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ON THE TOPOLOGY OF FULL NON-DEGENERATE COMPLETE INTERSECTION VARIETY

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§ 1. Introduction

Let $h_1(\mathbf{u}), \cdots, h_k(\mathbf{u})$ be Laurent polynomials of m-variables and let

$$Z^* = \{\mathbf{u} \in \mathbf{C}^{*m};\ h_1(\mathbf{u}) = \cdots = h_k(\mathbf{u}) = 0\}$$

be a non-degenerate complete intersection variety. Such an intersection variety appears as an exceptional divisor of a resolution of non-degenerate complete intersection varieties with an isolated singularity at the origin (Ok4]). We say that Z^* is *full* if $\dim(\Delta(h_\alpha)) = m$ for any $\alpha = 1, \cdots, k$. Let I be a subset of $\{1, \cdots, m\}$. We say that Z^* is *I-full* if (i) for each $\alpha = 1, \cdots, k$, $h_\alpha(\mathbf{u})$ is a polynomial in the variables $\{u_i;\ i \in I\}$ (fixing other variables) and (ii) for any $J \supset I^c$, the polynomials $\{h^J_\alpha(\mathbf{u}_J);\ \alpha = 1, \cdots, k\}$ are not constantly zero and the *variety* $\{\mathbf{u}^J \in \mathbf{C}^{*J};\ h^J_1(\mathbf{u}_J) = \cdots = h^J_k(\mathbf{u}_J) = 0\}$ is full in the above sense where h^J_α is the restriction of h_α to the coordinate subspace $\mathbf{C}^J = \{\mathbf{u} \in \mathbf{C}^m;\ u_i = 0 \text{ if } i \notin J\}$ and I^c is the complement of I in $\{1, \cdots, m\}$. Thus any full non-degenerate complete intersection variety is $\emptyset$-full. Assume that Z^* is I-full and let

$$Z = \{\mathbf{u} \in \mathbf{C}^I \times \mathbf{C}^{*I^c};\ h_1(\mathbf{u}) = \cdots = h_k(\mathbf{u}) = 0\}.$$

Here we identify $\mathbf{C}^I \times \mathbf{C}^{*I^c}$ with the subspace of $\mathbf{C}^m$ by $\mathbf{C}^I \times \mathbf{C}^{*I^c} = \{\mathbf{z} \in \mathbf{C}^m;\ z_i \neq 0,\ i \in I^c\}$. In the case that $I = \{1, \cdots, m\}$, the I-fullness condition implies that each h_α has a non-zero constant term and each $h_\alpha(\mathbf{u})$ is a convenient polynomial. Here the polynomial h_α is called convenient if and only if $h^{\{i\}}_\alpha$ is not constantly zero for any $1 \leq i \leq m$. In particular, $\vec{0} \notin Z$ in this case. The purpose of this paper is to study the topology of a full non-degenerate complete intersection variety. We will prove

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MAIN THEOREM (1.1). *Let Z be a I-full non-degenerate complete intersection variety and let ι: $Z \to \mathbf{C}^I \times \mathbf{C}^{*I^c}$ be the inclusion map. Then ι is an $(m-k)$-equivalence i.e., the homomorphism $\iota_\#$: $\pi_i(Z) \to \pi_i(\mathbf{C}^I \times \mathbf{C}^{*I^c})$ is an isomorphism for $i < m-k$.*

In the case of $m-k>1$, the above theorem says that the fundamental group $\pi_1(Z)$ is a free abelian group of rank $m-|I|$ and the higher homotopy groups $\pi_j(Z)$ vanish for $1<j<m-k$. For the proof we use an induction on k. An essential step is to show that the mapping h_k: $Z_{k-1} \to \mathbf{C}$ has no critical points at infinity where $Z_{k-1} = \{\mathbf{u} \in \mathbf{C}^I \times \mathbf{C}^{*I^c}; h_1(\mathbf{u}) = \cdots = h_{k-1}(\mathbf{u}) = 0\}$ (Lemma (3.2), § 3). To see this, we use a toric compactification X of $\mathbf{C}^I \times \mathbf{C}^{*I^c}$ and we consider the family of compact varieties $\{\overline{h_k^{-1}(t)};\ t \in \mathbf{C}\}$ in X. This is a new viewpoint comparing with those in [B1] and [Ok2]. By the Whitehead theorem ([S]), we have the corollary:

COROLLARY (1.1.1). ι_*: $H_i(Z; \mathbf{Z}) \to H_i(\mathbf{C}^I \times \mathbf{C}^{*I^c}; \mathbf{Z}) \cong \mathbf{Z}^{\binom{|I^c|}{i}}$ is *an isomorphism for $i < m-k$ and a surjection for $i = m-k$.*

