Y. Teranishi
Nagoya Math. J.
Vol. 121 (1991), 15-34

YOUNG DIAGRAMMATIC METHODS IN NON-COMMUTATIVE INVARIANT THEORY

YASUO TERANISHI

Introduction

In this paper we will study some aspects of non-commutative invariant theory. Let V be a finite-dimensional vector space over a field K of characteristic zero and let

$$K[V] = K \oplus V \oplus S^2(V) \oplus \cdots, \text{ and}$$
$$K\langle V\rangle = K \oplus V \oplus \otimes^2 V \oplus \otimes^3 V \oplus \cdots$$

be respectively the symmetric algebra and the tensor algebra over V. Let G be a subgroup of $GL(V)$. Then G acts on $K[V]$ and $K\langle V\rangle$. Much of this paper is devoted to the study of the (non-commutative) invariant ring $K\langle V\rangle^G$ of G acting on $K\langle V\rangle$.

In the first part of this paper, we shall study the invariant ring in the following situation.

Take a classical group G (i.e., $G = SL(n, K)$, $O(n, K)$ or $Sp(n, K)$) and the standard G-module K^n. Let V be the d-th symmetric power of K^n. Then G acts on V and we get $K\langle V\rangle^G$.

By the Lane-Kharchenko theorem ([L], [Kh]), the invariant ring $K\langle V\rangle^G$ is a free algebra. For the construction of explicit free generators, we will develop a symbolic method along the lines of Kung-Rota [K-R].

In the second part of this paper, we will study S-algebras in the sence of A.N. Koryukin. Koryukin [Ko] has proved that if V is a finite-dimensional K-vector space and G is a reductive subgroup of $GL(V)$ then $K\langle V\rangle^G$ is finitely generated as an S-algebra. We will prove that a homogeneous system of generators for the (commutative) invariant ring $K[\Lambda^2 V \oplus V]^G$ gives rise to a system of generators for the invariant ring $K\langle V\rangle^G$ as an S-algebra.

In the final part of this paper, we will study (non-commutative) in-

Received February 6, 1990.

variants of finite linear groups acting on the ring of 2 by 2 generic matrices with zero trace. In this case, rings of invariants are finitely generated and Cohen-Macaulay modules over their centrers. We will give a formula for the Poincare series of the invariant rings. The formula is analogous to the classical formula of Molien in the commutative case, but more complicated.

§ 1. Umbral derivation of tensor invariants of n-ary forms

1.1. We consider the generic n-ary forms of degree d,

$$f(\xi_1, \xi_2, \cdots, \xi_n) = \sum_{\substack{\alpha \in \mathbf{N}^n \\ |\alpha| = d}} \binom{d}{\alpha} a_\alpha \xi^\alpha$$

with coefficients a_α which are indeterminates over a field K of characteristic zero. Here, for an $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\xi^\alpha = \xi^{\alpha_1} \cdots \xi^{\alpha_n}$ and $\binom{d}{\alpha} = \dfrac{d!}{\alpha_1! \cdots \alpha_n!}$. Then each transformation

$$\xi_i = \sum_{1 \leq k \leq n} a_{ki} \xi'_k ,$$

carries the generic n-ary form $f(\xi_1, \cdots, \xi_n)$ into another n-ary form

$$f'(\xi_1, \xi_2, \cdots, \xi_n) = \sum_{\alpha \in \mathbf{N}^n} \binom{d}{\alpha} a'_\alpha \xi^\alpha .$$

The map $a_\alpha \to a'_\alpha$ defines the d-th symmetric tensor representation of the general linear group $GL(n, K)$. Further let $d_1, d_2, \cdots, d_r$ be positive integers and consider a system of generic n-ary forms $f_1, f_2, \cdots, f_r$ of degree $d_1, d_2, \cdots, d_r$, respectively:

$$f_1 = \sum_{\substack{\alpha \in \mathbf{N}^n \\ |\alpha| = d_1}} \binom{d_1}{\alpha} a_\alpha^{(1)} \xi^\alpha, \; f_2 = \sum_{\substack{\beta = \mathbf{N}^n \\ |\beta| = d_2}} \binom{d_2}{\beta} a_\beta^{(2)} \xi^\beta, \; \cdots, \; f_r = \sum_{\substack{\gamma \in \mathbf{N}^n \\ |\gamma| = d_r}} \binom{d_r}{\gamma} a_\gamma^{(r)} \xi^\gamma .$$

Viewing the coefficients $a_\alpha^{(1)}, a_\beta^{(2)}, \cdots, a_\gamma^{(r)}$ as independent variables over K, we get a linear action of $GL(n, K)$ on the polynomial ring

$$S_{n,\underline{d}} = K[a_\alpha^{(1)}, a_\beta^{(2)}, \cdots, a_\gamma^{(r)}] .$$

Let G be a classical subgroup (i.e., $G = SL(n, K)$, $O(n, K)$, or $Sp(n, K)$). The invariant ring $S_{n,\underline{d}}^G$ under the group action of G is called the ring of simultaneous G-invariants of n-ary forms $f_1, f_2, \cdots, f_r$. The polynomial ring $S_{n,\underline{d}}$ is $\mathbf{N}^r$-graded by giving $a_\alpha^{(i)}$ multi-degree $\underline{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, the

i-th unit vector of $\mathbf{N}^r$, the grading on $S_{n,\underline{d}}$ induces the same grading on $S^G_{n,\underline{d}}$.

For each $\underline{m} = (m_1, \cdots, m_r) \in \mathbf{N}^r$, we denote by $(S_{n,\underline{d}})^G_{\underline{m}}$ the vector space of degree $\underline{m}$. If $\underline{m} = (1, 1, \cdots, 1)$, the space is called the space of multi-linear G-invariants of type $d = (d_1, \cdots, d_r)$.

Let $x^{(1)} = {}^t(x^{(1)}_1, \cdots, x^{(1)}_n)$, $x^{(2)} = {}^t(x^{(2)}_1, \cdots, x^{(2)}_n)$, $\cdots$, $x^{(r)} = {}^t(x^{(r)}_1, \cdots, x^{(r)}_n)$ be the n-dimensional column vectors whose entries $x^{(i)}_j$ are independent commuting variables. We call these variable vectors $x^{(1)}, x^{(2)}, \cdots, x^{(r)}$ umbral vectors and we call the polynomial ring $K[x^{(i)}_j; 1 \leq i \leq r, 1 \leq j \leq n]$ the umbral space. The umbral operator U is the linear operator from the umbral space to the polynomial ring $S_{n,\underline{d}}$ defined by

$$U(x^{(i)\alpha}) = \begin{cases} a^{(i)}_\alpha, \text{ if } |\alpha| = d_i \\ 0, \text{ otherwise}, \end{cases}$$

where $x^{(i)\alpha_1} = x^{(i)\alpha_1}_1 \cdots x^{(i)\alpha_n}_n$, for $\alpha \in \mathbf{N}^n$. For a monomial, we set

$$U(x^{(i_1)\alpha_1} \cdots x^{(i_t)\alpha_t}) = U(x^{(i_1)\alpha_1}) \cdots U(x^{(i_t)\alpha_t}) .$$

1.2. We associate to an n-tuple $\underline{i} = (i_1, i_2, \cdots, i_n)$ of positive integers satisfying $1 \leq i_1 < i_2 < \cdots < i_n \leq r$, an indeterminate $p_{\underline{i}} (= p_{i_2 i_1 \cdots i_n})$. Let I be the ideal of the polynomial ring $K[\cdots, p_{\underline{i}}, \cdots]$ generated by the Plücker relations

$$\sum_{1 \leq k \leq n+1} (-1)^{k+1} p_{j_1 j_2 \cdots \hat{j_k} \cdots j_{n+1}} p_{i_1 i_2 \cdots i_{n-1} j_k} .$$

