X. Wang
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THE HECKE ALGEBRA ON THE COHOMOLOGY OF $\Gamma_0(p_0)$

XIANGDONG WANG

§ 1. Introduction

Let p_0 be a prime, $p_0 > 3$ and $\Gamma_0(p_0)$, $\Gamma_1(p_0)$, as usual, the congruence subgroups of $\Gamma = PSL_2(\mathbb{Z})$.

$$\Gamma_0(p_0) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \bmod p_0\right\},$$
$$\Gamma_1(p_0) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p_0) \,\middle|\, d \equiv 1 \bmod p_0\right\}.$$

Denote

$$\Delta = \left\{r \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z},\ \gcd(a, b, c, d) = 1,\ \det(r) \not\equiv 0 \bmod p_0\right\},$$
$$\Delta_0 = \left\{r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \,\middle|\, c \equiv 0 \bmod p_0\right\},$$
$$\Delta_1 = \left\{r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0 \,\middle|\, d \equiv 1 \bmod p_0\right\}$$

with $\Delta_1 \subset \Delta_0 \subset \Delta$ and $\Delta_0/\Delta_1 \cong (\mathbb{Z}/p_0)^*$. Let $R = \mathbb{Z}[\frac{1}{6}]$. We consider the following R-module $M_n = \{\sum_{v=0}^{n} a_v x^v y^{n-v} \,|\, a_v \in R\}$. The semigroup Δ acts on M_n via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^v y^{n-v} = (ax + cy)^v (bx + dy)^{n-v}.$$

Let η: $\Gamma_0(p_0)/\Gamma_1(p_0) \cong (\mathbb{Z}/p_0)^* \to R^*$ be the Legendre-symbol. We extend η to Δ_0 such that η acts trivially on Δ_1, i.e. η is a character from Δ_0/Δ_1 to R^*. Denote by R_η the R-module of rank 1 with a Δ_0-operation given by $s_0.1 = \eta(s_0)\cdot 1$, $\forall s_0 \in \Delta_0$. Set $M_{n,\eta} = M_n \otimes R_\eta$. This is then a $R[\Delta_0]$-module. The goal of the present paper is to investigate the Hecke algebra on the cohomology group $H^*(\Gamma_0(p_0), M_{n,\eta})$. Let $S_k(\Gamma_0(p_0), \eta)$, as usual, be the

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cusp forms with the weight k. Then the Eichler-Shimura theorem says that the following sequence

$$0 \to S_{n+2}(\Gamma_0(p_0), \eta) \oplus \overline{S_{n+2}(\Gamma_0(p_0), \eta)} \to H^1(\Gamma_0(p_0), M_{n,\eta} \otimes \mathbb{C}) \xrightarrow{r^*} \bigoplus_s H^1(\Gamma_0(p_0)_s, M_{n,\eta} \otimes \mathbb{C}) \to 0$$

is exact, where s runs over cusps of $\Gamma_0(p_0)$ and $\Gamma_0(p_0)_s := \{r \in \Gamma_0(p_0) \mid r.s = s\} = \langle T_s \rangle$ is an infinite cyclic group. It is well known that $\Gamma_0(p_0)$ has two cusps 0, ∞. The dimension of

$$H^1(\Gamma_0(p_0)_s, M_{n,\eta} \otimes \mathbb{C}) \cong M_{n,\eta}/(1 - T_s)M_{n,\eta}$$

is 1, which follows in particular that

$$\dim(H^1(\Gamma_0(p_0), M_{n,\eta} \otimes \mathbb{C})) = 2\dim(S_{n+2}(\Gamma_0(p_0), \eta)) + 2$$

(cf. [Hab] p. 284). By the above identification, we see that the study of the Hecke algebra on the cusp forms is equivalent to that on the cohomology $H^1(\Gamma_0(p_0), M_{n,\eta})$, see Chap. 1 in [Hab] for more details and backgrounds. Applying the Shapiro lemma to the cohomology group of $\Gamma_0(p_0)$ we get in Section 5 a basis for the cohomology $H^1(\Gamma_0(p_0), M_{n,\eta})$. Using this basis we obtain an algorithm that can be used to compute the Hecke operator T_l on the cohomology $H^1(\Gamma_0(p_0), M_{n,\eta})$. Finally the characteristic polynomials of T_2, T_3, T_5 and T_7 are given in Table 1 for small p_0 and n.

§ 2. The Shapiro-Lemma

In order to determine the cohomology of $\Gamma_0(p_0)$, we first recall the Shapiro-Lemma. Denote by $W_{n,\eta}$ the induced module of $M_{n,\eta}$ on Γ:

$$W_{n,\eta} = \mathrm{Ind}_{\Gamma_0(p_0)}^{\Gamma} M_{n,\eta} = \{f \colon \Gamma \to M_{n,\eta} \mid f(r_0 r) = r_0.f(r), \ \forall r_0 \in \Gamma_0(p_0)\}$$

The operation of Γ on $W_{n,\eta}$ is defined by $(a.f)(r) := f(ra)$, $a, r \in \Gamma$. We extend now this operation to an operation of Δ on $W_{n,\eta}$. For $a \in \Delta$, $r \in \Gamma$, there exist always $a' \in \Delta_0$, $r' \in \Gamma$, such that $ra = a'r'$. We define $(a.f)(r) := a'.f(r')$. It is obvious that this definition coincides with the above definition if $a \in \Gamma$. Now on the cohomology groups

$$H^1(\Gamma_0(p_0), M_{n,\eta}) \quad \text{and} \quad H^1(\Gamma, W_{n,\eta})$$

we can define the Hecke algebra (cf. [Hab] Chap. 1). By the Shapiro-Lemma (cf. [Bro] or [AS] § 1) there is a canonical isomorphism between

$$H^1(\Gamma_0(p_0), M_{n,\eta}) \cong H^1(\Gamma, W_{n,\eta})$$

as modules under the Hecke algebra.

§ 3. The dimension of the cohomology $H^1(\Gamma, W_{n,\eta})$

To get startet, we consider the Γ-module $W_{n,\eta}$. Let

$$a_i = \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix}, \quad i = 0, 1, \cdots, p_0 - 1, \quad a_{p_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\{a_i\}$ is then a set of representatives of Γ with respect to $\Gamma_0(p_0)$:

$$\Gamma = \bigcup_{i=0}^{p_0} \Gamma_0(p_0) a_i .$$

An element $f \in W_{n,\eta}$ is uniquely determined by the values $f(a_0), f(a_1), \cdots, f(a_{p_0})$ by using the condition $f(r_0 r) = r_0 f(r)$. The dimension of $W_{n,\eta}$ over R is $(p_0 + 1)\cdot \dim(M_{n,\eta}) = (p_0 + 1)(n + 1)$. In other words, $W_{n,\eta}$ is generated by the elements $(w_0, w_1, \cdots, w_{p_0})$ with $w_i \in M_{n,\eta}$.

Now we consider the cohomology $H^1(\Gamma, W_{n,\eta})$. The structure of cohomology $H^1(\Gamma, W_{n,\eta})$ is well known (cf. [Wan] § 1):

$$H^1(\Gamma, W_{n,\eta}) \cong W_{n,\eta}/(W^S_{n,\eta} + W^Q_{n,\eta})$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $W^r_{n,\eta} := \{w \in W_{n,\eta} \mid r.w = w\}$ for $r \in \Gamma$. We begin with the description of $W^S_{n,\eta}$. It is easy to show that

$$\begin{cases} a_0 S = a_{p_0} \\ a_i S = S_i a_j \quad i\cdot j \equiv -1 \bmod p_0, \quad S_i = \begin{pmatrix} -j & -1 \\ 1 + ij & i \end{pmatrix} \in \Gamma_0(p_0) \\ a_{p_0} S = a_0 \end{cases}$$

and by the definition we obtain

$$\begin{cases} (S.f)(a_0) = f(a_{p_0}) \\ (S.f)(a_i) = S_i.f(a_j), \quad i = 1, \cdots, p_0 - 1 \\ (S.f)(a_{p_0}) = f(a_0) . \end{cases}$$

Therefore, $W^S_{n,\eta}$ has the expression:

$$\begin{aligned} W^S_{n,\eta} &= \{f \in W_{n,\eta} \mid f(a_0) = f(a_{p_0}),\ f(a_i) = S_i.f(a_j)\} \\ &= \{(w_0, \cdots, w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \mid w_0 = w_{p_0},\ w_i = S_i.w_j\} \end{aligned}$$

Let $T = SQ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. One shows immediately that

$$\begin{cases} a_i T = a_{i+1}, & i = 0, 1, \cdots, p_0 - 2 \\ a_{p_0-1} T = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} a_0 \\ a_{p_0} T = T a_{p_0} \end{cases}$$

and

$$\begin{cases} a_0 Q = T a_{p_0} \\ a_1 Q = T^{-1} a_0 \\ a_i Q = S_i a_{j+1}, & i = 2, 3, \cdots, p_0 - 1 \\ a_{p_0} Q = a_1 \end{cases}$$

from which it follows

$$W_{n,\eta}^T = \left\{ (w_0, \cdots, w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \,\middle|\, w_0 = \cdots = w_{p_0-1} = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} w_0,\ w_{p_0} = T w_{p_0} \right\}$$

$$W_{n,\eta}^Q = \{ (w_0, \cdots, w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \mid T w_{p_0} = w_0, w_1 = w_{p_0}, S_i w_{j+1} = w_i \}.$$

For the purpose of determining the dimension of $H^1(\Gamma, W_{n,\eta})$ we show now

3.1 LEMMA. $W_{n,\eta}^S \cap W_{n,\eta}^Q = \{0\}$.

Proof. Let $f = (w_0, \cdots, w_{p_0}) \in W_{n,\eta}^S \cap W_{n,\eta}^Q$. It implies that $f \in W_{n,\eta}^T$ i.e.,

$$w_0 = w_1 = \cdots = w_{p_0-1} = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} w_0, \quad \text{and} \quad w_{p_0} \in M_{n,\eta}^T .$$

Hence it follows that $w_{p_0} = ax^n$, $w_0 = by^n$ for some a, b. For $f \in W_{n,\eta}^S$ we have $w_0 = w_{p_0}$, i.e. $ax^n = by^n$, which implies that $a = b = 0$. □

Therefore, the dimension of the cohomology $W_{n,\eta}$ is

$$\dim (H^1(\Gamma, W_{n,\eta})) = \dim (W_{n,\eta}) - \dim (W_{n,\eta}^S) - \dim (W_{n,\eta}^Q) .$$

Now we compute the dimensions of $W_{n,\eta}^S$ and $W_{n,\eta}^Q$.