The Euler characteristic $\chi(Z)$ can be computed by a result of Khovanskii (Kh2]). Therefore the cohomology group of Z can be completely computed by Corollary (1.1.1) and the result of Khovanskii as Z has a homotopy type of CW-complex of dimension $m-k$. Taking $I=\varnothing$, or $\{1, \cdots, m\}$, we have the following corollaries:

COROLLARY (1.1.2). *Let Z^* be a full non-degenerate complete intersection variety and let ι: $Z^* \to \mathbf{C}^{*m}$ be the inclusion map. Then ι is an $(m-k)$-equivalence and Z^* has a homotopy type of CW-complex of dimension $m-k$.*

The above assertion has been essentially proved in [Ok2] for $k=1$.

COROLLARY (1.1.3) *Assume that Z is a $\{1, \cdots, m\}$-full non-degenerate complete intersection variety. Then Z is $(m-k-1)$-connected and thus Z is homotopic to a bouquet of spheres of dimension $(m-k)$.*

Note that $h_1(\mathbf{u}), \cdots, h(\mathbf{u})$ are convenient polynomials with non-zero constant terms if Z is a $\{1, \cdots, m\}$-full non-degenerate complete intersection variety. The topology of affine hypersurfaces (the case of $k=1$) have been studied by many people. See for instance [M], [V], [K], [B1], [Ok1], [Ok2] and [B2].

Let Z^* be a full non-degenerate complete intersection variety and let Y be a smooth compactification in a suitable toric variety X. (See § 2). There are many beautiful works on the (algebraic) geometry of X and Y. See [De], [Da], [E], [Da-Kh], [Kh1], [Kh2], [K-K-M-S], [Od1] and [Od2]. Their works are done mainly from the viewpoint of algebraic geometry. Our essential tool is a Morse theory in a toric variety. As an application, we will prove that *the fundamental group of* Y is *an abelian group which is generated by at most* k *elements* (Theorem (4.2)). This is a generalization of a result in [Ok3] and [Ok4] for the case $k = 1$.

§ 2. Toric compactification

This paper is a continuation of the previous paper [Ok4]. Unless otherwise stated, we use the same notations. Let $Z^* = \{\mathbf{u} \in \mathbf{C}^{*m};\ h_1(\mathbf{u}) = \cdots = h_k(\mathbf{u}) = 0\}$ be as in § 1. Let P be a covector. It defines a linear function on the respective Newton diagrams $\Delta(h_i)$. Z^* is called a *non-degenerate complete intersection variety* if for any covector P, the variety $Z^*(P) \underset{\text{def}}{=} \{\mathbf{u} \in \mathbf{C}^{*m};\ h_{1P}(\mathbf{u}) = \cdots = h_{kP}(\mathbf{u}) = 0\}$ is a reduced smooth complete intersection variety (Kh1], [Ok4]). Note that Z^* is itself a smooth complete intersection variety as we can easily see it by taking $P = \vec{0}$. We recall the construction of a smooth toric compactification of Z^* as in [Kh1] or [Ok4]. Let N be the space of covectors. Let $P, Q \in N$. We define an equivalence relation by $P \sim Q$ if and only if $\Delta(P; h_i) = \Delta(Q; h_i)$ for $i = 1, \cdots, k$. Here $\Delta(P; h_i)$ is the face of $\Delta(h_i)$ where the covector P takes its minimal value which we denote by $d(P; h_i)$. This gives a polyhedral cone subdivision of N which we call the dual Newton diagram of Z^* and we denote it by $\Gamma^*(h_1, \cdots, h_k)$. Let Σ^* be a unimodular simplicial subdivision of $\Gamma^*(h_1, \cdots, h_k)$ and let X be the corresponding toric variety. Then the closure $\bar{Z}^*$ of Z^* in X is a smooth variety. Let us denote this compactification by $Y = \bar{Z}^*$. The irreducible divisors contained in $X - \mathbf{C}^{*m}$ is in a bijective correspondence with the vertices $P \in \mathrm{Vertex}(\Sigma^*)$. The corresponding divisor is denoted by $\hat{E}(P)$. We denote the divisor $\hat{E}(P) \cap Y$ of Y by $E(P)$. The irreducible components of $Y - Z^*$ are divisors of Y and they correspond bijectively to the vertices $P \in \mathrm{Vertex}(\Sigma^*)$ which satisfy the (A_0)-condition:

$$(\mathrm{A}_0) \qquad \dim\Big(\sum_{\alpha \in J} \Delta(P; h_\alpha)\Big) \geq |J| \qquad \text{for any non-empty } J \subset \{1, \cdots, k\}.$$

This condition is a necessary and sufficient condition for $E(P) \neq \varnothing$ ([Ok4]).