The quotient ring

$$K[\cdots, p_{\underline{i}}, \cdots]/I$$

is the coordinate ring $K[\mathrm{Gr}(n, r)]$ of the Grassmann variety $\mathrm{Gr}(n, r)$. The ring $K[\cdots, p_{\underline{i}}, \cdots]$ (resp. $K[\mathrm{Gr}(n, r)]$) is an $\mathbf{N}^r$-graded ring by giving each $p_{\underline{i}}$ degree $\underline{e}_{i_1} + \cdots + \underline{e}_{i_n} \in \mathbf{N}^r$. We associate to each monomial

$$p_{\underline{i}} \cdot p_{\underline{j}} \cdots p_{\underline{k}} \ (i = (i_1, \cdots, i_n), j = (j_1, \cdots, j_n), \cdots, k = (k_1, \cdots, k_n))$$

of degree $\underline{d} = (d_1, \cdots, d_r) \in \mathbf{N}^r$, a multi-linear form in $a^{(1)}_\alpha, a^{(2)}_\beta, \cdots, a^{(r)}_\gamma$ in the following way. We replace each factor $p_{m_1 \cdots m_n}$ of a monomial $p_{\underline{i}} \cdot p_{\underline{j}} \cdots p_{\underline{k}}$ by the determinant $|x^{(m_1)} \cdots x^{(m_n)}|$ of the n by n matrix

$$\begin{pmatrix} x^{(m_1)}_1 & \cdots & x^{(m_n)}_n \\ \vdots & & \vdots \\ x^{(m_1)}_1 & \cdots & x^{(m_n)}_n \end{pmatrix} .$$

Then expanding the product of these determinants, we find that

$$U(|x^{(i_1)}\cdots x^{(i_n)}|\cdot|x^{(j_1)}\cdots x^{(j_n)}|\cdots|x^{(k_1)}\cdots x^{(k_n)}|)$$

is a **Z**-linear combination of terms of the form

$$a_\alpha^{(1)}\cdot a_\beta^{(2)}\cdots a_\gamma^{(r)}\ (\alpha, \beta, \cdots, \gamma \in \mathbf{N}^n) \quad \text{with } |\alpha| = d_1, |\beta| = d_2, \cdots, |\gamma| = d_r.$$

Therefore we can define a K-linear map

$$U_{n,r,\underline{d}} : K[\cdots, p_{\underline{i}}, \cdots]_{\underline{d}} \longrightarrow (S_{n,\underline{d}})_{(1\cdots1)}$$

by

$$\begin{aligned} &U_{n,r,\underline{d}}(p_{i_1\cdots i_n}\cdot p_{j_1\cdots j_n}\cdots p_{k_1\cdots k_n}) \\ &\quad = U(|x^{(i_1)}\cdots x^{(i_n)}|\cdot|x^{(j_1)}\cdots x^{(j_n)}|\cdots|x^{(k_1)}\cdots x^{(k_n)}|\,. \end{aligned}$$

THEOREM 1.1. *The image of $K[\cdots, p_{\underline{i}}, \cdots]_{\underline{d}}$ by the K-linear map $U_{n,r,\underline{d}}$ is the K-vector space $(S_{n,\underline{d}})_{(1\cdots1)}^{SL(n,K)}$ of multi-linear $SL(n, K)$ invariants of type $\underline{d}$ and the kernel is $I \cap K[\cdots, p_{\underline{i}}, \cdots]_{\underline{d}}$.*

In other words, the map $U_{n,r,\underline{d}}$ induces a K-linear isomorphism

$$K[\mathrm{Gr}(n, r)]_{\underline{d}} \simeq (S_{n,r,\underline{d}})_{(1\cdots1)}^{SL(n,K)}\,.$$

Proof. Consider the standard action of $SL(n, K)$ on the umbral vectors $x^{(1)}, x^{(2)}, \cdots, x^{(r)}$. Then the fundamental theorem of vector invariants (cf. [W] Chap. 2) says that the ring $K[\mathrm{Gr}(n, r)]$ is isomorphic to the ring of $SL(n, K)$-invariants of the umbral space, via the map

$$p_{i_1\cdots i_n} \longrightarrow |x^{(i_1)}\cdots x^{(i_n)}|\,.$$

The umbral space is $\mathbf{N}^r$-graded by giving each $x_j^{(i)}$ degree $\underline{e}_i \in \mathbf{N}^r$. Then it is clear that, for each $\underline{d} = (d_1, \cdots, d_r) \in \mathbf{N}^r$, the umbral operator

$$U: K[x_j^{(i)}; 1 \le i \le r, 1 \le j \le n]_{\underline{d}} \longrightarrow (S_{n,\underline{d}})_{(1\cdots1)}$$

is an $SL(n, K)$-isomorphism of vector spaces and hence we obtain K-linear isomorphisms,

$$\begin{aligned} K[\mathrm{Gr}(n, r)]_{\underline{d}} &\simeq K[x_j^{(i)}; 1 \le i \le r, 1 \le j \le n]_{(1\cdots1)}^{SL(n,K)} \\ &\simeq (S_{n,\underline{d}})_{(1\cdots1)}^{SL(n,K)}\,. \end{aligned}$$

This completes the proof.

For every $d = (d_1, \cdots, d_r) \in \mathbf{N}^r$, we set

$$k = |\underline{d}|/n \quad \text{and} \quad \underline{d}^{\sim} = (k - d_1, \cdots, k - d_r)\,.$$

Then it can be easily seen that if $\dim_K(S_{n,\underline{d}})^{SL(n,K)}_{(1\cdots 1)} \geq 1$, $\underline{d}^{\sim} \in \mathbf{N}^r$. For an n-tuple $(i_1, \cdots, i_n)$, $1 \leq i_1 \leq i_2 < \cdots < i_n \leq r$, let $(i'_1, \cdots, i'_{r-n})$ denote the complement of $(i_1, \cdots, i_n)$ in $(1, 2, \cdots, r)$.

To each monomial

$$p = p_{i_1\cdots i_n} \cdot p_{j_1\cdots j_n} \cdot \cdots \cdot p_{k_1\cdots k_n}$$

we associate the monomial

$$\hat{p} = p_{i'_1\cdots i'_{r-n}} \cdot p_{j'_1\cdots j'_{r-n}} \cdot \cdots \cdot p_{k'_1\cdots k'_{r-n}} .$$

Then the map $p \to \hat{p}$ defines a K-linear isomorphism

$$K[\mathrm{Gr}(n, r)]_{\underline{d}} \simeq K[\mathrm{Gr}(r-n, r)]_{\underline{d}^{\sim}} .$$

By Theorem 1.1, we obtain

COROLLARY. *If* $\dim_K(S_{n,\underline{d}})^{SL(n,K)}_{(1\cdots 1)} \geq 1$, *then*

$$\dim_K(S_{n,\underline{d}})^{SL(n,K)}_{(1\cdots 1)} = \dim_K(S_{r-n,\underline{d}^{\sim}})^{SL(n,K)}_{(1\cdots 1)} .$$

Let us recall some notations and definitions on Young diagrams. Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a partition. We identify λ with the corresponding Young diagram (denoted also by λ). If $\lambda_n > 0$ and $\lambda_{n+1} = 0$, for some n, we call n the length of λ and denote it by $l(\lambda)$. A Young diagram whose squares are filled with some positive integers is called a numbered diagram. If a numbered diagram is column strict, i.e., the numbers in each row are non-decreasing from left to right and numbers in each column are strictly increasing from top down, it is called a Young tableau. If a Young talbeau T has i_1 1's, i_2 2's, etc, then the sequence $(i_1, i_2, \cdots)$ is called the weight of T. For a Young diagram λ, we denote its transpose by ${}^t\lambda$.