Let ν_2, ν_3 the number of $\Gamma_0(p_0)$-inequivalent elliptic points of the order 2, 3 respectively.

$$\nu_2 = 0 \text{ or } 2 \equiv p_0 + 1 \bmod 4, \qquad \nu_3 = 0 \text{ or } 2 \equiv p_0 + 1 \bmod 3$$

It is obvious that

$$\nu_2 = 2 \Leftrightarrow p_0 \equiv 1 \bmod 4 \Leftrightarrow \eta(-1) = 1 \Leftrightarrow \text{there is a } i_0 \text{ with } i_0^2 \equiv -1 \bmod p_0 .$$

In that case one has $\eta(i_0) = i_0^{(p_0-1)/2} = (-1)^{(p_0-1)/4}$ and $a_{i_0} S = S_{i_0} a_{i_0}$. Furthermore it is easy to show that

$$\nu_3 = 2 \Leftrightarrow p \equiv 1 \bmod 3 \Leftrightarrow 6 \mid p_0 - 1 \Leftrightarrow \text{there is a } i_0 \text{ of order 6 in } (\mathbb{Z}/p_0)^*$$
$$\Leftrightarrow i_0^3 \equiv -1 \bmod p_0 \Leftrightarrow i_0(i_0 - 1) \equiv -1 \bmod p_0 .$$

It follows that $a_{i_0}Q = S_{i_0}a_{i_0}$. Since $(i_0 - 1)^2 \equiv -i_0$ one has $\eta(i_0) = \eta(-1)\eta(i_0 - 1)^2 = \eta(-1) = (-1)^{(p_0-1)/2}$.

3.2 LEMMA.

$$\dim(W^S_{n,\eta}) = 2\left[\frac{p_0+1}{4}\right](n+1) + 2d_S$$

$$\dim(W^Q_{n,\eta}) = \left[\frac{p_0+1}{3}\right](n+1) + 2d_Q$$

where

$$d_S = \begin{cases} 0 & p_0 \equiv 3 \bmod 4 \\ 2\left[\frac{n}{4}\right] + 1 & p_0 \equiv 1 \bmod 8, \\ 2\left[\frac{n+2}{4}\right] & p_0 \equiv 5 \bmod 8 \end{cases} \qquad d_Q = \begin{cases} 0 & p_0 \equiv 2 \bmod 3 \\ 2\left[\frac{n}{6}\right] + 1 & p_0 \equiv 1 \bmod 12. \\ 2\left[\frac{n+3}{6}\right] & p_0 \equiv 7 \bmod 12 \end{cases}$$

In particular,

$$\dim(H^1(\Gamma, W_{n,\eta})) = \left(p_0 + 1 - 2\left[\frac{p_0+1}{4}\right] - \left[\frac{p_0+1}{3}\right]\right)(n+1) - 2d_S - 2d_Q$$

$$\dim(S_{n+2}(\Gamma_0(p_0), \eta)) = \frac{1}{2}\left(p_0 + 1 - 2\left[\frac{p_0+1}{4}\right] - \left[\frac{p_0+1}{3}\right]\right)(n+1) - d_S - d_Q - 1 .$$

Proof. For $f = (w_0, \cdots, w_{p_0}) \in W^S_{n,\eta}$ we have $w_i = S_i w_j$ and $S_j = S_i^{-1}$. If $j \neq i$, then w_j is uniquely determined by w_i. The number of such pair (i, j) is $2[(p_0 + 1)/4]$. If $j = i$, that means $p_0 \equiv 1 \bmod 4$, one has $w \in \operatorname{Ker}(1 - S_i)$. We calculate the dimension of $\operatorname{Ker}(1 - S_i)$. Let $m \otimes 1 \in M_{n,\eta}$, then $S_i.(m \otimes 1) = \eta(i)(S_i m \otimes 1)$. For $S_i = \begin{pmatrix} -i & -1 \\ 1 + i^2 & i \end{pmatrix}$ there is a regular matrix P with $S_i = PSP^{-1}$. It follows that

$$d_S = \dim(M^{S_i}_{n,\eta}) = \dim(\operatorname{Ker}(1 - S_i)) = \dim(\operatorname{Ker}(1 - \eta(i)S)) .$$

For $p_0 \equiv 1 \bmod 8$ one has $\eta(i) = (-1)^{(p_0-1)/4} = 1$. The dimension of $\operatorname{Ker}(1 - S)$ can be easily determined, $\dim \operatorname{Ker}(1 - S) = 2[n/4] + 1$. Since there are two i with $i^2 \equiv -1$, dimension of $W^S_{n,\eta}$ has the expression:

$$\dim(W^s_{n,\eta}) = 2\left[\frac{p_0+1}{4}\right](n+1) + 4\left[\frac{n}{4}\right] + 2$$

The other cases can be proved in the same manner. □

§ 4. The dimension of $H^1(\Gamma, W_{n,\eta})_\pm$

Let $\Gamma_\infty = \langle T \rangle$ be the stabilizer of the cusp ∞ in Γ. We have an exact sequence:

$$0 \to H^0(\Gamma_\infty, W_{n,\eta}) \to H^1_c(\Gamma, W_{n,\eta}) \to H^1(\Gamma, W_{n,\eta}) \to H^1(\Gamma_\infty, W_{n,\eta}) \to \cdots$$

where $H^1_c(\cdot, \cdot)$ is the cohomology with the compact support, referring to [Hab] Chap. 1 for details and backgrounds. It has been shown in [Wan] § 1 that the cohomology

$$H^1(\Gamma_\infty, W_{n,\eta}) \cong W_{n,\eta}/(1-T)W_{n,\eta}.$$

4.1 LEMMA.

$$H^1(\Gamma_\infty, W_{n,\eta} \otimes \mathbb{Q}) \cong \mathbb{Q}\phi_0 + \mathbb{Q}\phi_\infty$$

where $\phi_0(T) = (x^n, 0, \cdots, 0) \in W_{n,\eta}$, $\phi_\infty(T) = (0, \cdots 0, y^n) \in W_{n,\eta}$.

Proof. For each $w = (w_0, \cdots, w_{p_0}) \in W_{n,\eta}$, we consider the equation

$$(*) \qquad w = a(x^n, 0, \cdots, 0) + b(0, \cdots, 0, y^n) + (T-1)v$$

with $v = (v_0, \cdots, v_{p_0}) \in W_{n,\eta}$, which means:

$$\begin{aligned} w_0 &= ax^n + v_1 - v_0 \\ w_i &= v_{i+1} - v_i, \quad 0 < i < p_0 - 1 \\ w_{p_0-1} &= \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} v_0 - v_{p_0-1} \\ w_{p_0} &= by^n + (T-1)v_{p_0}, \end{aligned}$$

it follows that

$$\left(1 - \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix}\right) v_0 = ax^n - \sum_{j=0}^{p_0-1} w_j.$$

We take a as the coefficient of x^n in $\sum_{j=0}^{p_0-1} w_j$ and b as the coefficient of y^n in w_{p_0}. The equations

$$\begin{aligned} \left(1 - \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix}\right) v_0 &= c_0 y^n + c_1 x y^{n-1} + \cdots + c_{n-1} x^{n-1} y \\ (1-T)v_{p_0} &= d_1 x y^{n-1} + d_2 x^2 y^{n-2} + \cdots + d_n x^n \end{aligned}$$

are always solvable in $M_{n,\eta}\otimes\mathbb{Q}$ for any $c_0, \cdots, c_{n-1}, d_1, \cdots, d_n \in \mathbb{Q}$. Therefore the equation (*) is solvable in $M_{n,\eta}\otimes\mathbb{Q}$. □

Let $\varepsilon=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. We define for a cocycle $\phi\in Z^1(\Gamma, W_{n,\eta})$

$$(\varepsilon\phi)(r):=\varepsilon\phi(\varepsilon^{-1}r\varepsilon) \qquad \forall r\in\Gamma .$$