Assume that Z is I-full. We may assume and we assume that $I=\{1,\cdots,s\}$ for bervity's sake. Then the s-dimensional simplicial cone, say τ_I, with vertices $R_1,\cdots,R_s$ is compatible with $\Gamma^*(h_1,\cdots,h_k)$ where $R_i={}^t(0,\cdots\overset{i}{\breve{1}},\cdots,0)$. This results from the assumption that $h_i^{I^c}$ is non-trivial. (In fact, let $P=\sum_{i=1}^s r_iR_i\in N$ with $r_1,\cdots,r_s\geq 0$ and $P\neq 0$. Then $d(P;h_i)=0$ and $\Delta(P;h_i)=\Delta(h_i^{I^c})$.) Thus we may assume that τ_I is *a simplicial cone of* Σ^*. For this, we apply the subdivision method in § 3 of [Ok3]. X is covered by the union of the coordinate space $\mathbf{C}_\sigma^m$ where σ moves in the m-dimensional simplicial cones of Σ^*. Let $\sigma=(P_1,\cdots,P_m)$ be an m-dimensional simplicial cone. As we did in [Ok3], we use the notation $\sigma=(P_1,\cdots,P_m)$ in two ways. First $P_1,\cdots,P_m$ are primitive integral vectors expressed as column vectors $P_i={}^t(p_{1i},\cdots,p_{mi})$, $i=1,\cdots,m$. As a cone in N, $\sigma=\{\sum_{i=1}^m t_iP_i\in\mathbf{R}^n;\ t_1,\cdots,t_m\geq 0\}$. As a matrix, σ denote the matrix (p_{ij}). The isomorphism $\pi_\sigma\colon \mathbf{C}_\sigma^{*m}\to\mathbf{C}^{*m}$ is defined by $\pi_\sigma(\boldsymbol{y}_\sigma)=(\prod_{j=1}^m y_{\sigma j}^{p_{1j}},\cdots,\prod_{j=1}^m y_{\sigma j}^{p_{mj}})$. Remember that in the coordinate space C_σ^m, $Y\cap\mathbf{C}_\sigma^m$ is defined by $\overline{\pi_\sigma^{-1}(Z^*)}=\{\boldsymbol{y}_\sigma\in\mathbf{C}_\sigma^m;\ h_{1\sigma}(\mathbf{y}_\sigma)=\cdots=h_{k\sigma}(\boldsymbol{y}_\sigma)=0\}$ where $h_{\alpha\sigma}(\boldsymbol{y}_\sigma)$ is defined by the equality $h_\alpha(\pi_\sigma(\boldsymbol{y}_\sigma))=h_{\alpha\sigma}(\boldsymbol{y}_\sigma)\cdot\prod_{j=1}^m y_{\sigma j}^{d(P_j;h_\alpha)}$. Here we use the same notation as in [Ok3], [Ok4]. The closure of $\{y_{\sigma i}=0\}$ in X is the divisor $\hat{E}(P)$ by definition and the closure of $Y\cap\mathbf{C}_\sigma^m\cap\{y_{\sigma i}=0\}$ is a smooth divisor (if not empty) of Y and this is nothing but the divisor $E(P_i)$ in the above correspondence. By the assumption on Σ^*, we can find a simplex $\xi=(Q_1,\cdots,Q_m)\in\Sigma^*$ such that $Q_i=R_i$ for $i=1,\cdots,s$. That is, τ_I is a face of ξ. Then as a matrix, ξ can be written as

$$\xi=\begin{pmatrix} I_s & A\\ 0 & B\end{pmatrix}$$

where I_s is the unit $s\times s$ matrix. As ξ is a unimodular matrix, B is also a unimodular matrix of size $m-s$. Thus π_ξ gives a holomorphic diffeomorphism of $\mathbf{C}_\xi^s\times\mathbf{C}_\xi^{*(m-s)}=\{\boldsymbol{y}_\xi;\ y_{\xi s+1}\cdots y_{\xi m}\neq 0\}$ with $\mathbf{C}^s\times\mathbf{C}^{*(m-s)}$ and therefore Z can be identified with $Y\cap(\mathbf{C}_\xi^s\times\mathbf{C}_\xi^{*(m-s)})$. Thus Y can be considered as a smooth compactification of Z. Note that $Z=Z^*\bigcup_{i=1}^s E(R_i)-\bigcup_{Q\neq R_1,\cdots,R_s}E(Q)$ under this identification.