A monomial $p_{i_1\cdots i_n} \cdot p_{j_1\cdots j_n} \cdots p_{k_1\cdots k_n}$ is called a standard monomial if the associated numbered diagram

$$\begin{bmatrix} i_1 & j_1 & \cdots & k_1 \\ \vdots & \vdots & & \vdots \\ i_n & j_n & \cdots & k_n \end{bmatrix}$$

is a Young tableau. A Young tableau is called an $SL(n, K)$-tableau if each column has n squares. Let T be an $SL(n, K)$-tableau with weight $\underline{d} = (d_1, d_2, \cdots, d_r) \in \mathbf{N}^r$. We denote the associated monomial in $K[\mathrm{Gr}(n, r)]$ by $p(T)$. Then $p(T)$ has degree $\underline{d}$.

PROPOSITION 1.1 ([D-R-S] Theorem 1). *For each* $\underline{d} \in \mathbf{N}^r$, *the set of* *monomials $\{p(T);\ T$ is an $SL(n, K)$-tableau of weight $\underline{d}\}$ is a K-basis of $K[\mathrm{Gr}(n, r)]_{\underline{d}}$.*

By Theorem 1.1 and Proposition 1.1 we then obtain the following

THEOREM 1.2. *For each $\underline{d} = (d_1, \cdots, d_r) \in \mathbf{N}^r$, the set of elements $\{U_{n,r,\underline{d}}(p(T));\ T$ is an $SL(n, K)$-tableau of weight $\underline{d}\}$ is a K-basis of the vector space of multi-linear $SL(n, K)$-invariants with type $\underline{d}$.*

Consider a free algebra $K\langle a_\alpha;\ \alpha \in \mathbf{N}^n$ and $|\alpha| = d\rangle$ generated by a_α. Then this algebra is $\mathbf{N}$-graded by giving each a_α degree one. For each (non-commutative) monomial $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_r}$ of degree r, we set $\Psi_r(a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_r}) = a^{(1)}_{\alpha_1}a^{(2)}_{\alpha_2}\cdots a^{(r)}_{\alpha_r}$, then we obtain a K-linear isomorphism

$$\Psi_r : K\langle a_\alpha; \alpha \in \mathbf{N}^n, |\alpha| = d\rangle_r \longrightarrow (S_{n,\langle d\rangle})_{\langle 1\rangle}\,,$$

where $\langle d\rangle = (d\cdots d) \in \mathbf{N}^r$. Further we set

$$\hat{U}_{n,r,d} = \Psi_r^{-1} U_{n,r,\langle d\rangle}\,.$$

Then from Theorem 1.2, we obtain

PROPOSITION 1.2. *For each $r \in \mathbf{N}$, the set of elements $\{\hat{U}_{n,r,d}(p(T));\ T$ is an $SL(n, K)$-tableau of weight $\langle d\rangle \in \mathbf{N}^r\}$ constitutes a K-basis of $K\langle a_\alpha;\ \alpha \in \mathbf{N}^n, |\alpha| = d\rangle_r$.*

Let T be a Young tableau with, say, s columns and let t be a positive integer with $t < s$. Then we denote by T_t the Young tableau taken from the first t columns of T. An $SL(n, K)$-tableau T with, say, s columns and weight $(d\cdots d) \in \mathbf{N}^r$ is called indecomposable, if there is no positive integer t, $t < s$, such that T_t is an $SL(n, K)$-tableau of weight $(d\cdots d) \in \mathbf{N}^k$ for some k, $0 < k < r$. Then the following result follows from Proposition 1.2 and the Lane-Kharchenko theorem.

THEOREM 1.3 ([Te2] Theorem 3.3). *The set $\{\hat{U}_{n,r,d}(p(T));\ r \in \mathbf{N}$ and T is an indecomposable $SL(n, K)$-tableau of weight $(d\cdots d) \in \mathbf{N}^r\}$ constitutes a set of free generators of the non-commutative) invariant ring $K\langle a_\alpha;\ \alpha \in \mathbf{N}^n, |\alpha| = d\rangle^{ST(n,K)}$.*

Let $A(n, d, r) = \dim_K K[a_\alpha; \alpha \in \mathbf{N}^n, |\alpha| = d]^{SL(n,K)}$ and
$\hat{A}(n, d, r) = \dim_K K\langle a_\alpha; \alpha \in \mathbf{N}^n, |\alpha| = d\rangle^{SL(n,K)}$.

In the commutative case, the classical Hermite reciprocity theorem says that $A(2, d, r) = A(2, r, d)$ for all d and r. On the other hand, in the non-commutative case, we obtain the following reciprocity theorem.

PROPOSITION 1.3. *If* $r > n$, then

$$\hat{A}(n, d, r) = \hat{A}(r - n, d^{\sim}, r),$$

where $d^{\sim} = rd/n - d$.

Proof. This follows from the corollary of Theorem 1.1.

1.3. In this section we shall be concerned with simultaneous invariants of the orthogonal group $O(n, K)$. Let n and r be positive integers with $n \leq r$ and x_{ij}, $1 \leq i, j \leq r$, independent variables. Let I be an ideal of the polynomial ring $K[x_{ij}; 1 \leq i, j \leq r]$ generated by the following elements:

(1) $x_{ij} - x_{ji}$, $1 \leq i, j \leq r$, and
(2) the $(n+1) \times (n+1)$ minors of the $r \times r$ matrix $X = (x_{ij})$, $1 \leq i, j \leq r$.

The polynomial ring $K[x_{ij}; 1 \leq i, j \leq r]$ has an $\mathbf{N}^r$-graded structure by giving each x_{ij} degree $\underset{\sim}{e}_i + \underset{\sim}{e}_j$. Here, as before, $\underset{\sim}{e}_i$ and $\underset{\sim}{e}_j$ denote respectively the i-th and j-th unit vectors of $\mathbf{N}^r$.

For each monomial $x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_k j_k}$ of degree $\underset{\sim}{d} \in \mathbf{N}^r$, we set

$$U_{n,r,\underset{\sim}{d}}(x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_k j_k}) = U((x^{(i_1)}, x^{(j_1)})(x^{(i_2)}, x^{(j_2)}) \cdots (x^{(i_k)}, x^{(j_k)})),$$

where $x^{(1)}, \cdots, x^{(r)}$ are umbral vectors and U the umbral operator, and $(x, y) = \sum_{1 \leq i \leq n} x_i y_i$, the standard inner product.

Then we get a K-linear map

$$U_{n,r,\underset{\sim}{d}} : K[x_{ij}; 1 \leq i, j \leq r] \longrightarrow (S_{n,\underset{\sim}{d}})_{(1 \cdots 1)}.$$

The fundamental theorem of vector invariants (cf. [W] Chap. 2) for the orthogonal group $O(n, K)$ says that the ring $K[x_{ij}; 1 \leq i, j \leq r]/I$ is isomorphic to the ring $K[x_j^{(i)}; 1 \leq i \leq r, 1 \leq j \leq n]^{O(n,K)}$ of orthogonal vector invariants, via the map $x_{ij} \to (x^{(i)} x^{(j)})$. By the same argument as in the proof of Theorem 1.1, we then obtain the following result.