It induces an automorphism of the order 2 on the cohomologies (cf. [Wan] §1). Hence we obtain two exact sequences:

$$0\to H^0(\Gamma_\infty, W_{n,\eta})_+ \to H^1_c(\Gamma, W_{n,\eta})_+ \to H^1(\Gamma, W_{n,\eta})_+ \xrightarrow{r^*} H^1(\Gamma_\infty, W_{n,\eta})_+ \to \cdots$$

$$0\to H^0(\Gamma_\infty, W_{n,\eta})_- \to H^1_c(\Gamma, W_{n,\eta})_- \to H^1(\Gamma, W_{n,\eta})_- \xrightarrow{r^*} H^1(\Gamma_\infty, W_{n,\eta})_- \to \cdots$$

where $H^1(\Gamma, W_{n,\eta})_\pm := \{\phi\in H^1(\Gamma, W_{n,\eta}) \mid \varepsilon.\phi = \pm\phi\}$. Since

$$\begin{cases} a_0\varepsilon = \varepsilon a_0 \\ a_i\varepsilon = E.a_{p_0-i}, \quad i=1,2,\cdots,p_0-1 \\ a_{p_0}\varepsilon = \varepsilon a_{p_0} \end{cases}$$

where $E:=\begin{pmatrix}-1 & 0\\ p_0 & 1\end{pmatrix}$. The operation of ε on $W_{n,\eta}$ is

$$\begin{cases} (\varepsilon u)(a_0) = \varepsilon.u(a_0) \\ (\varepsilon u)(a_i) = E.u(a_{p_0-i}) \\ (\varepsilon u)(a_{p_0}) = \varepsilon.u(a_{p_0}) . \end{cases}$$

In particular, it follows that

$$\varepsilon.\phi_\infty(T)=\phi_\infty(T), \qquad \varepsilon.\phi_0(T)=(-1)^n\phi_0(T) .$$

4.2 LEMMA.

a. $\phi_\infty\in H^1(\Gamma_\infty, W_{n,\eta})_-$

b. $\phi_0\in H^1(\Gamma_\infty, W_{n,\eta})_-$ *for n even;* $\phi_0\in H^1(\Gamma, W_{n,\eta})_+$ *for n odd.*

Proof.

$$\begin{aligned}(\varepsilon\phi_\infty)(T) &= \varepsilon.\phi_\infty(\varepsilon^{-1}T\varepsilon) = \varepsilon.\phi_\infty(T^{-1}) = -\varepsilon T^{-1}\phi_\infty(T) = -T\varepsilon.\phi_\infty(T)\\ &= -\varepsilon.\phi_\infty(T) + (1-T)\varepsilon.\phi_\infty(T) \sim -\varepsilon.\phi_\infty(T) = -\phi_\infty(T) .\end{aligned}$$

It means that $\varepsilon.\phi_\infty = -\phi_\infty$. (b) can be proved in the same way. □

By applying the Eichler-Shimura isomorphism, together with the observation above, we obtain

4.3 COROLLARY.

a. *For n even we have*

$$\dim(H^1(\Gamma, W_{n,\eta})_-) = \frac{1}{2}\dim(H^1(\Gamma, W_{n,\eta})) + 1$$

$$\dim(H^1(\Gamma, W_{n,\eta})_+) = \frac{1}{2}\dim(H^1(\Gamma, W_{n,\eta})) - 1\,.$$

b. *For n odd we have*

$$\dim(H^1(\Gamma, W_{n,\eta})_-) = \frac{1}{2}\dim(H^1(\Gamma, W_{n,\eta}))$$

$$\dim(H^1(\Gamma, W_{n,\eta})_+) = \frac{1}{2}\dim(H^1(\Gamma, W_{n,\eta}))\,.$$

§ 5. The basis of $H^1(\Gamma, W_{n,\eta})$

It is well known that

$$H^1(\Gamma, W_{n,\eta}) \cong W_{n,\eta}/(W^S_{n,\eta} + W^Q_{n,\eta})\,.$$

Our goal in this section is to choose a subset V of $W_{n,\eta}$ such that $W_{n,\eta} = W^S_{n,\eta} \oplus W^Q_{n,\eta} \oplus V$. Since the group Γ is generated by S, Q with the relations $S^2 = 1$, $Q^3 = 1$ (cf. [Ser]), the cohomology

$$\begin{aligned} H^1(\Gamma, W_{n,\eta}) &= \frac{\{(\phi(S), \phi(Q)) \mid \phi(S) \in (1-S)W_{n,\eta}, \phi(Q) \in (1-Q)W_{n,\eta}\}}{\{((1-S)u, (1-Q)u) \mid u \in W_{n,\eta}\}} \\ &\cong \frac{\{\phi(Q) \mid \phi(S) = 0, \phi(Q) \in (1-Q)W_{n,\eta}\}}{\{(1-Q)u \mid u \in W^S_{n,\eta}\}} \\ &\cong \{(1-Q)v \mid v \in V\}\,, \end{aligned}$$

i.e, every class $\phi \in H^1(\Gamma, W_{n,\eta})$ has the form

$$\begin{cases} \phi(S) = 0 \\ \phi(Q) = (1-Q)u, \quad u \in V\,. \end{cases}$$

Defining by α_i (resp. β_i) the permutation of $\{0, 1, \cdots, p_0\}$ induced by the operation of S (resp. Q) on $\{a_0, a_1, \cdots, a_{p_0}\}$. We have (cf. § 3)

$$\alpha_i \cdot i \equiv -1 \bmod p_0, \quad 0 < i < p_0$$
$$\beta_i = \alpha_i + 1, \quad 1 < i < p_0$$

5.1 DEFINITION. For $i, j, k \in \{1, 2, \cdots, p_0 - 1\}$

a. A pair (i, j) is called a α-pair if $j = \alpha_i$ $i = \alpha_j$, or equivalently, $i \cdot j \equiv -1 \bmod p_0$;

b. A triple (i, j, k) is called a β-triple if $j = \beta_i$, $k = \beta_j$, $i = \beta_k$, or equivalently, $i \cdot j \cdot k \equiv -1 \bmod p_0$;

c. Let B be a subset of $\{1, 2, \cdots, p_0 - 1\}$. We denote by $\langle B \rangle$ the subset of $\{1, 2, \cdots, p_0 - 1\}$ determined by the following conditions:

i. $B \subset \langle B \rangle$;

ii. if (i, j) is an α-pair and $j \in \langle B \rangle$ then $i \in \langle B \rangle$;

iii. if (i, j, k) is a β-triple and $j, k \in \langle B \rangle$ then $i \in \langle B \rangle$;

d. A subset B of $\{1, 2, \cdots, p_0 - 1\}$ is called a basis set if it satisfies:

i. $\langle B \rangle = \{1, 2, \cdots, p_0 - 1\}$;

ii. $\forall i \in B$, $\langle B \backslash \{i\} \rangle \neq \{1, 2,, \cdots, p_0 - 1\}$.

It follows immediately from the definition that the number of the α-pair is $2[(p_0 + 1)/4] - 1$ and the number of the β-triple is $[(p_0 + 1)/3] - 1$. Therefore the number of the elements in B is

$$\begin{aligned} \sharp B &= (p_0 - 1) - \left(2\left[\frac{p_0 + 1}{4}\right] - 1\right) - \left(\left[\frac{p_0 + 1}{3}\right] - 1\right) \\ &= p_0 + 1 - 2\left[\frac{p_0 + 1}{4}\right] - \left[\frac{p_0 + 1}{3}\right]. \end{aligned}$$

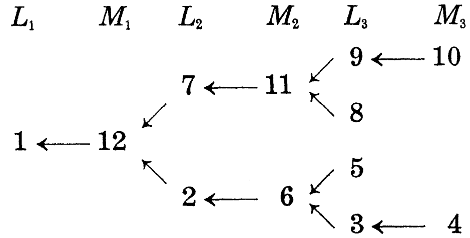
We define inductively two series of subsets of $\{1, 2, \cdots, p_0 - 1\}$.

$$\begin{aligned} &L_1 = \{1\} \\ &M_r = \{\alpha_i \mid i \in L_r\} \backslash L_r, \quad r > 0 \\ &L_{r+1} = \{j = \beta_i, \beta_j \mid i \in M_r\} \backslash M_r \end{aligned}$$

5.2 EXAMPLE. $p_0 = 13$. In that case $\nu_2 = 2$, $\nu_3 = 2$. The permutations of $\{a_0, a_1, \cdots, a_{p_0}\}$ induced by the operation of S and Q are:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
S	13	12	6	4	3	5	2	11	8	10	9	7	1	0
Q	13	0	7	5	4	6	3	12	9	11	10	8	2	1

The sets L_r and M_r can be described by the diagram:



5.3 LEMMA.

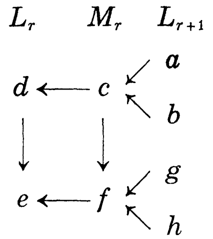
a. $\{1, 2, \cdots, p_0 - 1\} = \bigcup_{r=1}^{N}(L_r \cup M_r)$ *for some* $N < p_0$;

b. *For each* $i \in L_r$ *there exists a* $j \in L_r$ *with* $i \cdot j \equiv 1 \bmod p_0$;

c. *For each* $i \in M_r$ *there exists a* $j \in M_r$ *with* $i \cdot j \equiv 1 \bmod p_0$.