PROPOSITION (2.1). *Let Z be a 1-full non-degenerate complete intersection variety. Then Z is non-singular.*

Proof. This results immediately from the smoothness of Y and the inclusion property: $Z\subset Y$.

§ 3. Proof of the Main Theorem

In this section, we prove Main Theorem (1.1) stated in § 1 by the induction on k. The case $k=0$ is obvious. Thus we assume that $k \geq 1$ and $I=\{1, \cdots, s\}$ $(0 \leq s \leq m)$, for brevity's sake. Let a be a positive integer and let $\Phi_a\colon \mathbf{C}^s \times \mathbf{C}^{*(m-s)} \to \mathbf{C}^s \times \mathbf{C}^{*(m-s)}$ be the map which is defined by $\Phi_a(\mathbf{u}) = (u_1, \cdots, u_s, u_{s+1}^a, \cdots, u_m^a)$. Φ_a gives an $a^{(m-s)}$-fold covering map. We first prove the following lifting principle ([Ok2]).

LEMMA (3.1). *Let $Z^{(a)} = \Phi_a^{-1}(Z)$ and let $\iota_a\colon Z^{(a)} \to \mathbf{C}^s \times \mathbf{C}^{*(m-s)}$ be the inclusion map. Then $\iota\colon Z \to \mathbf{C}^s \times \mathbf{C}^{*(m-s)}$ is a $(m-k)$-equivalence if (and only if) ι_a is a $(m-k)$-equivalence.*

Proof. We assume first that $m-k \geq 2$. We consider the following commutative diagrams of the fundamental groups where the horizontal sequences are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(Z^{(a)}) & \xrightarrow{\Phi_{a\#}} & \pi_1(Z) & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_{a\#}} & & \downarrow{\scriptstyle \iota_\#} & & \| & & \\
0 & \longrightarrow & \pi_1(\mathbf{C}^s \times \mathbf{C}^{*(m-s)}) & \xrightarrow{\Phi_{a\#}} & \pi_1(\mathbf{C}^s \times \mathbf{C}^{*(m-s)}) & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

Here G is a finite group which is isomorphic to $(\mathbf{Z}/a\mathbf{Z})^{m-s}$. By the assumption, $\iota_{a\#}$ is an isomorphism. Therefore by Five Lemma, $\iota_\#$ is an isomorphism. As $\pi_i(Z^{(a)}) = \pi_i(Z)$ for $i>1$, we have that $\pi_i(Z)=0$ for $i=2, \cdots, m-k-1$. This proves the assertion in the case of $m-k \geq 2$. Assume that $m-k=1$. We have to show that Z is connected. But this is obvious from the assumption that $Z^{(a)}$ is connected. This completes the proof.

Note that $Z^{(a)}$ is also a I-full non-degenerate complete intersection variety if Z is a I-full non-degenerate complete intersection variety. Thus to prove Main theorem, we may replace Z by a suitable $Z^{(a)}$ if necessary and we may assume that $\Delta(h_k^{I^c})$ contains an interior integral point $Q = (0, \cdots, 0, q_{s+1}, \cdots, q_m)$ if $s \neq m$ ($\Leftrightarrow I \neq \{1, \cdots, m\}$). In this case, multiplying the monomial $\mathbf{u}^{-Q}$ to h_k, we may assume (and we assume) that

(B) *The origin $\vec{0}$ is an interior point of $\Delta(h_k^{I^c})$ if $I \neq \{1, \cdots, m\}$.*

We consider the variety $Z_{k-1} = \{\mathbf{u} \in \mathbf{C}^s \times \mathbf{C}^{*(m-s)};\ h_1(\mathbf{u}) = \cdots = h_{k-1}(\mathbf{u}) = 0\}$. We may also assume that Z_{k-1} is an I-full nondegenerate complete intersection variety, by moving the coefficients of $h_1, \cdots, h_{k-1}$ slightly if necessary. This does not change the diffeomorphism class of Z. In fact, they are isotopic in $\mathbf{C}^s \times \mathbf{C}^{*(m-s)}$. (The proof of this assertion is parallel to that of Lemma (3.2) below.) We consider the holomorphic function $h \underset{\text{def}}{=} h_k | Z_{k-1}\colon Z_{k-1} \to \mathbf{C}$. Let $C(h)$ be the critical points of h and let $\Sigma(h)$ be the set of the critical values of h. The following is a generalization of Lemma (5.9) of [Ok2].

