring of $B(2, m)$ generated by the elements:

$$\mathrm{Tr}\,((X_i^0)^2) , \qquad 1 \leq i \leq m ,$$

and

$$p_k^0 = \textstyle\sum_{i+j=k} \mathrm{Tr}\,(X_i^0 X_j^0) , \qquad 3 \leq k \leq 2m - 1 .$$

A semi-standard monomial is called an S-standard monomial of $T^0(2, m)$ if the Young tableau associated with its trace part is S-standard.

We now prove by induction on degree that $T^0(2, m)$ is a $B^0(2, m)$-module generated by the S-standard monomials of $T^0(2, m)$. We assume that any element of $T^0(2, m)$ with degree $< d$ is a linear combination of S-standard monomials over $B^0(2, m)$. We then claim that any element of degree d is a linear combination of S-standard monomials over $B^0(2, m)$. By the induction hypothesis, it is enough to prove our claim for elements of the form

$$\mathrm{Tr}\,(X^0_{i_1}X^0_{j_1}) \cdots \mathrm{Tr}\,(X^0_{i_r}X^0_{j_r})X^0_{k_1} \cdots X^0_{k_t}\,, \tag{5.1}$$

with $\quad i_1 + j_1 \leq i_2 + i_2 \leq \cdots \leq i_r + j_r\,, \qquad 1 \leq k_1 < \cdots < k_t \leq m\,.$

Take such an element f and consider the sequence of numbers

$$(i_1 + j_1, \cdots, i_r + j_r, 2k_1, \cdots, 2k_t)\,.$$

Permutating the numbers in the sequence above, we get a sequence of numbers

$$(a_1, a_2, \cdots, a_{r+t})\,, \quad \text{with} \quad a_1 \leq a_2 \leq \cdots \leq a_{r+t}\,. \tag{5.2}$$

The sequence (5.2) of numbers is called the weight of f (denoted by $w(f)$). For example, if

$$f = \mathrm{Tr}\,(X^0_1X^0_4)\,\mathrm{Tr}\,(X^0_2X^0_3)X^0_1X^0_2X^0_4\,,$$

we have $w(f) = (2, 4, 5, 5, 8)$.

Suppose now that

$$i_k < i_{k+1} < j_{k+1} < j_k \quad \text{or} \quad i_{k+1} < i_k < j_k < j_{k+1} \qquad \text{for some } k\,.$$

Then by using (4.6) and a similar argument as in the proof of Theorem 1, it is easily verified that f is a linear combination of monomials with lexikographically smaller weight than $w(f)$. Then clearly the process terminates within finitely many steps and hence any element of $T^0(2, m)$ is a $B^0(2, m)$-linear combination of standard monomials of the form

$$\mathrm{Tr}\,(X^0_{\alpha_1}X^0_{\beta_1}) \cdots \mathrm{Tr}\,(X^0_{\alpha_s}X^0_{\beta_s})X^0_{\gamma_1} \cdots X^0_{\gamma_u}\,, \tag{5.3}$$

with $\quad \alpha_1 \leq \cdots \leq \alpha_s\,, \quad \beta_1 \leq \cdots \leq \beta_s, \quad 1 \leq \gamma_1 \leq \cdots \leq \gamma_u \leq m\,.$

Furthermore using the relation

$$p^0_k = \textstyle\sum_{i+j=k} \mathrm{Tr}\,(X^0_iX^0_j)\,, \tag{5.4}$$

and repeating the process used above, we may assume that $\beta_k > \alpha_k + 2$ for all k, $1 \leq k \leq s$. If $\alpha_t = \alpha_{t+1}$ for some t, then using the relation (5.3), we replace the factor $\mathrm{Tr}\,(X^0_{\alpha_t}X^0_{\beta_t})$ by

$$p_k^0 - \sum_{\substack{i+j=k \\ i\neq\alpha_t}} \mathrm{Tr}\,(X_i^0 X_j^0)\,, \qquad k = \alpha_t + \beta_t\,,$$

Similarly if $\beta_t = \beta_{t+1}$ for some t, we replace the factor $\mathrm{Tr}\,(X^0_{\alpha_{t+1}} X^0_{\beta_{t+1}})$ by

$$p_k^0 - \sum_{\substack{i+j=k \\ i\neq\alpha_{t+1}}} \mathrm{Tr}\,(X_i^0 X_j^0)\,, \qquad k = \alpha_{t+1} + \beta_{t+1}\,.$$