Proof. a. Assume that a is the smallest element in $\{1, 2, \cdots, p_0 - 1\}$ with the property $a \notin \bigcup_{r=1}^{\infty}(L_r \cup M_r)$. Let $a = \beta_b$ for some $b \in \{1, 2, \cdots, p_0 - 1\}$. Then $a = \beta_b = \alpha_b + 1$ and $\alpha_b < a$. By the assumption it implies $\alpha_b \in \bigcup_{r=1}^{\infty}(L_r \cup M_r)$, which follow that $b \in \bigcup_{r=1}^{\infty}(L_r \cup M_r)$ and $a \in \bigcup_{r=1}^{\infty}(L_r \cup M_r)$ by the definition of $\langle B \rangle$. It contradicts the assumption.

b. We prove the assertion by the induction. The assertion for $r = 1$ is obvious. Let a be an element in L_{r+1}, then there is an element $c \in M_r$ such that $a = \beta_c$ or $c = \beta_a$. We treat only the case $a = \beta_c$. Let $b = \beta_a \in L_{r+1}$ and $d = \alpha_c \in L_r$. By the induction assumption there is a $e \in L_r$ with $d \cdot e \equiv 1 \bmod p_0$. Let $f = \alpha_e$, we see immediately that $f \cdot c \equiv 1 \bmod p_0$. Let $g = \beta_f$, $h = \beta_g \in L_{r+1}$, we look at the following diagram:



and assert that $a \cdot h \equiv 1 \bmod p_0$. Indeed,

$$\begin{aligned} a = \beta_c = \alpha_c + 1 = d + 1 &\equiv (d + 1)\cdot(-ef) \equiv (e + 1)\cdot(-f) \\ &\equiv 1 - f = 1 - \beta_h = -\alpha_h \end{aligned}$$

i.e., $a \cdot h \equiv -\alpha_h \cdot h \equiv 1$.

c. It follows immediately from (b).

5.4 LEMMA. *There is a basis set* B *with the property*: *if* $a \in B$ *then* $p_0 - a \in B$.

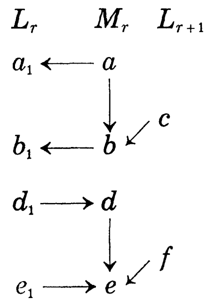
The proof of the lemma presents in fact an algorithm to compute the basis set B.

Proof. First note that $\langle L_r \rangle \subset \langle M_r \rangle \subset \langle L_{r+1} \rangle$.

Case 1: If $a, \alpha_a \in L_r$ and $a \notin \langle B \rangle$, there is an elements $b \in L_r$ with $ab \equiv 1$, which yields $\alpha_a \cdot \alpha_b \equiv 1$. Since $(a + \alpha_b)b = ab + \alpha_b \cdot b \equiv 1 +$

$= 0$, one has $a + \alpha_b = p_0$ and $\{a, b, \alpha_a, \alpha_b\} \subset \langle\{a, \alpha_b\}\rangle$. Hence we add a, α_b to B.

Case 2. (a, b, c) is a β-triple, $a, b \in M_r$, $c \in L_{r+1}$ and $a, b \notin B$. For $a, b \in M_r$ there are $d, e \in M_r$ with $ad \equiv 1$, $be \equiv 1$. We consider the following diagram:

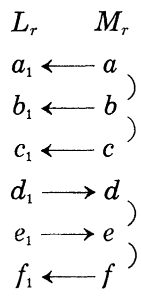


one verifies trivially that $a + d_1 = p_0$ and

$$\{a, b, c, d, e, f, a_1, b_1, d_1, e_1\} \subset \langle\{a, d_1, c, f\}\rangle .$$

Therefore we add a, d_1 to B.

Case 3: (a, b, c) is a β-triple, $a, b, c \in M_r$ and $a, b, c \notin \langle B\rangle$. There are $d, e, f \in M_r$ with $ad \equiv 1$, $be \equiv 1$, $cf \equiv 1$. We consider the following diagram:



It is obvious, that $a + d_1 = p_0$, $b + e_1 = p_0$ and

$$\{a, b, c, d, e, f, a_1, b_1, c_1, d_1, e_1, f_1\} \subset \langle\{a, b, d_1, e_1\}\rangle .$$

We add thus a, b, d_1, e_1 to B.

In such a way we obtain a basis set B. □

In the example 5.2 we can take the basis set $B = \{5, 8, 4, 9\}$.

We study now the cohomology $H^1(\Gamma, W_{n,\eta}) = W_{n,\eta}/(W^S_{n,\eta} + W^Q_{n,\eta})$. Let B be a basis set. Then each element $(0, w_1, \cdots, w_{p_0-1}, 0) \in W_{n,\eta}$ is congruent mod $W^S_{n,\eta} + W^Q_{n,\eta}$ to an element $g = (v_0, \cdots, v_{p_0}) \in W_{n,\eta}$ with $v_i = 0$ for $i \notin B$.

If $\nu_2 = 2$, there is a $i_0 \in B$ such that $i_0^2 \equiv -1$. If $w_{i_0} \in \operatorname{Ker}(1 - S_{i_0}) = M^{S_{i_0}}_{n,\eta}$, then $(0, \cdots, 0, w_{i_0}, 0, \cdots, 0) \in W^S_{n,\eta}$. Therefore

$$\begin{aligned}\{(0, \cdots, w_{i_0}, \cdots, 0) \mid w_{i_0} \in M_{n,\eta}\}/(W^S_{n,\eta} + W^Q_{n,\eta})\\ \cong \{(0, \cdots, v_{i_0}, \cdots, 0) \mid v_{i_0} \in M_{n,\eta}/M^{S_{i_0}}_{n,\eta}\}.\end{aligned}$$

Similarly, if $\nu_3 = 2$ and $i_0 \in B$, $i_0^3 \equiv -1$, then

$$\begin{aligned}\{(0, \cdots, w_{i_0}, \cdots, 0) \mid w_{i_0} \in M_{n,\eta}\}/(W^S_{n,\eta} + W^Q_{n,\eta})\\ \cong \{(0, \cdots, v_{i_0}, \cdots, 0) \mid v_{i_0} \in M_{n,\eta}/M^{S_{i_0}}_{n,\eta}\}.\end{aligned}$$

Now we consider the index 0, p_0. Since

$$(0, \cdots, 0, w_{p_0}) = (-w_{p_0}, 0, \cdots, 0) \bmod W^S_{n,\eta} + W^Q_{n,\eta},$$

we need only to consider only the index 0. Let

$$(w_0, 0, \cdots, 0) = \underbrace{(a, 0, \cdots, 0, a)}_{\in W^S_{n,\eta}} + \underbrace{(Tb, b, 0, \cdots, 0, b)}_{\in W^Q_{n,\eta}} + (0, c, 0, \cdots, 0)$$

for some a, b, c, then $b = -a$, $c = a$, $(1 - T)a = \omega_0$. The equation $(1 - T)a = w_0$ can be solved only for $w_0 = c_1 x y^{n-1} + c_2 x^2 y^{n-2} + \cdots + c_n x^n$. Therefore the element $(y^n, 0, \cdots, 0)$ is linear independent to

$$\begin{aligned}\{(0, \cdots, 0, v_i, 0, \cdots, 0) \mid i \in B,\ v_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta}\\ \text{if } i^2 \equiv -1 \text{ or } i^3 \equiv -1\} \bmod W^S_{n,\eta} + W^Q_{n,\eta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(0, x^n, 0, \cdots, 0) = (-x^n, 0, \cdots, 0, -x^n) + (Tx^n, x^n, 0, \cdots, 0, x^n)\\ \in W^S_{n,n} + W^Q_{n,\eta}\end{aligned}$$

and $(0, x^n, 0, \cdots, 0)$ can be represented by the elements of

$$\{(0, \cdots, 0, v_i, 0, \cdots, 0) \mid i \in B,\ v_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta} \text{ if } i^2 \equiv -1 \text{ or } i^3 \equiv -1\},$$

which implies that the elements of

$$\{(0, \cdots, 0, v_i, 0, \cdots, 0) \mid i \in B,\ v_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta} \text{ if } i^2 \equiv -1 \text{ or } i^3 \equiv -1\}$$

are linear dependent mod $W^S_{n,\eta} + W^Q_{n,\eta}$. A basis of $H^1(\Gamma, W_{n,\eta})$ is then $(y^n, 0, \cdots, 0)$ and

$$\{(0, \cdots, 0, v_i, 0, \cdots, 0) \mid i \in B,\ v_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M_{n,\eta}^{S_i} \\ \text{if } i^2 \equiv -1 \text{ or } i^3 \equiv -1\} \bmod \sim ,$$

where the relation $\sim$ is given by the equation

$$(0, x^n, 0, \cdots, 0) \equiv 0 \bmod W_{n,\eta}^S + W_{n,\eta}^Q$$

§ 6. The basis of $H^1(\Gamma, W_{n,\eta})_\pm$

We shall first deal with the operation of ε on $H^1(\Gamma, W_{n,\eta})$. From the definition in § 4 we have for a class $\phi \in H^1(\Gamma, W_{n,\eta})$, $\phi(S) = 0$, $\phi(Q) = (1 - Q)u$,

$$(\varepsilon\phi)(S) = \varepsilon.\phi(\varepsilon S\varepsilon) = \varepsilon.\phi(S^{-1}) = 0$$

$$(\varepsilon\phi)(Q) = \varepsilon.\phi(\varepsilon Q\varepsilon) = \varepsilon.\phi(SQ^{-1}S) = -\varepsilon SQ^{-1}\phi(Q) = -\varepsilon SQ^{-1}(1 - Q)u \\ = (1 - Q)\varepsilon Su = (1 - Q)S\varepsilon u .$$

If $\phi \in H^1(\Gamma, W_{n,\eta})_-$, i.e. $\varepsilon\phi + \phi = 0$, it follows that $S\varepsilon u + u \in W_{n,\eta}^S + W_{n,\eta}^Q$. Since $S\varepsilon u + u = (S + 1)\varepsilon u + u - \varepsilon u$ and $(S + 1)\varepsilon u \in W_{n,\eta}^S$, we obtain

$$\phi \in H^1(\Gamma, W_{n,\eta})_- \iff u - \varepsilon u \in W_{n,\eta}^S + W_{n,\eta}^Q .$$