LEMMA (3.2). (i) *$C(h)$ and $\Sigma(h)$ are finite sets.*

(ii) *$h\colon Z_{k-1} - h^{-1}(\Sigma(h)) \to \mathbf{C} - \Sigma(h)$ is a locally trivial fibration. The fiber is diffeomorphic to Z.*

(iii) *$h\colon Z_{k-1} \to \mathbf{C}$ has no critical point at the infinity.*

Proof. First we fix a compactification X of $\mathbf{C}^s \times \mathbf{C}^{*(m-s)}$ as in § 2. Let $X_t = h^{-1}(t)$ and let $\overline{X}_t$ be the closure of X_t in X and let $\partial \overline{X}_t \underset{\text{def}}{=} \overline{X}_t - X_t$. Note that $Z = X_0$. We first prove that $\partial \overline{X}_t = \partial \overline{X}_0$ and thus it has no singularity at infinity.

ASSERTION (3.2.1). *Let P be a covector. Assume that $\{\Delta(P; h_i);\ i = 1, \cdots, k\}$ satisfies (A_0) condition and that $\vec{0} \in \Delta(P; h_k)$. Then $P \in \mathrm{Cone}(R_1, \cdots, R_s)$.*

Proof of Assertion (3.2.1). Suppose that $P \in \mathrm{Vertex}(\Sigma^*)$ be a vertex such that $\{\Delta(P; h_i);\ i = 1, \cdots, k\}$ satisfies (A_0) condition and assume that $\vec{0} \in \Delta(P; h_k)$. Assume first that $I \neq \{1, \cdots, m\}$. Then the assumption (B) implies that $\Delta(h^{I^c}) \subset \Delta(P; h_k)$. On the other hand, by the I-fullness condition we have that $\dim(\Delta(h_k^{I^c})) = m - s$. The inclusion $\Delta(h_k^{I^c}) \subset \Delta(P; h_k)$ is possible only if the I^c-component of P is zero. Therefore we can write P as $P = {}^t(p_1, \cdots, p_s, 0, \cdots, 0)$. Now we assert $p_i \geq 0$ for any $i = 1, \cdots, s$. In fact, assume that $p_{i_0} < 0$ for some $i_0 \in I$. Let $J = \{i \in I;\ p_i < 0\} \cup I^c$ and we consider h_k^J. By the I-fullness condition, we must have that $d(P; h_k) < 0$ and in particular $\Delta(P; h_k) \cap \Delta(h_k^{I^c}) = \emptyset$ which is a contradiction. Thus we have proved that $p_i \geq 0$ $(i = 1, \cdots, s)$. In other word, we have $P \in \tau_I \underset{\text{def}}{=} \mathrm{Cone}(R_1, \cdots, R_s)$. This completes the proof.

Suppose that $P \in \mathrm{Vertex}(\Sigma^*)$ be a vertex such that $\{\Delta(P; h_i); i = 1, \cdots, k\}$ satisfies (A_0) condition. Assume that $\vec{0} \in \Delta(P; h_k)$. By the assumption on the subdivision Σ^*, τ_I is a simplicial cone in Σ^*. This implies that $P = R_i$ for some $1 \leq i \leq s$. Thus if $P \neq R_1, \cdots, R_s$ and $\{\Delta(P; h_i);$ $i = 1, \cdots, k\}$ satisfies (A_0) condition, we must have $\vec{0} \notin \Delta(P; h_k)$. Let $h_{k,t}$ be the function defined by $h_{k,t}(\mathbf{u}) = h_k(\mathbf{u}) - t$ with t being considered as a constant. Then the assumption (B) implies that $\Delta(h_{k,t}) = \Delta(h_k)$ for any