THEOREM 1.4. *For each* $\underset{\sim}{d} \in \mathbf{N}^r$, *the image of the K-linear map* $U_{n,r,\underset{\sim}{d}}$ *is the vector space* $(S_{n,\underset{\sim}{d}})^{O(n,K)}$ *of multi-linear* $O(n, K)$*-invariants of type* $\underset{\sim}{d}$, *and*

$$\operatorname{Ker} U_{n,r,\underset{\sim}{d}} = I \cap K[x_{ij}; 1 \leq i, j \leq r]_{\underset{\sim}{d}}.$$

In other words, the K-linear map $U_{n,r,\underset{\sim}{d}}$ induces a K-linear isomorphism from the K-vector space $(K[x_{ij}; 1 \leq i, j \leq r]/I)_{\underline{d}}$ to the K-vector space of multi-linear $O(n, K)$-invariants of type $\underline{d}$.

Let, as before, $\langle d \rangle = (d \cdots d) \in \mathbf{N}^r$ and $\hat{U}_{n,r,d} = \Psi_r^{-1} U_{n,r,\langle d \rangle}$.

COROLLARY. *For all* $d, r \in \mathbf{N}$,

$$\hat{U}_{n,r,d} : (K[x_{ij}; 1 \leq i, j \leq r]/I)_{\langle d \rangle} \longrightarrow K\langle a_\alpha; \alpha \in \mathbf{N}^n, |\alpha| = d \rangle_r^{O(n,K)}$$

is a K-linear isomorphism.

Let λ be a Young diagram. A Young tableau T with shape λ of length $\leq n$ is called an $O(n, K)$-tableau if λ is an even partition. Given $(i_1, i_2, \cdots, i_m) \in \mathbf{N}^m$ and $(j_1, j_2, \cdots, j_m) \in \mathbf{N}^m$ with $1 \leq i_k, j_k \leq r$, we denote by $(i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_m)$ the determinant of the minor of the r by r symmetric matrix

$$X = (x_{ij}; x_{ij} = x_{ji})$$

with row indices $(i_1, i_2, \cdots, i_m)$ and column indices $(j_1, j_2, \cdots, j_m)$.

To each $O(n, K)$-tableau of weight $\underline{d} \in \mathbf{N}^r$;

$$T = \begin{bmatrix} a_{11} & b_{11} & a_{21} & b_{21} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ a_{1m_1} & b_{1m_1} & a_{2m_2} & b_{2m_2} & \cdots \end{bmatrix},$$

we associate an element $x(T)$ of $K[x_{ij}; 1 \leq i, j \leq r]$ by

$$x(T) = \prod_{i \geq 1} (a_{i1} a_{i2} \cdots a_{im_i} | b_{i1} b_{i2} \cdots b_{im_i}) .$$

Then by Theorem 5.1 of [D-P], the set

$$\{x(T);\ T \text{ is an } O(n, K)\text{-tableau of weight } \underline{d}\}$$

constitutes a K-basis of $(K[x_{ij}; 1 \leq i, j \leq r]/I)_{\underline{d}}$. Combining this with the fundamental theorem of vector invariants for the orthogonal group $O(n, K)$, we obtain

PROPOSITION 1.4. *The set*

$$\{U_{n,r,\underline{d}}(x(T));\ T \text{ is an } O(n, K)\text{-tableau of weight } \underline{d}\}$$

is a K-basis of the vector space of simultaneous $O(n, K)$-invariants of type $\underline{d}$.

In particular, we have the following

PROPOSITION 1.5. *The set*

$\{\hat{U}_{n,r,d}(x(T));$ *T is an* $O(n, K)$*-tableau of weight* $(d \cdots d) \in \mathbf{N}^r\}$

is a K-basis of the vector space $K\langle a_\alpha;\ \alpha \in \mathbf{N}^n,\ |\alpha| = d\rangle_r^{O(n,K)}$.

An $O(n, K)$-tableau T of weight $(d \cdots d) \in \mathbf{N}^r$ with, say, s columns is called indecomposable if, for any $0 < t < s$, the sub-tableau T_t is not an $O(n, K)$-tableau of weight $(d \cdots d) \in \mathbf{N}^k$, $0 < k < r$. Then the following theorem follows from Proposition 1.5 and the Lane-Kharchenko theorem.

THEOREM 1.5. *The set*
$\{\hat{U}_{n,r,d}(x(T));\ r \in \mathbf{N}$ *and T is an indecomposable* $O(n, K)$*-tableau of weight* $(d \cdots d) \in \mathbf{N}^r\}$ *constitutes a set of free generators of the (non-commutative) invariant ring* $K\langle a_\alpha;\ \alpha \in \mathbf{N}^n,\ |\alpha| = d\rangle^{O(n,K)}$.

1.4. In this section we shall be concerned with simultaneous invariants for the symplectic group $\mathrm{Sp}(n, K)$. Let n be an even positive integer and r an integer with $r > n$. Let x_{ij}, $1 \le i, j \le r$, $i = j$, be independent commutative variables and let I be an ideal of the polynomial ring $K[x_{ij};\ 1 \le i, j \le r]$ generated by

(1) $x_{ij} + x_{ji}$, $1 \le i,\ j \le r$, and

(2) the Pfaffians of the $(n + 2) \times (n + 2)$ principal minors taken from the upper corner of the skew-symmetric matrix

$$\begin{bmatrix} 0 & x_{12} \cdots x_{1r} \\ -x_{12} & 0 \ \cdots\ x_{2r} \\ \cdot & \cdot \ \cdots \cdot \\ -x_{1r} & \cdot \ \cdots\ 0 \end{bmatrix}.$$

By giving each x_{ij} degree $\underline{e}_i + \underline{e}_j \in \mathbf{N}^r$, $K[x_{ij};\ 1 \le i, j \le r]$ has an $\mathbf{N}^r$-graded structure. For each monomial $x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_k j_k}$ of degree $\underline{d} \in \mathbf{N}^r$, we set

$$U_{n,r,\underline{d}}(x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_k j_k}) = U([x^{(i_1)}, x^{(j_1)}] \cdots [x^{(i_k)}, x^{(j_k)}]),$$

where U is the umbral operator and

$$[x, y] = (x_1 y_1' - x_1' y_1) + \cdots + (x_m y_m' - x_m' y_m), \quad n = 2m, \quad \text{with}$$
$$x = (x_1 x_1' x_2 x_2' \cdots x_m x_m'), \quad y = (y_1 y_1' y_2 y_2 \cdots y_m y_m').$$

Then we obtain a K-linear map

$$U_{n,r,\underline{d}} : K[x_{ij};\ 1 \le i, j \le r]_{\underline{d}} \longrightarrow (S_{n,\underline{d}})_{(1 \cdots 1)},$$

and, by using the fundamental theorem of vector invariants for the symplectic group Sp(n, K), we obtain the following

THEOREM 1.6. *For each $\underline{d} \in \mathbf{N}^r$, the image of $U_{n,r,\underline{d}}$ is the vector space of simultaneous $Sp(n, K)$-invariants of type $\underline{d}$ and*

$$\text{Ker } U_{n,r,\underline{d}} = I \cap K[x_{ij};\ 1 \leq i, j \leq r]_{\underline{d}}\ .$$

In other words the K-linear map $U_{n,r,\underline{d}}$ induces a K-linear isomorphism from the space $(K[x_{ij}; 1 \leq i, j \leq r]/I)_{\underline{d}}$ to the space of all multi-linear simultaneous Sp(n, K)-invariants of type $\underline{d}$.