Similarly,

$$\phi \in H^1(\Gamma, W_{n,\eta})_+ \iff u + \varepsilon u \in W_{n,\eta}^S + W_{n,\eta}^Q .$$

In order to determine a basis of $H^1(\Gamma, W_{n,\eta})_-$ we consider the vector space

$$U := \{u = (u_0, \cdots, u_{p_0}) \in W_{n,\eta} \mid u - \varepsilon.u = 0\} .$$

U has a basis consisting of the elements $(u_0, \cdots, u_{p_0})$ which satisfy one of the following conditions (cf. § 4):

$$1.\quad \begin{cases} u_0 = x^j y^{n-j}, & j \text{ even} \\ u_i = 0, & i > 0 \end{cases}$$

$$2.\quad \begin{cases} u_{p_0} = x^j y^{n-j}, & j \text{ even} \\ u_i = 0, & i < p_0 \end{cases}$$

$$3.\quad \begin{cases} u_i = x^j y^{n-j} \\ u_{p_0-i} = E.u_i \\ u_k = 0, & k \neq i,\ p_0 - i . \end{cases}$$

In particular, the classes $\phi \in H^1(\Gamma, W_{n,\eta})$, $\phi(S) = 0$, $\phi(Q) = (1 - Q)u$ are classes in $H^1(\Gamma, W_{n,\eta})_-$ for n even, where $u = (u_0, \cdots, u_{p_0}) \in W_{n,\eta}$ with

$$1. \quad \begin{cases} u_0 = y^n \\ u_j = 0, \quad j > 0 \end{cases}$$

or

$$2. \quad \begin{cases} u_i \in W_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta}, \quad i \in B,\ i < p_0/2 \\ u_{p_0-i} = E.u_i \\ u_j = 0, \quad j \neq i,\ p_0 - i\,. \end{cases}$$

The number of the above classes is

$$1 + \tfrac{1}{2}{}^{\#}B \dim (M_{n,\eta}) - d_S - d_Q = \dim (H^1(\Gamma, W_{n,\eta})_-)\,.$$

By using the fact that the basis set B consists of the pair (i_1, i_2) with $i_1 + i_2 = p_0$ we find that the above classes generate the cohomology $H^1(\Gamma, W_{n,\eta})_-$. Therefore this set of classes is a basis of $H^1(\Gamma, W_{n,\eta})_-$ for n even.

Similarly, we choose a basis of $H^1(\Gamma, W_{n,\eta})_+$ for n odd: $\begin{cases} \phi(S) = 0 \\ \phi(Q) = (1 - Q)u \end{cases}$ with

$$\begin{cases} u_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta}, \quad i \in B,\ i < p_0/2 \\ u_{p_0-i} = -E.u_i \\ u_j = 0, \quad j \neq i,\ p_0 - i\,. \end{cases}$$

6.1. *Remark.* In general it is very difficult to determine the basis of $H^1(\Gamma, W_{n,\eta})_+$ for n even, because the dimension of $H^1(\Gamma, W_{n,\eta})_+$ is $\frac{1}{2}\dim (H^1(\Gamma, W_{n,\eta})) - 1$, and the dimension of the vector space generated by the set

$$\begin{cases} u_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta}, \quad i \in B,\ i < p_0/2 \\ u_{p_0-i} = E.u_i \end{cases}$$

is $\frac{1}{2}\dim (H^1(\Gamma, W_{n,\eta}))$. It implies that there is a relation between the above elements. The case $H^1(\Gamma, W_{n,\eta})_-$ for n odd is similar.

We are now interested in the boundary map r^* on the basis.

6.2 Lemma. *For a class $\phi \in H^1(\Gamma, W_{n,\eta})$ with $\phi(S) = 0$, $\phi(Q) = (1 - Q)u$,*

a. *if $u = (y^n, 0, \cdots, 0)$ then $r^*\phi = \phi_\infty$;*

b. *if $u = (0, \cdots, 0, u_i, 0, \cdots, 0)$, $0 < i < p_0$ then $r^*\phi = a\phi_0$ for some a.*

Proof. a.

$$\begin{aligned}(r^*\phi)(T) &= \phi(T) = S\phi(Q) = S(1 - Q)u = (S - T)u \\ &= (S - 1)u + (1 - T)u \sim (S - 1)u = (-y^n, 0, \cdots, 0, y^n).\end{aligned}$$

The solution of the equation (*) in § 4.1 is $a = 0$, $b = 1$, i.e., $r^*\phi = \phi_\infty$.

b. $r^*\phi(T) \sim (S-1)u = (0, \cdots, -u_i, 0, \cdots, S_i^{-1}u_i, 0, \cdots, 0)$. It is obvious that $b = 0$ (cf. the proof of § 4.1). Hence $r^*\phi = a\phi_0$ for some a. □

§ 7. The Hecke operator T_l on $H^1(\Gamma, W_{n,\eta})$

To get startet, we recall the definition of the Hecke operator T_l on $H^1(\Gamma, W_{n,\eta})$, where l is a prime, $l \neq p_0$. Let

$$b_i = \begin{pmatrix} 1 & i \\ 0 & l \end{pmatrix}, \quad i = 0, 1, \cdots, l-1 \quad \text{and} \quad b_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix},$$

they are a complete set of representatives of $\Gamma\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\Gamma$ with respect to Γ:

$$\Gamma\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\Gamma = \bigcup_{i=0}^{l} \Gamma b_i$$

For each $r \in \Gamma$ there is a $s_i \in \Gamma$ such that $b_i r = s_i b_j$ for some j. Define for a cocycle $f \in Z^1(\Gamma, W_{n,\eta})$

$$(T_l f)(r) := \sum_{i=0}^{l} b_i' f(s_i)$$

where $b_i' := \det(b_i) b_i^{-1}$.

All this is discussed in more detail in [AS] § 1 or [Wan] § 1.2.

7.1. Example. $l = 2$, $p_0 = 5$, $n = 4$

For $l = 2$ the representatives are

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

A simple calculation shows that

$$\begin{cases} b_0 S = S b_2 \\ b_1 S = SQ^{-1}SQS b_1 \\ b_2 S = S b_0 \end{cases} \quad \begin{cases} b_0 T = b_1 \\ b_1 T = T b_0 \\ b_2 T = T^2 b_2 \end{cases} \quad \begin{cases} b_0 Q = QSQ b_2 \\ b_1 Q = SQ^{-1}SQ^{-1} b_0 \\ b_2 Q = S b_1 . \end{cases}$$

By the definition we get for a class $\phi \in H^1(\Gamma, W_{n,\eta})$

$$(T_2\phi)(S) = b_0'\phi(S) + b_1'\phi(SQ^{-1}SQS) + b_2'\phi(S) = (S-1)b_1' SQ^{-1}\phi(Q)$$

$$(T_2\phi)(Q) = b_0'\phi(QSQ) + b_1'\phi(SQ^{-1}SQ^{-1}) + b_2'\phi(S) = (1-Q)(b_0' + b_0' SQ)\phi(Q).$$

Hence the cocycle $T_2\phi$ is cohomology to

$$T_2\phi \sim \begin{cases}(T_2\phi)(S) = 0 \\ (T_2\phi)(Q) = (1-Q)(b_0' + b_0'SQ + b_1'SQ^{-1})\phi(Q)\,.\end{cases}$$

It is easy to see that

$$\begin{aligned}(1-Q)(b_0' + b_0'QS + b_1'SQ^{-1}) &= (1-Q)(b_0' + (Q+Q^2)b_2'Q^{-1}) \\ &= (1-Q)(b_0' - b_2'Q^{-1})\,,\end{aligned}$$

we obtain then

$$(T_2\phi)(Q) = (1-Q)(b_0' - b_2'Q^{-1})\phi(Q)\,.$$

For $p_0 = 5$ we choose a basis set $B = \{2, 3\}$. The basis of $H^1(\Gamma, W_{n,\eta})_-$ is then $(y^n, 0, 0, 0, 0, 0)$ and $(0, 0, w_2, Ew_2, 0, 0)$ $w_2 \in M_{n,\eta}/M_{n,\eta}^{S_2}$. For $n = 5$ the numerical computation shows that $M_{n,\eta}/M_{n,\eta}^{S_2} = Rv_1 + Rv_2 + Rv_3$ with

$$\begin{aligned}v_1 &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4 \\ v_2 &= x^3y - 6x^2y^2 + 12xy^3 - 8y^4 \\ v_3 &= x^2y^2 - 4xy^3 + 4y^4\,.\end{aligned}$$

Let $v_0 = y^4$, then the basis of $H^1(\Gamma, W_{n,\eta})_-$ is ϕ_i, $i = 0, 1, 2, 3$ with $\phi_i(S) = 0$, $\phi_i(Q) = (1-Q)v_i$. The operation of T_2 is

$$T_2(v_0, v_1, v_2, v_3) = (v_0, v_1, v_2, v_3) = \begin{pmatrix} -31 & 0 & 0 & 0 \\ * & 31 & 0 & 0 \\ * & 0 & -10 & 18 \\ * & 0 & -8 & 10 \end{pmatrix}.$$

The characteristic polynomial of T_2 on $H^1(\Gamma, W_{n,\eta})_-$ is

$$\chi_2(x) = (x+31)(x-31)(x^2+44)\,.$$

The factors $(x+31)$ and $(x-31)$ come from the operation of T_2 on the boundary cohomology $H^1(\Gamma_\infty, W_{n,\eta} \otimes \mathbb{Q}) \cong \mathbb{Q}\phi_0 + \mathbb{Q}\phi_\infty$. More precise,

$$T_2\phi_\infty = -31\phi_\infty, \qquad T_2\phi_0 = 31\phi_0\,.$$

Therefore the characteristic polynomial of T_2 on $S_6(\Gamma_0(p_0), \eta)$ is $x^2 + 44$. The numerical computations of T_2, T_3, T_5 and T_7 for small p_0 and n are given in the table 1.