t if $I \neq \{1, \cdots, m\}$. If $I = \{1, \cdots, m\}$, $\Delta(h_{k,t}) = \Delta(h_k)$ for any $t \neq h_k(\vec{0})$ and $\Delta(h_{k,h_k(\vec{0})}) \subset \Delta(h_k)$. Recall that $h_k(\mathbf{u})$ is a convenient *polynomial* with a non-zero constant in this case. Thus $\vec{0} \in \partial\Delta(h_k)$ and $\vec{0} \notin (h_{k,h_k(\vec{0})})$. However even though their outside boundaries are same i.e., if a face $\Xi \subset \Delta(h_k)$ does not contain the origin, Ξ is also a face of $\Delta(h_{k,t})$. Therefore *the divisors $\hat{E}(P) \cap \bar{X}_t$ are independent of the variable t if $P \neq R_1, \cdots, R_s$ in any cases.* Therefore we have that $\partial\bar{X}_t = \partial\bar{X}_0$ and $\partial\bar{X}_t$ is smooth by the non-degeneracy assumption on Z. This proves the assertion (iii). It is obvious that the base points of the family $\{\bar{X}_t\}$ are contained in $\partial\bar{X}_0$. Thus the assertion (i) is now follows from the Bertini's theorem ([Grf-Hr]). The assertion (ii) can be proved easily using the controlled vector field argument as follows. Let $W = \{(x, t) \in \bar{Z}_{k-1} \times C; x \in \bar{X}_t\}$ and let $\pi\colon W \to \mathbf{C}$ be the projection. W is a smooth codimension one submanifold of $\bar{Z}_{k-1} \times \mathbf{C}$. By the above argument, $\partial W \underset{\text{def}}{=} \{(x, t) \in W;\ x \in \partial\bar{X}_t\}$ is simply the product $\partial\bar{X}_0 \times \mathbf{C}$. Thus by the Ehreman's fibering theorem ([W]), $\pi\colon (W, \partial W) \cap \pi^{-1}(\mathbf{C} - \Sigma(h)) \to \mathbf{C} - \Sigma(h)$ is a locally trivial fibration. In particular, $\pi\colon (W - \partial W) \cap \pi^{-1}(\mathbf{C} - \Sigma(h)) \to \mathbf{C} - \Sigma(h)$ is also a locally trivial fibration. Define a mapping $\Psi\colon Z_{k-1} \to W - \partial W$ by $\Psi(x) = (x, h(x))$. This is obviously a diffeomorphism and we have the commutativity $\pi \circ \Psi = h$. Therefore we can pull back by Ψ the fibering structure of $\pi\colon (W - \partial W) \cap \pi^{-1}(\mathbf{C} - \Sigma(h)) \to \mathbf{C} - \Sigma(h)$ to $h\colon Z_{k-1} - h^{-1}(\Sigma(h)) \to \mathbf{C} - \Sigma(h)$. This completes the proof of the assertion (ii).

LEMMA (3.3). *Let $\iota_k\colon Z_k - Z_{k-1}$ be the inclusion map. Then ι_k is an $(m - k)$-equivalence.*

Proof. Let $C(h) = \{\rho_1, \cdots, \rho_\nu\}$ be the critical points of h and let $\Sigma(h) = \{\eta_1, \cdots, \eta_{\nu'}\}$ be the critical values of h ($\nu \geq \nu'$) and let $\eta_0 = 0$. Note that 0 is a regular value of h as $Z = h^{-1}(0)$. Take sufficiently small $\varepsilon > 0$ and $\delta > 0$ ($\varepsilon \gg \delta$) and let B_a be the closed ball of radius ε with center ρ_a and let D_j be the closed disk of radius δ with center η_j. We assume that the closed balls $B_1, \cdots, B_\nu$ (respectively the disks $D_1, \cdots, D_{\nu'}$) are mutually disjoint. Assume that $\eta_j = h(\rho_a)$. Let $E_a = h^{-1}(D_j) \cap B_a$ and $E_a^* = E_a - h^{-1}(\eta_j)$. Let $h\colon E_a^* \to D_j - \{\vec{0}\}$ is the local Milnor fibration. Let η_j' be a fixed point of ∂D_j and let $F_a = h^{-1}(\eta_j') \cap E_a$ (the local Milnor fiber). It is well known that F_a is homotopic to a bouquet of $(m - k)$-dimensional spheres ([M]). We also know that the pair (E_a, F_a) is homotopic to the pair (CF_a, F_a) where CF_a is the cone of F_a. Let l_j be a simple path from

η_0 to η'_j and let $L_j = l_j \cup D_j$ and let $L = \bigcup_{j=1}^{\nu'} L_j$. We may assume that L is contractible to η_0. We can see that (i) $h^{-1}(L)$ is a strong deformation retract of Z_{k-1} by the fibering structure. (ii) For each $h^{-1}(L_j)$ is homotopic to the space $Z \bigcup_{\{a; h(p_a)=\eta j\}} CF_a$. The cone CF_a are glued along F_a. This can be proved using the product structure of h: $h^{-1}(D_j) - \bigcup_{\{a; h(\rho_a)=\eta j\}} E_a \to D_j$. If ρ_a is a simple critical point, CF_a is homotopically a cell of dimension $m - k - 1$. In general, to add CF_a along F_a does not change the homotopy groups of dimension less than $m - k$. (iii) Thus $h^{-1}(L)$ is homotopic to the space $h^{-1}(\eta_0) \bigcup_{a=1}^{\nu} CF_a$ by (i) and (ii). See Figure (3.3.1). See also § 5, [Ok2] for a similar argument.