And we repeat the same process as above. Then we finally find that any element of $T^0(2, m)$ is a $B^0(2, m)$-linear combination of standard monomials of the form

$$\mathrm{Tr}\,(X_{\alpha_1} X_{\beta_1}) \cdots \mathrm{Tr}\,(X_{\alpha_s} X_{\beta_s}) X_{\gamma_1} \cdots X_{\gamma_u}\,, \tag{5.5}$$

with $\alpha_1 < \cdots < \alpha_s\,,\quad \beta_1 < \cdots < \beta_s\,,\quad \gamma_1 < \cdots < \gamma_u\,,$

and $\beta_p > \alpha_p + 2 \qquad \text{for all } p,\ 1 \leq p \leq s\,.$

Clearly the condition in (5.5) says that the associated Young tableau

$$\begin{bmatrix} \alpha_1 & \cdots & \alpha_s \\ \beta_1 & \cdots & \beta_s \end{bmatrix}$$

is S-standard. Therefore we have proved that any element of $T^0(2, m)$ is a $B^0(2, m)$-linear combination of S-standard monomials of $T^0(2, m)$. Then by (4.1), any element of $T(2, m)$ is a $B(2, m)$-linear combination of standard monomials of $T(2, m)$. Since

$$P(B(2, m), t) = \frac{1}{(1-t)^m (1-t^2)^{3m-3}}\,,$$

we, in particular, obtain

$$P(T(2, m), t) \leq \frac{1}{(1-t)^m (1-t^2)^{3m-3}} \sum_{d\geq 0} \left\{\begin{matrix} S\text{-standard mono-} \\ \text{mials of degree } d \end{matrix}\right\} t^d\,.$$

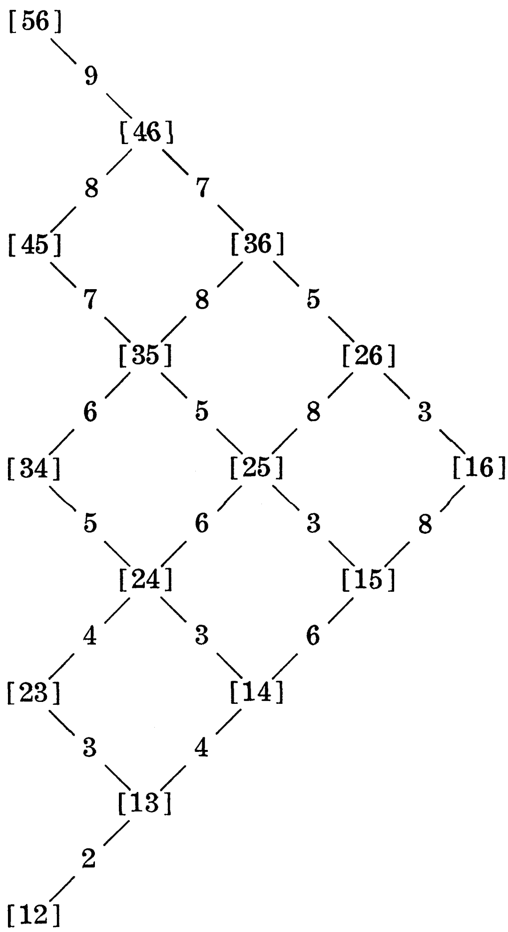
By Proposition 2, we have

$$P(T(2, m), t) \leq \frac{1}{(1-t)^{2m}} P(K[\mathrm{Gr}\,(2, m)], t)\,,$$

where the equality holds if and only if the S-standard monomials of $T(2, m)$ are $B(2, m)$-linearly independent. By Procesi's identity, this completes the proof of Theorem 2.

§6. Example

Consider the Hasse diagram for Λ_6:



S-standard Young tableaux:

$$\phi,\ \begin{bmatrix}1\\4\end{bmatrix},\ \begin{bmatrix}1\\5\end{bmatrix},\ \begin{bmatrix}1\\6\end{bmatrix},\ \begin{bmatrix}2\\5\end{bmatrix},\ \begin{bmatrix}2\\6\end{bmatrix},\ \begin{bmatrix}3\\6\end{bmatrix},\ \begin{bmatrix}1&2\\4&5\end{bmatrix},\ \begin{bmatrix}1&2\\4&6\end{bmatrix},\ \begin{bmatrix}1&3\\4&6\end{bmatrix},$$

$$\begin{bmatrix}1&2\\5&6\end{bmatrix},\ \begin{bmatrix}1&3\\5&6\end{bmatrix},\ \begin{bmatrix}2&3\\5&6\end{bmatrix},\ \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}.$$

$T(2, 6)$ is a free module over the polynomial ring $B(2, 6)$ with generators

$$\mathrm{Tr}\,(Y)X_{k_1}X_{k_2}\cdots X_{k_t}\,,$$

where Y is an S-standard Young tableau associated with Λ_6 and $X_{k_1}X_{k_2}\cdots X_{k_t}$ is 1, if $t = 0$, or a monic in the generic 2 by 2 matrices

$$X_1, \cdots, X_6 \quad \text{with} \quad k_1 < k_2 < \cdots < k_t\,.$$

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