7.2 *Remark.* The space $S_{n+2}(\Gamma_0(p_0), \eta)$ carries the Petersson product, a non-degenerate Hermitian product on $S_{n+2}(\Gamma_0(p_0), \eta)$. If ^{t} denotes "transpose" with repsect to this product, then $T_l^t = \eta(l)T_l$. Let now λ be an eigenvalue of T_l, we have then $\bar{\lambda} = \eta(l)\lambda$ (cf. [Rib] § 1). Therefore, if $\eta(l) = -1$, then $\lambda = ia$ with $a \in \mathbb{R}$. If $\eta(l) = 1$, $\lambda \in \mathbb{R}$.

(1) $p_0 \equiv 1 \bmod 4$. In that case the dimension of $S_{n+2}(\Gamma_0(p_0), \eta)$ is even.

i. $\eta(l) = -1$. The characteristic polynomial of T_l is

$$\begin{aligned}\chi_l(x) &= (x - ia_1)(x + ia_1)(x - ia_2)(x + ia_2)\cdots(x - ia_r)(x + ia_r)\\ &= (x^2 + a_1^2)(x^2 + a_2^2)\cdots(x^2 + a_r^2)\\ &= x^{2r} + b_1 x^{2r-2} + \cdots + b_r\end{aligned}$$

with $b_1, \cdots, b_r \geq 0$.

ii. $\eta(l) = 1$. The characteristic polynomial of T_l is

$$\chi_l(x) = g(x)^2$$

for some polynomial $g(x)$. The roots of $g(x)$ are all real.

(2) $p_0 \equiv 3 \bmod 4$. In that case the dimension of $S_{n+2}(\Gamma_0(p_0), \eta)$ is odd.

i. $\eta(l) = -1$. There are zero eigenvalues. The characteristic polynomial is

$$\chi_l(x) = x^h(x^{2s} + b_1 x^{2s-2} + \cdots + b_s)$$

where h is the class number of the field $\mathbb{Q}(\sqrt{-p_0})$.

ii. $\eta(l) = +1$. The characteristic polynomial is

$$\chi_l(x) = g(x)^2 \cdot f(x)$$

where $f(x)$ is a polynomial generated by the Theta series and $\deg(f(x)) = h$ (cf. [Shi]).

The results in the table 1 confirm the remark above.

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Table 1

The characteristical polynomials of the Hecke operators T_2, T_2, T_5, and T_7 on the the cusp forms $S_k(\Gamma_0(p_0), \eta)$, where η is the Legendre symbol.

P0=5, K=N+2: ETA(2, P0)=−1, ETA(3, P0)=−1, ETA(7, P0)=−1

N=4

T2:=X²+44

T3:=X²+396

T7:=X²+3564

N=6

T2:=X²+116

$T3:=X^2+1044$

$T7:=X^2+176436$

$N=8$

$T2:=X^4+1708*X^2+1216$

$T3:=X^4+33552*X^2+45529776$

$T7:=X^4+104167728*X^2+2144749073480496$

$N=10$

$T2:=X^4+4132*X^2+2496256$

$T3:=X^4+341568*X^2+18385718256$

$T7:=X^4+4904976672*X^2+20876912776215558896$

$N=12$

$T2:=X^6+41052*X^4+440779968*X^2+617678127104$

$T3:=X^6+8329788*X^4+17708569483248*X^2+1517182687182390336$

$T7:=X^6+213997084092*X^4+105266238382057776341488*X^2$
$+465280274031462077190382306765444$

$N=14$

$T2:=X^6+117588*X^4+2455515648*X^2+4160982695936$

$T3:=X^6+48755052*X^4+160831293357168*X^2+7991454328138726790 4$

$T7:=X^6+8435989101708*X+21799671533824901950559088*X^2$
$+17560391031732483266163471186728360256$

$N=16$

$T2:=X^8+813836*X^6+197805587136*X^4+15212877148553216*X^2$
$+338022604671796903936$

$T3:=X^8+634018824*X^6+123866741829162816*X^4$
$+805235918890685234435366 4*X^2+625567943601835645403415787752 96$

$T7:=X^8+1358809234759656*X^6+57158388543758280617652626937 6*X^4$
$+717436452532486774095893673842376779068757 76*X^2$
$+1225649387103886247126536790871068024121055114558759170816$

$N=18$

$T2:=X^8+2907524*X^6+2568216374016*X^4+678867689422782464*X^2$
$+830184914753253120409 6$

$T3:=X^8+4476366576*X^6+699861404494885161 6*X^4$
$+43941029251512575272762575 36*X^2$
$+859178610673769519506507390330864896$

$T7:=X^8+5116020974740094 4*X^6+649955449858844816462059274614176*X^4$
$+13642776884971222592423433568989050161252055370 24*X^2$
$+66924011678488440533280702972291236048420236926345768783612 4416$

N=20

$T2:=X^{10}+15122620*X^{8}+74461069946560*X^{6}+1433556362014045798 40*X^{4}$

$+92050796042892961151713280*X^{2}+1458436346125398943 7721829965824$

$T3:=X^{10}+75700218780*X^{8}+169029007312492987 0560*X$

$+1058903342349253509809490106 1760*X^{4}$

$+10613905392864453389568881849143800910080*X^{2}$

$+2839805815981800681177617222924350898397211646976$

$T7:=X^{10}+262394272658498 0220*X^{8}$

$+2393834166138243432310096381198875360*X^{6}$

$+897532555190091115792311471245276662831526251090803840*X^{4}$

$+1213007451524819813099438782239454129103485419809424242465232 8$

$8877748480*X^{2}$

$+461270427686957096731880415655240852829431421851015326877617 30$

$599377381274672137928 20224$

N=22

$T2:=X^{10}+62579380*X^{8}+1269587477762560*X^{6}+96207678237122455 96160*X^{4}$

$+196483989919341170123394252 80*X^{2}$

$+3574276364739503586982992256434176$

$T3:=X^{10}+565341209820*X^{8}+1180334060923497145 04160*X^{6}$

$+10931210697192722327499640220787840*X^{4}$

$+406914738264133623534685754882233338060775680*X^{2}$

$+29229826732701725659783065593808079294208126 26129050624$

$T7:=X^{10}+11119655538478799 4780*X^{8}$

$+3435262712787547437076787484432246075360*X^{6}$

$+364120383334530873891786567367338677735603850387227691 96160*X^{4}$

$+8558668425083758505281071570824419379150496830306258534706568 2$

$3315813705484*X^{2}$

$+3487991783420007514334751545972468602202864516626473288704440 8$

$189830741158988702077 3579776$

P0=7, K=N+2: ETA(2, P0)=1, ETA(3, P0)=-1, ETA(5, P0)=-1

N=1

$T2:=X+3$

$T3:=X$

$T5:=X$

N=3

$T2:=X-1$

$T3:=X$

$T5:=X$

$N=5$

$T2:=(X+8)^2*(X-9)$

$T3:=X*(X^2+2040)$

$T5:=X*(X^2+2040)$

$N=7$

$T2:=(X^2-16*X-120)^2*(X+31)$

$T3:=X*(X^4+17184*X^2+40430880)$

$T5:=X*(X^4+1809120*X^2+736852788000)$

$N=9$

$T2:=(X^2+24*X-592)^2*(X-57)$

$T3:=X*(X^4+132480*X^2+4381776000)$

$T5:=X*(X^4+11349120*X^2+25531635024000)$

$N=11$

$T2:=(X^3-10216*X+172800)^2*(X+47)$

$T3:=X*(X^6+2434704*X^4+1858882957920*X^2+429665499302054400)$

$T5:=X*(X^6+1290415440*X^4+544550093091324000*X^2$
$+75252114900743951016000000)$

$N=13$

$T2:=(X^4-88*X^3-49600*X^2+3161344*X+199833600)^2*(X+87)$

$T3:=X*(X^8+32897856*X^6+307339393288320*X^4+678298556041314969600*X^2$
$+319723290962957097216000\0)$

$T5:=X*(X^8+30327873600*X^6+30349045935845547840\0*X^4$
$+120328279654140363917091456000\0*X^2$
$+16439949075700498841503687941264000000\0)$

$N=15$

$T2:=(X^4+272*X^3-98776*X^2-15713792*X+773514240)^2*(X-449)$

$T3:=X*(X^8+193153824*X^6+13542540815792160*X^4$
$+407914538508420139929600*X^2+4459777119693624095941077504000)$

$T5:=X*(X^8+579368436960*X^6+8173036213126267035600\0*X^4$
$+29484212493946548532543171200000000*X^2$
$+19389586132717877222418373488026640000000\0)$

$N=17$

$T2:=(X^5-456*X^4-716336*X^3+195823104*X^2+124785737728*X$
$-13438656184320)^2*(X+999)$

$T3:=X*(X^{10}+2541979176*X^8+19811946765805142\40*X^6$

$+470805560399816932850265600*X^4$
$+339149679554179917955160682762240000*X^2$
$+4635985871891340222247738276018384896000000)$

$T5:=X*(X^{10}+26205473373480*X^8+22951348729014501081133920 0*X^6$
$+771146143064265537788863097464793280000*X^4$
$+727302726371763893278482096922796468827756800000000*X^2$
$+2583961355010735340802227338768611093525002616160000000000000)$