For a $2m$-tuple $(i_1, i_2, \cdots, i_{2m})$ of positive integers with $1 \leq i_1 < i_2 < \cdots < i_{2m} \leq r$, we denote by $[i_1 i_2 \cdots i_{2m}]$ the Pfaffian of the principal minor taken from the upper corner of the r by r skew-symmetric matrix $X = (x_{ij}; x_{ij} = -x_{ji})$, with row and column indices $i_1, i_2, \cdots, i_{2m}$. Let λ be a partition of length $\leq n$. A Young tableau T of shape λ is called an *$Sp(n, K)$-tableau* if the transpose ${}^t\lambda$ of λ is an even partition. To each $Sp(n, K)$-tableau

$$T = \begin{bmatrix} a_{11} & a_{21} & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \\ a_{1k_1} & a_{2k_2} & \cdots \end{bmatrix}$$

of weight $\underline{d} \in N^r$, we associate an element $x(T)$ of $K[x_{ij}; 1 \leq i, j \leq r]$ by

$$x(T) = [a_{11} \cdots a_{1k_1}][a_{21} \cdots a_{2k_2}] \cdots\ .$$

Note that, since ${}^t\lambda$ is an even partition, $k_1, k_2, \cdots$ are even integer. Then it follows from Theorem 6.5 of [D-P] the set

$$\{x(T);\ T \text{ is an } Sp(n, K)\text{-tableau of weight } \underline{d}\}$$

is a K-basis of the vector space $(K[x_{ij}; 1 \leq i, j \leq r]/I)_{\underline{d}}$. Therefore by the fundamental theorem of vector invariants for the symplectic group $Sp(n, K)$, we obtain the following two propositions.

PROPOSITION 1.6. *The set*

$$\{U_{n,r,\underline{d}}(x(T));\ T \text{ is an } Sp(n, K)\text{-tableau of weight } \underline{d}\}$$

constitutes a K-basis of the vector space of all simulataneous multi-linear $Sp(n, K)$-invariants of type $\underline{d}$.

PROPOSITION 1.7. *For $d \in \mathbf{N}$, let $\hat{U}_{n,r,d}$ be the K-linear map defined by $\hat{U}_{n,r,d} = \Psi_r\ U_{n,r,(d\cdots d)}$. Then the set*

$$\{\hat{U}_{n,r,d}(x(T));\ T \text{ is an } Sp(n, K)\text{-tableau of weight } (d\cdots d) \in \mathbf{N}^r\}$$

is a K-basis of the vector space $K\langle a_\alpha;\ \alpha \in \mathbf{N}^n,\ |\alpha| = d\rangle_r$.

An $Sp(n, K)$-tableau of weight $(d\cdots d) \in \mathbf{N}^r$ with, say, s columns is called indecomposable if, for any $0 < t < s$, the sub-tableau T_t is not an $Sp(n, K)$-tableau. Then we, as before, obtain

THEOREM 1.7. *The set* $\{\hat{U}_{n,r,d}(x(T));\ r \in \mathbf{N}$ *and* T *is an indecomposable* $Sp(n, K)$*-tableau of weight* $(d\cdots d) \in \mathbf{N}^r\}$ *is a set of free generators of the (non-commutative) invariant ring* $K\langle a_\alpha;\ \alpha \in \mathbf{N}^n,\ |\alpha| = d\rangle^{Sp(n,K)}$.

§ 2. *S*-Generators of tensor invariants

2.1. Let V be a finite dimensional K-vector space and G a subgroup of $GL(V)$ acting on $K\langle V\rangle$ as a group of graded algebra homomorphisms on $K\langle V\rangle$. For each $m \in N$, the symmetric group S_m acts on the space $\otimes^m V$ by

$$\sigma(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}, \quad \sigma \in S_m .$$

In general a graded sub-algebra $R = \oplus_{m\geq 0} R_m$ of $K\langle V\rangle$ is called an S-algebra if each R_m is a sub-S_m-module of $\otimes^m V$. The invariant ring $K\langle V\rangle^G$ is an S-algebra, since the actions of $GL(n, K)$ and S_m on $\otimes^m V$ centralize each other. Let $\{f_i\}_{i\in I}$ be a system of homogeneous elements of $K\langle V\rangle^G$. We denote by $SK\langle f_i;\ i \in I\rangle$ the algebra generated by the f_i, $i \in I$, together with the actions of the symmetric groups. If $SK\langle f_i;\ i \in I\rangle = K\langle V\rangle^G$, then $\{f_i;\ i \in I\}$ is called a homogeneous system of S-generators. If $K\langle V\rangle^G$ has a homogeneous system of S-generators consisting of finitely many tensor invariants, then $K\langle V\rangle^G$ is called finitely generated as an S-algebra. A. N. Koryukin [Ko] proved that if G is a reductive algebraic subgroup of $GL(V)$, the invariant ring $K\langle V\rangle^G$ is finitely generated as an S-algebra. We now consider the commutative ring $K[\oplus^n V]^G$, $n = \dim V$, of all simultaneous polynomial invariants. To each homogeneous element f of $K[\oplus^n V]^G$, we can associate an element $\hat{f}$, called complete polarization, of $K\langle V\rangle^G$. For details, consult [Te1].

THEOREM 2.1. (Theorem 2.1 [Te1]). *Let G be a subgroup of GL(V) and* $\{f_i\}_{i\in I}$ *a homogeneous system of generators of the (commutative) invariant ring* $K[\oplus^n V]^G$, $n = \dim V$. *Then* $\{\hat{f}\}_{i\in I}$ *is a homogeneous system of S-generators of* $K\langle V\rangle^G$.

Theorem 2.1 enables us to find such a number $N_{\tilde{G},V}$ that the invariant ring $K\langle V\rangle^G$ is generated as an S-algebra by invariants of degree $\leq N_{\tilde{G},V}$.

THEOREM 2.2. *If the field K is algebraically closed and G is an algebraic subgroup of $GL(V)$, then*

(1) *if G is a finite group, we can take $N_{\tilde{G},V} = \sharp G$,*

(2) *if G is a torus, we can take $N_{\tilde{G},V} = n^2C(n^2 s!t^s)$,*

(3) *if G is semi-simple and connected, we can take*

$$N_{\tilde{G},V} = n^2 C\left(\frac{2^{r+s}n^{2(s+1)}(n^2-1)^{s-r}t^r(s+1)!}{3^s(((s-r)/2))!)^2}\right).$$

Here $n = \dim V$, $s = \dim G$, and $r =$ rank of G. For a positive integer k, $C(k) =$ L.C.M.$\{a \in \mathbf{N}; 0 < a \leq k\}$. For the definition of t, see [P1] *Theorem* 2.

Proof. By Theorem 2.1, the problem can be reduced to the commutative case, and we obtain the desired result by Theorem 2 of [P1].

Remark. T. Tambour (Theorem 7 [T]) proved (1) by a different method. In the commutative case, the proof of (1) was given by E. Noether [N], of (2) by G. Kempf [K], and of (3) by V. L. Popov [P1].