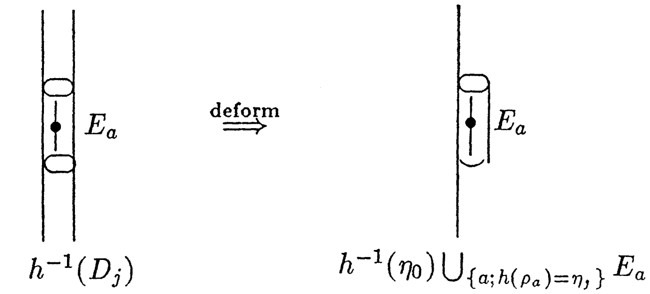

Figure (3.3.1)

Thus we can conclude that $Z = h^{-1}(\eta_0) \hookrightarrow Z_{k-1}$ is an $(m - k)$-equivalence.

Now we are ready to prove our Main Theorem (1.1) in § 1. $Z_{k-1} \hookrightarrow \mathbf{C}^s \times \mathbf{C}^{*(m-k)}$ is an $(m - k + 1)$-equivalence by the induction's assumption. Thus the composition ι: $Z \to Z_{k-1} \to \mathbf{C}^s \times \mathbf{C}^{*(m-k)}$ is an $(m - k)$-equivalence by Lemma (3.3). This completes the proof.

Remark (3.4). In this paper, we have only studied the full non-degenerate complete intersection variety. Let Z^* be a non-degenerate complete intersection variety which is not necessarily full. We say that Z^* satisfies (A_i)-condition if

$$(\mathrm{A}_i) \quad \dim\left(\sum_{j \in J} \Delta(h_j)\right) \geq \min(|J| + i, m) \quad \text{for any non-empty } J \subset \{1, \cdots, k\}.$$

A full non-degenerate complete intersection variety satisfies (A_{m-1})-condition. We finish this section by the following question: Is ι: $Z^* \to \mathbf{C}^{*m}$ a $\min(m - k, i)$-equivalence? (True also for $i = 0$. See [Ok4]).

§ 4. Fundamental group of the compactification Y

Let Z^* be a full non-degenerate complete intersection variety as before and let $Y = \bar{Z}^*$ be the closure of Z^* in X. We are going to show that the fundamental group of Y is an abelian group which is generated by at most k elements. This gives a generalization of Theorem (7.3) of [Ok3]. The main difficulty is that the configuration of the irreducible divisors $\{E(P)\}$ which are in the complement of Z^* in Y is not so clear. We say that a simplex $\sigma = (P_1, \cdots, P_m) \in \Sigma^*$ is *good* if there exist $1 \leq i_1 < \cdots < i_{m-k} \leq m$ so that $(m-k)$ divisors $\{E(P_{i_1}), \cdots, E(P_{i_{m-k}})\}$ have a non-empty intersection in this coordinate chart $\mathbf{C}^m_\sigma$. For the next lemma, it is not necessary to assume the fullness of Z^*.

LEMMA (4.1). *There exists a good simplex of Σ^*.*

Proof. Note that Z^* is compact if and only if $m - k = 0$. Let $S = \{P; E(P) \neq \emptyset, P \in \mathrm{Vertex}(\Sigma^*)\}$. Note that $Z^* = Y - \bigcup_{P \in S} E(P)$. We prove the assertion by the induction on $m - k$. The assertion is obvious if $m - k = 0$. Assume that $m - k \geq 1$. As Z^* is compact, there is a vertex $P \in \mathrm{Vertex}(\Sigma^*)$ such that $E(P)$ is non-empty i.e., $P \in \mathcal{S}$. Now we consider the variety $E(P)$ as a non-degenerate complete intersection variety in the toric variety $\hat{E}(P)$. Replacing X by the divisor $\hat{E}(P)$ and Y by $E(P)$, we apply the induction's assumption. We can find vertices $P_1, \cdots, P_{m-k-1}$ of $\mathrm{Vertex}(\Sigma^*)$ such that the divisors $\{E(P_i) \cap E(P); i = 1, \cdots, m-k-1\}$ have a non-empty intersection. That is $\bigcap_{i=1}^{m-k-1} E(P_i) \cap E(P) \neq \emptyset$. Thus the divisors $E(P), E(P_1), \cdots, E(P_{m-k-1})$ has a non-empty intersection. Thus we can find a m-simplex σ which contains the vertices $P, P_1, \cdots, P_{m-k-1}$. This complete the proof.