$P0=11,\ K=N+2$: $ETA(2, P0)=-1$, $ETA(3, P0)=1$, $ETA(5, P0)=1$,
$ETA(7, P0)=-1$

$N=1$
$T2:=X$
$T3:=X+5$
$T5:=X+1$
$T7:=X$

$N=3$
$T2:=X*(X^2+30)$
$T3:=(X+3)^2*(X-7)$
$T5:=(X-31)^2*(X+49)$
$T7:=X*(X^2+3000)$

$N=5$
$T2:=X*(X^4+270*X^2+16680)$
$T3:=(X^2-12*X-1509)^2*(X-10)$
$T5:=(X+65)^4*(X-74)$
$T7:=X*(X^4+393000*X^2+38537472000)$

$N=7$
$T2:=X*(X^6+1374*X^4+436560*X^2+40320000)$
$T3:=(X^3+18*X^2-6285*X-201150)^2*(X+113)$
$T5:=(X^3+224*X^2-525475*X+31988350)^2*(X-1151)$
$T7:=X*(X^6+22327704*X^4+102738589578240*X^2+134544048242688000000)$

$N=9$
$T2:=X*(X^8+6030*X^6+11712120*X^4+7669330560*X^2+564269690880)$
$T3:=(X^4+201*X^3-98919*X^2-1150929*X+1149750126)^2*(X-475)$
$T5:=(X^4-1215*X^3-21311915*X^2-2265218325*X+17429871112150)^2$
$*(X+3001)$
$T7:=X*(X^8+767889840*X^6+102582267767649600*X^4$
$+1566249894398109763584000*X^2+6330325858079634845966794752000)$

N=11

$T2:=X*(X^{10}+30654*X^8+318945120*X^6+1305642637440*X^4$
$+2049564619929600*X^2+957721368231936000)$

$T3:=(X^5-1218*X^4-775914*X^3+838214892*X^2+189020241225*X$
$+120422340866250)^2*(X+1358)$

$T5:=(X^5-13246*X^4-413004050*X^3+7878939523400*X^2$
$-322982308880243750*X+12308222362848968750)^2*(X+25774)$

$T7:=X*(X^{10}+72369291504*X^8+1579588871009845139520*X^6$
$+12964051646785030759215833088000*X^4$
$+37709819138673185762480264655566929920000*X^2$
$+23187441850664232142842389272548747887247360000000)$

P0=13, K=N+2: ETA(2, P0)=−1, ETA(3, P0)=1, ETA(5, P0)=−1, ETA(7, P0)=−1

N=2

$T2:=X^2+9$

$T3:=(X+1)^2$

$T5:=X^2+81$

$T7:=X^2+225$

N=4

$T2:=X^6+161*X^4+5856*X^2+18864$

$T3:=(X^3-8*X^2-549*X+4068)^2$

$T5:=X^6+8018*X^4+13754433*X^2+2485690416$

$T7:=X^6+82950*X^4+1662348177*X^2+423560602764$

N=6

$T2:=X^6+449*X^4+37224*X^2+205776$

$T3:=(X^3+28*X^2-2601*X-71748)^2$

$T5:=X^6+243506*X^4+1206410625*X^2+93756690000$

$T7:=X^6+847206*X^4+231424342425*X^2+20471634652072500$

N=8

$T2:=X^{10}+3841*X^8+5134480*X^6+2823572208*X^4+614223235584*X^2$
$+43308450164736$

$T3:=(X^5+X^4-66033*X^3+1260423*X^2+530326440*X+14266185264)^2$

$T5:=X^{10}+14820283*X^8+74785768290163*X^6+146559998245698565881*X^4$
$+87330504466586448091944000*X^2+1206547810951912951716600624000 0$

$T7:=X^{10}+252125259*X^8+23724928789729587*X^6$
$+1025407325324195954977977*X^4$

$+1966112980588748350440452608473 6*X^2$

$+121307703706137674344780717867862132400$

N=10

$T2:=X^{12}+18433*X^{10}+121088056*X^8+340607607312*X^6+380893885719552*X^4$

$+134825856231997440*X^2+1497425476589715456$

$T3:=(X^6+244*X^5-665334*X^4-129598956*X^3+109163403621*X^2$

$+14522233287672*X-255121008509808)^2$

$T5:=X^{12}+289917556*X^{10}+32326953002900950*X^8$

$+1726712418063587931532500*X^6$

$+44108094881553049831926298640625*X^4$

$+430033290962195234920750132329450000000*X^2$

$+1088610564567377499456977013060519750000000 0$

$T7:=X^{12}+13650769356*X^{10}+64465836700280921262*X^8$

$+139418894150631875357617076028*X^6$

$+1437852685114805251501687890700179314 01*X^4$

$+6375395482700460954877632200665513395285810000 0*X^2$

$+905450737640119482890234370767629262159657021349375000 0$

P0=17, K=N+2: ETA(2, P0)=1, ETA(3, P0)=−1, ETA(5, P0)=−1, ETA(7, P0)=−1

N=2

$T2:=(X^2+X-8)^2$

$T3:=X^4+74*X^2+1072$

$T5:=X^4+480*X^2+38592$

$T7:=X^4+530*X^2+68608$

N=4

$T2:=(X^3+X^2-68*X-36)^2$

$T3:=X^6+668*X^4+145216*X^2+10185984$

$T5:=X^6+9488*X^4+8442048*X^2+40743936$

$T7:=X^6+71708*X^4+104887424*X^2+2346850713600$

N=6

$T2:=(X^5+9*X^4-452*X^3-2988*X^2+27904*X+83616)^2$

$T3:=X^{10}+16832*X^8+93191572*X^6+192821327856*X^4+116860780245888*X^2$

$+9421474370420736$

$T5:=X^{10}+351440*X^8+44989957632*X^6+2580556932172800*X^4$

$+65876023734658560000*X^2+602974359706927104000000$

$T7:=X^{10}+4233136*X^8+5824132863636*X^6+2871845375371443376*X^4$

$+4979960158315609564714247*X^{2}+1485656601736989519288985 1904$

$N=8$

$T2:=(X^{6}-15*X^{5}-1892*X^{4}+20460*X^{3}+770176*X^{2}-3195840*X-6636441)^{2}$

$T3:=X^{12}+122690*X^{10}+5157152560*X^{8}+87983684680032*X^{6}$
$+612743619071665152*X^{4}+1335826553351738886144*X^{2}$
$+2039493995689321986785 28$

$T5:=X^{12}+13939648*X^{10}+67854209805568*X^{8}+136905662805154384896*X^{6}$
$+1030306388452341366721536 00*X^{4}$
$+2387304787569289595946012672 0000*X^{2}$
$+2132021607333312666110860984320 00000$

$T7:=X^{12}+181444282*X^{10}+8551923317087424*X^{8}$
$1450159646516084259152 32*X^{6}+92207271653680381090505440 8448*X^{4}$
$+2318685324256549381944604148484046848*X^{2}$
$+19676765857885095912859495322700667158528 00$

$P0=19,\ K=N+2:\ ETA(2,P0)=-1,\ ETA(3,P0)=-1,\ ETA(5,P0)=1,$
$ETA(7,P0)=1$

$N=1$

$T2:=X*(X^{2}+13)$

$T3:=X*(X^{2}+13)$

$T5:=(X-4)^{2}*(X+9)$

$T7:=(X+5)^{3}$

$N=3$

$T2:=X*(X^{4}+35*X^{2}+142)$

$T3:=X*(X^{4}+301*X^{2}+5112)$

$T5:=(X^{2}+21*X+92)^{2}*(X-31)$

$T7:=(X^{2}-68*X+499)^{2}*(X+73)$

$N=5$

$T2:=X*(X^{8}+483*X^{6}+75582*X^{4}+4242376*X^{2}+71047680)$

$T3:=X*(X^{8}+3442*X^{6}+4292649*X^{4}+2281096296*X^{2}+432254085120)$

$T5:=(X^{4}-54*X^{3}-49415*X^{2}+3367200*X+292006000)^{2}*(X+54)$

$T7:=(X^{4}+70*X^{3}-157380*X^{2}-29481334*X-1276939885)^{2}*(X-610)$

$N=7$

$T2:=X*(X^{12}+2323*X^{10}+2010462*X^{8}+803113072*X^{6}+150633270400*X^{4}$
$+12173735396352*X^{2}+333034797957120)$

$T3:=X*(X^{12}+59719*X^{10}+1354569075*X^{8}+14270784462117*X^{6}$
$+666708553053203 76*X^{4}+990717037048715051 52*X^{2}$