2.2. T. Tambour [T] has investigated a generating function associated with the graded S-algebra $K\langle V\rangle^G$ and proved that the generating function is equal to the Poincare series of the graded ring $K[\Lambda^2 V \oplus V]^G$, $\Lambda^2 =$ the exterior square. Then one can naturally expect some relationship between structure of the S-algebra and that of $K[\Lambda^2 V \oplus V]^G$. In this section we will establish a relationship between them. For a partition λ, we denote by $s_\lambda(x_1, x_2, \cdots)$ the Schur function corresponding to λ. The Littlewood identity ([M] Chap. 1)

$$\textstyle\sum_\lambda s_\lambda(x_1, x_2, \cdots) = \Pi_i(1-x_i)^{-1}\Pi_{i<j}(1-x_i x_j)^{-1}$$

shows that the $GL(n, K)$ ($= GL(V)$)-module $K[\Lambda^2 V \oplus V]$ is decomposed into the irreducible parts

$$K[\Lambda^2 V \oplus V] = \oplus_\lambda W_\lambda\,,$$

where λ is over all the partitions of length $\leq n$ and W_λ denotes the irreducible $GL(n, K)$-submodule corresponding to λ. Let

$$x_{ij},\ 1 \leq i < j \leq n, \quad \text{and} \quad x_k,\ 1 \leq k \leq n\,,$$

be indeterminates, then

$$K[\Lambda^2 V \oplus V] = K[x_{ij}, x_k;\ 1 \leq i < j \leq n,\ 1 \leq k \leq n]\,.$$

For each m, $1 \leq m \leq n$, we define a polynomial J_m in x_{ij} and x_k by

$$J_m = \begin{cases} \sum_{i_1\cdots i_m} \varepsilon^{i_1\cdots i_m} x_{i_1 i_2} \cdots x_{i_{m-1} i_m}, & \text{if } m \text{ is even,} \\ \sum_{i_1\cdots i_m} \varepsilon^{i_1\cdots i_m} x_{i_1 i_2} \cdots x_{i_{m-2} i_{m-1}} x_{i_m}, & \text{if } m \text{ is odd,} \end{cases}$$

where

$$\varepsilon^{i_1\cdots i_m} = \begin{cases} 1, & \text{if } (i_1, \cdots, i_m) \text{ is an even permutation of } 1, \cdots, m \\ -1, & \text{if } (i_1, \cdots, i_m) \text{ is an odd permutation of } 1, \cdots, m \\ 0, & \text{otherwise.} \end{cases}$$

When m is even, J_m is the Pfaffian relative to the principal m by m minor taken from the upper corner of the n by n skew-symmetric matrix $X = (x_{ij};\ x_{ij} = -x_{ij})$.

For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of length $\leq n$, we set

$$f_\lambda(x_{ij}, x_k) = J_1^{l_1} J_2^{l_2} \cdots J_n^{l_n}\,,$$

where $l_i = \lambda_i - \lambda_{i+1}$, $1 \leq i \leq n$, with $\lambda_{n+1} = 0$. Then it is easily seen that $f_\lambda(x_{ij}, x_k)$ is an weight vector under the action of the group of all upper triangular n by n matrices and

$$\begin{pmatrix} t_1 & & \\ & \ddots & t_{ij} \\ & & t_n \end{pmatrix} f_\lambda(x_{ij}, x_k) = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n} f_\lambda(x_{ij}, x_k)\,.$$

Therefore $f_\lambda(x_{ij}, x_k)$ is the highest weight vector of the irreducible $GL(n, K)$-module W_λ and hence we have

$$W_\lambda = GL(n, K) \cdot f_\lambda(x_{ij}, x_k)\,.$$

We denote by e_λ the Young idempotent corresponding to a partition λ.

Let

$$T_\lambda = e_\lambda \cdot \otimes^m V\,.$$

Then T_λ is an irreducible $GL(n, K)$-submodule of $\otimes^m V$ and hence there exists a $GL(n, K)$-isomorphism

$$a_\lambda : W_\lambda \longrightarrow T_\lambda\,,$$

for each partition λ of length $\leq n$. We define an isomorphism of $GL(n, K)$-modules

$$a : K[\Lambda^2 V \oplus V] \longrightarrow \oplus_{l(\lambda) \leq n} T_\lambda ,$$

by $a = \oplus_{l(\lambda) \leq n} a_\lambda$.

For partitions λ and μ of length $\leq n$, consider the $GL(n, K)$-map Ψ and $\Psi' : W_\lambda \otimes W_\mu \to W_{\lambda+\mu}$, defined as follows: for $f_1 \in W_\lambda$ and $f_2 \in W_\mu$,

$$\Psi(f_1 \otimes f_2) = f_1 \cdot f_2 \text{ (usual multiplication of polynomials)}$$

and

$$\Psi'(f_1 \otimes f_2) = a_{\lambda+\mu}^{-1}(e_{\lambda+\mu} \cdot (a_\lambda(f_1) \otimes a_\mu(f_2))) ,$$

where $e_{\lambda+\mu}$ the Young idempotent associated with the partition $\lambda + \mu$. Since $W_\lambda = GL(n, K) \cdot f_\lambda(x_{ij}, x_k)$ and $f_\lambda(x_{ij}, x_k) \cdot f_\mu(x_{ij}, x_\mu) = f_{\lambda+\mu}(x_{ij}, x_k)$, the map Ψ is well-defined.

Hereafter we asssume that the field K is algebraically closed. Because W_λ and W_μ are irreducible $GL(n, K)$-modules and the decomposition of the tensor product $W_\lambda \otimes W_\mu$ into irreducible parts contains the irreducible $GL(n, K)$-module $W_{\lambda+\mu}$ with multiplicity one, it follows from Schur's lemma that Ψ and Ψ' coincide, up to a non-zero scalar in K. Therefore the following diagram of $GL(n, K)$-isomorphisms is commutative up to a non-zero scalar:

$$\begin{array}{ccc} W_\lambda \otimes W_\mu & \xrightarrow[\Psi]{} & W_{\lambda+\mu} \\ {\scriptstyle a_\lambda \otimes a_\mu} \downarrow & & \downarrow {\scriptstyle a_{\lambda+\mu}} \\ T_\lambda \otimes T_\mu & \xrightarrow[\psi]{} & T_{\lambda+\mu} , \end{array}$$

where ψ is defined by $\psi(x \otimes y) = e_{\lambda+\mu}(x \otimes y)$, $x \in T_\lambda$, $y \in T_\mu$.

THEOREM 2.3. *Let the field K be algebraically closed and G a subgroup of $GL(V)$. If $\{f_i\}_{i \in I}$ a homogeneous system of generators for the (commutative) invariant ring $K[\Lambda^2 V \oplus V]^G$, then $\{a(f_i)\}_{i \in I}$ is a homogeneous system of S-generators for the (non-commutative) invariant ring $K\langle V \rangle^G$.*

Proof. For each $k \in \mathbf{N}$, we regard $\otimes^k V$ as a $GL(n, K) \times S_k$-module. Then by H. Weyl's reciprocity theorem, it decomposes as

$$\otimes^k V = \oplus_{\substack{l(\lambda) \leq n \\ |\lambda| = k}} T_\lambda \otimes V_\lambda^{S_k}, \quad n = \dim_K V .$$

Here $V_\lambda^{S_k}$ denotes the irreducible S_k-module corresponding to the partition λ. Denoting by $K[S_k]$ the group ring of S_k, we have

$$T_\lambda \otimes V_\lambda^{S_k} \simeq (K[S_k] e_\lambda) \cdot T_\lambda ,$$

and hence

$$(\otimes^k V)^G = \oplus_{\substack{l(\lambda)\leq n\\ |\lambda|=k}}(K[S_k]e_\lambda)\cdot(T_\lambda)^G .$$

This together with the diagram above completes the proof.

§ 3. Non-commutative invariants of rings of 2 by 2 generic matrices with zero trace

In this section we will study invariant rings of 2 by 2 generic matrices with zero trace under linear actions of finite groups. Let K be a field of characteristic zero and let $X_1, X_2, \cdots, X_n$ $(n \geq 2)$ be 2 by 2 generic matrices with trace zero over K. That is

$$X_1 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{bmatrix},\ X_2 = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & -y_{11} \end{bmatrix}, \cdots,\ X_n = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & -z_{11} \end{bmatrix},$$

where $x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}, \cdots, z_{11}, z_{12}, z_{21}$ are commuting indeterminates over K. The K-subalgebra

$$R_n = K[X_1, X_2, \cdots, X_n]$$

generated by $X_1, X_2, \cdots, X_n$ is called the ring of n generic 2 by 2 matrices with zero trace. This is a K-subalgebra of the 2 by 2 matrix algebra $M_2(K[x_{ij}, y_{ij}, z_{ij}])$ over the polynomial ring $K[x_{ij}, y_{ij}, z_{ij}]$.