Now we fix a good coordinate $\sigma = (P_1, \cdots, P_m)$ such that $P_i \in \mathcal{S}$ for $i = 1, \cdots, m-k$. The calculation of the fundamental group can be done by the exact same way as § 7 of [Ok3]. As the canonical homomorphism of the fundamental groups $\pi_1(Z^*) \to \pi_1(Y)$ is a surjection, we can see easily by Van Kampen theorem that $\pi_1(Y)$ is isomorphic to the quotient group of $\pi_1(Z^*)$ by the subgroup H generated by $\{\rho(P); P \in \mathcal{S}\}$. Here $\rho(P)$ is the corresponding element of $\pi_1(Z^*)$ to a small loop $l(P)$ which goes around the divisor $E(P)$, $P \in \mathcal{S}$. See Figure (4.1.1).

By the commutativity of the fundamental group $\pi_1(Z^*)$, this element does not depend on the choice of the small loop $l(P)$. Let $P \in \mathrm{Vertex}(\Sigma^*)$. We can write P as $P = \sum_{i=1}^m a_i P_i$ for some integers $a_1, \cdots, a_m$. Then it

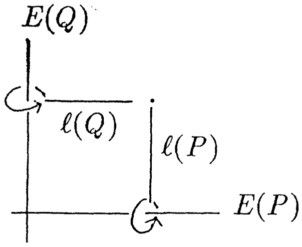

Figure (4.1.1)

is easy to see that the above element $\rho(P)$ of $\mathbf{Z}^{*m}$ is equal to $(a_1, \cdots, a_m)$ if we identify $\pi_1(Z)$ with $\mathbf{Z}^m$ through the isomorphism $\pi_1(Z^*) \cong \pi_1(\mathbf{C}_\sigma^{*m}) \cong \mathbf{Z}^m$. See [Ok3] for a similar calculation. By the assumption on P_i, we have that $\rho(P_i) = (0, \cdots, \overset{i}{\breve{1}}, \cdots, 0)$ for $i = 1, \cdots, m-k$. We define $A_\sigma(P) = (a_{m-k+1}, \cdots, a_m)$ as § 7 of [Ok2]. Then we have

THEOREM (4.2). *Assume that $m - k > 1$. Then the fundamental group of Y is an abelian group which is isomorphic to the quotient group of $\mathbf{Z}^k$ by the subgroup generated by $\{A_\sigma(P); P \in \mathscr{S}\}$.*

The fundamental group $\pi_1(Y)$ is finite if the subgroup H has rank m. It seems that this is always true but the proof involves some combinatorial problem and we will treat this somewhere else. This class of algebraic varieties may give many interesting examples of algebraic surfaces as was the case for the hypersurfaces in a toric variety ([Ok5]). Note also that the fundamental group does not depend on the choice of a smooth compactification Y.

EXAMPLE (4.3). Let $m = 4$ and $k = 2$. Let

$$h_\alpha(\mathbf{u}) = \sum_{i=1}^{4} a_{\alpha i} u_i^{-17} u_{i+1} u_{i+2} u_{i+3}^{19} + 1, \qquad \alpha = 1, 2$$

where $u_{i+4} = u_i$, $i = 1, \cdots, 4$. Let Y be as before. Here the coefficient $\{a_{\alpha i}; \alpha = 1, 2, i = 1, \cdots, 4\}$ are generically chosen. Then the Newton diagram $\Delta(h_\alpha)$ is a 4-dimensional simplex with vertices $A_0 \underset{\text{def}}{=} \vec{0}$, $A_1 = (-17, 1, 1, 19), \cdots, A_4 = (1, 1, 19, -17)$. Then dual Newton diagram $\Gamma^*(h_1, h_2)$ is a simplicial cone with vertices $P = {}^t(1, 1, 1, 1)$, $P_1 = (3, 4, 5, 6)$, $\cdots, P_4 = {}^t(4, 5, 6, 3)$. Note that $\det(P_i, P_{i+1}) = 1$ and $\det(P_i, P_{i+2}) = 2$. Thus on the two-dimensional cone (P_i, P_{i+2}) we have to add vertices $Q_i \underset{\text{def}}{=} (P_i + P_{i+2})/2$ $(i = 1, 2)$. An easy (but not so pleasant) calculation shows that $\pi_1(Y) \cong \mathbf{Z}_2 \times \mathbf{Z}_4$.

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