$+336641285069765325 61920)$

$T5:=(X^6-4*X^5-1446203*X^4+95652050*X^3+409166434600*X^2$

$-102103842940000*X+6563900254320000)^2*(X+289)$

$T7:=(X^6-1843*X^5-25578196*X^4+37453164210*X^3+157007096825425*X^2$

$+139069305605381375*X-2193374201222141 8750)^2*(X-527)$

$P0=23,\ K=N+2:\quad ETA(2, P0)=1,\ ETA(3, P0)=1,\ ETA(5, P0)=-1,$

$ETA(7, P0)=-1$

$N=1$

$T2:=X^3-12*X+7$

$T3:=X^3-27*X+38$

$T5:=X^3$

$T7:=X^3$

$N=3$

$T2:=(X^2+4*X-2)^2*(X^3-48*X+79)$

$T3:=(X^2+6*X-45)^2*(X^3-243*X+14)$

$T5:=X^3*(X^4+2556*X^2+1270188)$

$T7:=X^3*(X^4+11988*X+31754700)$

$N=5$

$T2:=(X^4-4*X^3-162*X^2+920*X+832)^2*(X+7)*(X^2-7*X-143)$

$T3:=(X^4-15*X^3-957*X^2+13293*X^2-12870)^2*(X+38)$

$*(X^2-38*X-743)$

$T5:=X^3*(X^8+95100*X^6+3184494300*X^4+44006549508000*X^2$

$+214214641502400000)$

$T7:=X^3*(X^8+660492*X^6+152231816700*X^4+13982809796769600*X^2$

$+4008342403210083840 00)$

$N=7$

$T2:=(X^6+4*X^5-850*X^4-3248*X^3+147872*X^2+268672*X-5317760)^2$

$*(X^3-768*X+1951)$

$T3:=(X^6+36*X^5-20508*X^4-1030644*X^3+86837139*X^2+5371429140*X$

$+55514443500)^2*(X^3-19683*X+1062686)$

$T5:=X^3*(X^{12}+3434556*X^{10}+4503520431468*X^8+2817283398730424640*X^6$

$+847955819735403719760000*X^4+103782437973914306469472512000*X^2$

$+2208782254549937077079536204800000)$

$T7:=X^3*(X^{12}+40494132*X^{10}+605958970060332*X^8$

$+4096821152215401422400*X^6+12087187496206708701149510400*X^4$

$+11540311691303117336118557810688000*X^2$

$+25260738891156651189861843844423680000)$

$P0=29\ K=N+2$: $ETA(2, P0)=-1$, $ETA(3, P0)=-1$, $ETA(5, P0)=1$, $ETA(7, P0)=1$

N=2

$T2:=X^6+38*X^4+301*X^2+560$

$T3:=X^6+61*X^4+791*X^2+875$

$T5:=(X^3-11*X^2-133*X+1071)^2$

$T7:=(X^3+14*X^2-108*X-1192)^2$

N=4

$T2:=X^{12}+278*X^8+28285+1260472*X^6+22944832*X^4+140087936*X^2+966400$

$T3:=X^{12}+2245*X^{10}+1884878*X^8+715200530*X^6+112977325989*X^4 +4281127461369*X^2+46577165867100$

$T5:=(X^6-23*X^5-12280*X^4+235866*X^3+33953337*X^2-384523443*X -2627317458)^2$

$T7:=(X^6-10*X^5-76080*X^4+1925088*X^3+1377655664*X^2-73626194400*X -519034134784)^2$

N=6

$T2:=X^{16}+1382*X^{14}+744077*X^{12}+200869632*X^{10}+28931822432*X^8 +2155663113216*X^6+71710495842560*X^4+663330761523200*X^2 +590388176896000$

$T3:=X^{16}+22051*X^{14}+187767701*X^{12}+793510274339*X^{10} +1809033803032599*X^8+2281021494195869649*X^6 +1527214705246483000335*X^4+458931418705423915202025*X^2 +30465487147014831010162500$

$T5:=(X^8+99*X^7-276993*X^6-31299849*X^5+17584369885*X^4 +1686634037625*X^2-196943514064875*X^2-14966521618921875*X -2002786842877312\,50)^2$

$T7:=(X^8-330*X^7-3716228*X^6+233875960*X^5+3438911219312*X^4 +1091616310004000*X^3-2938272171110586 24*X^2 -8987309234716285 8880*X+8119186526578407384064)^2$

$P0=31$, $K=N+2$: $ETA(2, P0)=1$, $ETA(3, P0)=-1$, $ETA(5, P0)=1$, $ETA(7, P0)=1$

N=1

$T2:=(X+1)^2*(X^3-12*X+15)$

$T3:=X^3*(X^2+26)$

$T5:=(X-2)^2*(X^3-75*X+246)$

$T7:=(X-8)^2*(X^3-147X+*430)$

$N=3$

$T2:=(X^3+X^2-30*X+6)^2*(X^3-48*X-97)$

$T3:=X^3*(X^6+398*X^4+49236*X^2+1934136)$

$T5:=(X^3+4*X^2-291*X+1014)^2*(X^3-1875*X-29266)$

$T7:=(X^3+66*X^2-1005*X-31688)*(X^3-7203*X^2+50398)$

$N=5$

$T2:=(X^6+X^5-222*X^4-370*X^3+9416*X^2+13440*X-90624)^2*(X+15)$
$*(X^2-15*X+33)$

$T3:=X^3*(X^{12}+7208*X^{10}+19859688*X^8+26566749360*X^6+17884354852944*X^4$
$+5570285336959680*X^2+590986232936064000)$

$T5:=(X^6+73*X^5-51615*X^4-3624325*X^3+522398750*X^2+25671172500*X$
$-103336)^2*(X^2-246*X+13641)*(X+246)$

$T7:=(X^6-3*X^5-207897*X^4-2308819*X^3+13269144858*X^2+215614693848*X$
$-247)^2*(X^2-430*X-168047)*(X+430)$

$P0=37,\ K=N+2:$ $ETA(2, P0)=-1,\ ETA(3, P0)=1,\ ETA(5, P0)=-1,$
$ETA(7, P0)=1$

$N=2$

$T2:=X^8+50*X^6+709*X^4+3000*X^2+1764$

$T3:=(X^4+3*X^3-50*X^2-57*X-427)^2$

$T5:=X^8+431*X^6+29521*X^4+588072*X^2+2039184$

$T7:=(X^4-2*X^3-587*X^2+2460*X+53892)^2$

$N=4$

$T2:=X^{16}+390*X^{14}+60701*X^{12}+4799932*X^{10}+203487156*X^8+4519465040*X^6$
$+48993644736*X^4+211923220224*X^2+178006118400$

$T3:=(X^8+9*X^7-1280*X^6-11016*X^5+422488*X^4+2751084*X^3-25673805*X^2$
$-30714957*X+141986196)^2$

$T5:=X^{16}+31026*X^{14}+373650779*X^{12}+2220056867434*X^{10}$
$+6834316986168825*X^8+10475224449621004436*X^6$
$+6885539411711705092656*X^4+1110302609356408225416384*X^2$
$+197269442423240263991101 44$

$T7:=(X^8-95*X^7-54561*X^6+2410919*X^5+907038560*X^4+534484632*X^3$
$-349499585616*X^2-2731120576272*X+3409346511153792)^2$

P0 = 41, K = N + 2: ETA(2, P0) = 1, ETA(3, P0) = −1, ETA(5, P0) = 1, ETA(7, P0) = −1

N = 2

$T2 := (X^5 + 3*X^4 - 25*X^2 - 51*X^2 + 104*X + 32)^2$

$T3 := X^{10} + 180*X^6 + 10910*X^6 + 276172*X^4 + 2531856*X^2 + 524672$

$T5 := (X^5 + 2*X^4 - 282*X^3 - 1400*X^2 + 9016*X + 43904)^2$

$T7 := X^{10} + 1912*X^8 + 1274822*X^6 + 344662636*X^4 + 30875879696*X^2 + 87767656832$

P0 = 43, K = N + 2: ETA(2, P0) = −1, ETA(3, P0) = −1, ETA(5, P0) = −1, ETA(7, P0) = −1

N = 1

$T2 := X*(X^6 + 20*X^4 + 121*X^2 + 214)$

$T3 := X*(X^6 + 45*X^4 + 431*X^2 + 214)$

$T5 := X*(X^6 + 117*X^4 + 3863*X^2 + 25894)$

$T7 := X*(X^6 + 150*X^4 + 4896*X^2 + 3424)$

P0 = 47, K = N + 2: ETA(2, P0) = 1, ETA(3, P0) = 1, ETA(5, P0) = −1, ETA(7, P0) = 1

N = 1

$T2 := (X + 1)^2*(X^5 - 20*X^3 + 80*X - 17)$

$T3 := (X + 2)^2*(X^5 - 45*X^3 + 405*X - 298)$

$T5 := X^5*(X^2 + 78)$

$T7 := (X + 4)^2*(X^5 - 245*X^3 + 12005*X - 31922)$

N = 3

$T2 := (X^5 + X^4 - 40*X^3 + 12*X^2 + 300*X - 316)^2*(X^5 - 80*X^3 + 1280*X + 1759)$

$T3 := (X^5 - 4*X^4 - 207*X^3 + 576*X^2 + 7803*X + 9558)^2*(X^5 - 405*X^3 + 32805*X + 29294)$

$T5 := X^5(X^{10} + 5490*X^8 + 10917588*X^6 + 9407020248*X^4 + 3230761626000*X^2 + 270690407718048)$

$T7 := (X^5 - 14*X^4 - 2905*X^3 - 45230*X^2 - 141377*X + 94796)^2*(X^5 - 12005*X^3 + 28824005*X - 45406386)$

P0 = 53, K = N + 2: ETA(2, P0) = −1, ETA(3, P0) = 1, ETA(5, P0) = −1, ETA(7, P0) = 1

N = 2

$T2 := X^{12} + 80*X^{10} + 2356*X^8 + 30996*X^6 + 176575*X^4 + 393232*X^2 + 285376$

$$T3:=X^{12}+215*X^{10}+16178*X^{8}+505118*X^{6}+5738621*X^{4}+15503831*X^{2}+673036$$

$$T5:=X^{12}+789*X^{10}+196604*X^{8}+18690640*X^{6}+682399088*X^{4}+6573121072*X^{2}+2960741056$$

$$T7:=(X^{6}-12*X^{5}-1052*X^{4}+10868*X^{3}+215348*X^{2}-624840*X-9386656)^{2}$$

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Mathematisches Institut
der Universität Bonn
Beringstr. 1
D-5300 Bonn
Germany