Let $M_2^0(K)$ denote the set of 2 by 2 matrices with zero trace. The group $GL(2, K)$ acts on $\oplus^n M_2^0(K)$ by

$$g\cdot(A_1, A_2, \cdots, A_n) = (gA_1g^{-1}, gA_2g^{-1}, \cdots, gA_ng^{-1}),\quad \text{with}$$
$$g \in GL(2, K) \quad \text{and} \quad (A_1, A_2, \cdots, A_n) \in \oplus^n M_2^0(K) .$$

Then in a natural manner (cf. [Pr]), R_n can be identified with the ring of polynomial $GL(2, K)$-concomitants

$$f: \oplus^n M_2^0(K) \longrightarrow M_2(K) .$$

We denote by C_n the invariant ring $K[\oplus^n M_2(K)]^{GL(2,K)}$. C_n can be identified with center of R_n (cf. [Pr] Sec. 2). The general linear group $GL(n, K)$ acts on R_n and C_n by the left multiplication on the column vector ${}^t(X_1, X_2, \cdots, X_n)$ of 2 by 2 generic matrices with zero trace $X_1, X_2, \cdots, X_n$.

THEOREM 3.1. *Let G be a reductive subgroup of $GL(n, K)$. Then the invariant ring R_n^G is a finitely generated K-algebra.*

Proof. By a well-known theorem in invariant theory, C_n^G is finitely generated K-algebra. Since R_n^G is a finitely generated C_n^G-module, R_n^G is finitely generated K-algebra.

We now prove that for any finite subgroup G of $GL(n, K)$, R_n^G is a Cohen-Macaulay module over C_n^G. First we recall a result of Le Bruyn.

THEOREM 3.2 ([L] Theorem 5.1). *R_n is Cohen-Macaulay over C_n.*

We are going to prove the following

THEOREM 3.3. *If G is a finite subgroup of $GL(n, K)$, then R_n^G is a Cohen-Macaulay C_n^G-module.*

Proof. Because

$$C_n^G = K[\oplus^n M_2(K)]^{G\times SL(2,K)},$$

C_n^G is a Cohen-Macaulay ring, by the fundamental theorem of Hochstar and Roberts. Let $(\theta_1, \cdots, \theta_s)$ be a homogeneous system of parameters of C_n^G. By a standard argument, we see that $(\theta_1, \cdots, \theta_s)$ is a homogeneous system of parameters for C_n. By Le Bruyn's theorem, R_n is a Cohen-Macaulay module over C_n. Hence $R_n/(\theta_1, \cdots, \theta_n)$ is a finite dimensional K-vector space. Since the group $G\times SL(2, K)$ is reductive, there exists a Raynord's operator

$$\sharp: R_n \longrightarrow R_n^G .$$

Let $W = \{f \in R_n; f^\sharp = 0\}$. Then W is an R_n^G-module and

$$R_n = R_n^G \oplus W .$$

We choose a basis $(\bar{f}_1, \cdots, \bar{f}_t)$ of $R_n/(\theta_1, \cdots, \theta_s)$ so that $(\bar{f}, \cdots, \bar{f}_u)$ is a basis of $R_n^G/(\theta_1, \cdots, \theta_s)$ and $\bar{f}_{u+1}, \cdots, \bar{f}_t$ is a basis of $W/(\theta_1, \cdots, \theta_s)W$. Let $f_1, \cdots, f_u$ be representative in R_n^G for $\bar{f}_1, \cdots, \bar{f}_u$, respectively. Then we have

$$R_n^G = \oplus_{i=1}^u f_i K[\theta_1, \cdots, \theta_s] .$$

This completes the proof.

For a Young diagram λ (possibly $\lambda = \phi$) of length ≤ 1 and a Young diagram μ, we define an integer $\kappa(\mu, \lambda) \in \{-1, 0, 1\}$ as follows:

(1) if $l(\mu) \leq 1$, $\kappa(\mu, \lambda) = \begin{cases} 1, & \text{if } \mu = \lambda. \\ 0, & \text{otherwise,} \end{cases}$

(2) if $l(\mu) > 1$ and μ has no skew-hook of length $2l(\mu) - 3$ through the node $(l(\mu), 1)$, then $\kappa(\mu, \lambda) = 0$,

(3) if $l(\mu) > 1$ and μ has a skew-hook h of length $2l(\mu) - 3$ through the node $(l(\mu), 1)$, then $\kappa(\mu, \lambda) = (-1)^{\omega(h)}\kappa(\mu\backslash h, \lambda)$, where $\omega(h)$ denotes the leg length of h.

Let G be a finite subgroup of $GL(n, K)$. In the commutative case, the Poincare series of the invariant ring $K[x_1, \cdots, x_n]^G$ is given by Molien's classical formula

$$P(K[x_1, \cdots, x_n]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1_n - g \cdot t)} .$$

The invariant ring R_n^G is an $\mathbf{N}$-graded ring by giving each X_i degree 1. We consider the Poincare series of R_n^G:

$$P(R_n^G, t) = \sum_{r \in N} \dim_K(R_n^G)_r t^r .$$

THEOREM 3.4. *Let G be a finite subgroup of $GL(n, K)$. Then the Poincare series of the invariant ring R_n^G is given by*

$$P(R_n^G, t) = \frac{1}{|G|} \sum_{g \in G} \sum_{\mu} \frac{(\kappa(\mu, \phi) + \kappa(\mu, \square))\mathrm{Tr}(\rho_\mu(g))t^{|\lambda|}}{\det(1_N - \rho_{\square\square}(g)t^2)} ,$$

where $N = n(n+1)/2$, μ is over all the partitions of length $\leq n$ and ρ_μ denotes the irreducible representation of $GL(n, K)$ corresponding to μ.

Proof. We denote by R_n^o the K-vector space of polynomial concomitants:

$$f\colon \oplus^n M_2^o(K) \longrightarrow M_2^o(K) .$$

Since $M_2(K) = M_2^o(K) \oplus K \cdot 1_2$, we have a direct decomposition

$$R_n = R_n^o \oplus C_n .$$

We can make R_n an N^n-graded ring by giving each X_i degree $\underline{e}_i \in \mathbf{N}^n$, and consider the Poincare series

$$P(R_n, t_1, t_2, \cdots, t_n) = \sum_{d \in \mathbf{N}^n} \dim_K(R_n)_{\underline{d}} t_1^{d_1} \cdots t_n^{d_n}$$

of R_n in this multi-gradation.

In general, let G be a group and let V and W be G-modules of finite rank. G acts on $\oplus^n V$, $n \in \mathbf{N}$, diagonaly. We denote by $K[\oplus^n V, W]^G$ the K-vector space of G-equivariant polynomial maps

$$f\colon \oplus^n V \longrightarrow W .$$

Let M $(= K^{3})$ be the standard $SO(3, K)$-module. Because $SL(2, K)$

and $SO(3, K)$ are isogenous, we have

$$\begin{aligned} R_n &= K[\oplus^n M_2^o(K), M_2(K)]^{SL(2,K)} \\ &= K[\oplus^n M_2^o(K), M_2^o(K)]^{SL(2,K)} + K[\oplus^n M_2^o(K)]^{SL(2,K)} \\ &= K[\oplus^n M, M]^{SO(3,K)} \oplus K[\oplus^n M]^{SO(3,K)} . \end{aligned}$$

Then by Theorem 5.3 [Te3], we obtain

$$P(R_n, t_1, \cdots, t_2) = \sum_\mu \frac{(\kappa(\mu, \phi) + \kappa(\mu, \square) s_\mu(t_1, \cdots, t_n)}{\Pi_{1 \le 1, j \le n}(1 - t_i t_j)} ,$$

where μ is over all the partition of length $\le n$.

Let, in general, V be a finite dimensional K-vector space and G a finite subgroup of $GL(V)$. If M is a $GL(V)$-module of finite rank, we denote by M^G the fixed subspace of M under the action of G. Then we have

$$\dim_K M^G = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(M, g) ,$$

where $\operatorname{Tr}(M, g)$ denotes the trace of g as a linear operator on M.

Therefore

$$\begin{aligned} P(R_n^G, t) &= \frac{1}{|G|} \sum_{g \in G} \sum_{l(\mu) \le n} \frac{(\kappa(\mu, \phi) + \kappa(\mu, \square) s_\mu(t_1 \cdots t_n)}{\Pi_{1 \le i, j \le n}(1 - t_i t_j t^2)} t^{|\mu|} , \\ &\qquad (t_1, \cdots, t_n \text{ are eigenvalues of } g) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{l(\mu) \le n} \frac{(\kappa(\mu, \phi) + \kappa(\mu, \square) \operatorname{Tr}(\rho_\mu(g))}{\det(1_N - \rho_{\square\square}(g) t^2)} t^{|\mu|} . \end{aligned}$$

This completes the proof.

By a result of L. Le Bruyn ([L] Chap. 4), the Poincare series of R_n satisfies the functional equation

$$P(R_n, 1/t) = (-1)^{n-1} t^{3n} P(R_n, t), \quad n \ge 3 .$$

It follows from Theorem 3.5 with an easy verification that the Poincare series of the invariant ring R_n^G satisfies the same functional equation as $P(R_n, t)$, if G is a finite subgroup of $SL(n, K)$.

PROPOSITION 3.1. *If G is a finite subgroup of $SL(n, K)$, then the Poincare series of R_n^G satisfies the functional equation*

$$P(R_n^G, 1/t) = \begin{cases} (-1)^{n-1} t^{3n} P(R_n^G, t), & \text{if } n \ge 3 \\ -t^4 P(R_n^G, t), & \text{if } n = 2 . \end{cases}$$

The following theorem is a generalization of [L] (Chap. 3, Theorem 4.2).

THEOREM 3.6. *Let G be a finite subgroup of $SL(n, K)$. Then the invariant ring R_n^G $(n \geq 2)$ has finite global dimension if and only if $n \leq 3$ and $G = \{e\}$.*

Proof. By [L] (Chap. 3. Theorem 4.2), R_n has finite global dimension if and only if $n \leq 3$. Hence it is enough to prove the "only if" part. Suppose that the invariant ring R_n^G has finite global dimension. Then its Poincase series $P(R_n^G, t)$ has the form

$$P(R_n^G, t) = \frac{1}{f(t)},$$

for some monic polynomial with integer coefficients (cf. [L], p. 87). Since R_n^G is a Cohen-Macaulay module over C_n^G, the Poincare series has the form

$$P(R_n^G, t) = \frac{F(t)}{(1-t^{\alpha_1})(1-t^{\alpha_2})\cdots(1-t^{\alpha_r})},$$

where $F(t)$ is a monic polynomial with no-negative integer coefficients and $\alpha_1, \cdots, \alpha_r$ are some positive integers. Therefore $f(t)$ is product of some cyclotomic polynomials. By the functional equation, we see that

$$\deg f(t) = \begin{cases} 3n, & \text{if } n \geq 3 \\ 4, & \text{if } n = 2 . \end{cases}$$

If $n \geq 3$, then one sees easily that $P(R_n^G, t)$ has a pole of order $3n-3$ at $t = 1$ and hence $f(t)$ has the form

$$f(t) = (1-t)^{3n-3} g(t),$$

for some $g(t) \in \mathbf{Z}[t]$ of degree 3 with $g(t) \neq 0$. Moreover, since $g(t)$ is product of cyclotomic polynomials, one sees that

$$g(t) = 1 + t^3, \quad (1+t)(1 \pm t + t^2), \quad \text{or} \quad (1+t)^3 .$$

This implies that $3n - 6 \leq \dim_K(R_n^G)_1$, $(R_n^G)_1$ is the vector space of invariants of degree one. Since, clearly, $\dim_K(R_n^G) \leq n$, we have $n \leq 3$. If $n = 3$, we have $\dim_K(R_n^G)_1 = \dim_K(R_n)_1 = 3$, and hence $G = \{e\}$. If $n = 2$, by the same argument as before, we find that

$$f(t) = (1-t)^3(1+t) .$$

This implies, $\dim_K(R_2^G)_1 = \dim_K(R_2)_1 = 2$, and hence $G = \{e\}$.

References

[D-P] C. De Contini and C. Procesi, A characteristic free approach to invariant theory, Adv. in Math., **21** (1976), 330–354.

[D-R-S] P. Doubilet, G. -C. Rota, and J. Stein, On the foundation of combinatorial theory: IX. Combinatorial method in invariant theory, Stud. Appl. Math., **53** (1974), 185–216.

[K] G. R. Kempf, Computing invariants, pp. 81–94 in "Invariant Theory", Lecture Notes in Math., No. **1278**, Springer, 1987.

[Kh] V. K. Kharchenko, Algebras of invariants of free algebras, Algebra i Logika, **17** (1978), 478–487 (Russian); English translation: Algebra and Logic **17** (1978), 316–321.

[Ko] A. N. Koryukin, Noncommutative invariants of reductive groups, Algebra i Logika, **23** (1984, 419–429 (Russian); English translation: Algebra and Logic, **23** (1984), 290–296.

[K-R] J. P. S. Kung and G. -C. Rota, The invariant theory of binary forms, Bull. Amer. Math. Soc. (New Series), **10** (1985), 27–85.

[L] D. R. Lane, Free algebras of rank two and their automorphisms, Ph. D. Thesis, Betford College, London, 1976.

[Le] L. Le Bruyn, Trace rings of generic 2 by 2 matrices, Mem. Ams Amer. Math. Soc. **363** (1987).

[M] I. G. MacDonald, "Symmetric Functions and Hall Polynomials", Oxford University Press, Oxford, 1979.

[N] E. Noether, Der Endlichkeitessatz der Invarianten endlichen Gruppen, Math. Ann., **77** (1916), 89–92.

[P1] V. Popov, Modern development in invariant theory, pp. 394–406 in "Proc. of I. C. M. Barkeley 1986".

[P2] ——, Constructive invariant theory, Asterisque, **87–88** (1981), 303–334.

[Pr] C. Procesi, The invariant theory of $n \times n$ matrices, Adv. of Math., **19** (1976), 306–381.

[T] T. Tambour, Examples of S-algebras and generating functions, preprint.

[Te1] Y. Teranishi, Noncommutative invariant theory, 321–332, in "Perspectives in Ring Theory", Nato ASI Series 233, Kluwer Academic Publishers, 1988.

[Te2] ——, Noncommutative classical invariant theory, Nagoya Math. J., **112** (1988), 153–169.

[Te3] ——, Universal induced characters and restriction rules for the classical groups, Nagoya Math. J., **117** (1990), 173–205.

[W] H. Weyl, "The Classical Groups", Princeton University Press, Princeton, 1946.

Department of Mathematics
School of Science
Nagoya University
Chikusa-ku, Nagoya, 464-01
Japan