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CATEGORIES OF MULTIPLICATIVE FUNCTORS AND WEIL'S INFINITELY NEAR POINTS

Dedicated to C. Ehresmann in the commemoration of his 80th birthday

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§ 1. Introduction

At the early fifties A. Weil introduced [3] and algebraic approach to the theory of infinitesimal prolongations of smooth manifolds motivated by the theory of jets developed by Ch. Ehresmann on one side and also by the return to Fermat's methods on the infinitesimal calculus of first order, that makes use of nilpotent infinitesimals.

We start working with associative, commutative, unitary, finite dimensional $\boldsymbol{R}$-algebras A having a nilpotent ideal I complementary to $\boldsymbol{R}$, this identified with its image by the map $\boldsymbol{R} \to A$ $t \to t.1_A$. In this case $A = \boldsymbol{R} \oplus I$ and I is the unique maximal ideal of A. Following Weil we will call these algebras local algebras. The morphisms of $\boldsymbol{R}$-algebras will be the usual ones, that is, $\boldsymbol{R}$-linear and compatible with the subjacent ring structure.

The manifolds will be supposed C^∞, Hausdorff, second countable, with no restriction on the dimension of components, unless explicited stated, all maps between manifolds C^∞; this category of objects and morphisms will be denoted by $\mathfrak{M}$. Then in Weil's sense an A-near point over a manifold M is a morphism of $\boldsymbol{R}$-algebras $C^\infty(M) \xrightarrow{\eta} A$. It is a basic result that there is a canonical one-to-one correspondence between $\boldsymbol{R}$-near points over M and the points of M itself, namely $x \mapsto \varepsilon_x$, where ε_x is the evaluation morphism $C^\infty(M) \to \boldsymbol{R}$, $f \mapsto f(x)$. An A-near point η over M is said to be (infinitely) near to the point x of M if (and only if) $p \circ \eta$ corresponds to x by the bijection described above; here $A \xrightarrow{p} \boldsymbol{R}$ is the canonical projection. The Ehresmann's q^k-velocities over M originates in this point of view from the k-truncated polynomial algebra in q indeterminates with real coefficients.

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Now if $M \xrightarrow{\varphi} N$ is a map between manifolds we have the induced morphism of $\boldsymbol{R}$-algebras $C^\infty(N) \xrightarrow{\varphi^*} C^\infty(M)$, $g \mapsto g \circ \varphi$, which gives a map $\eta \mapsto \eta \circ \varphi^*$ associating A-near points over N to A-near points over M; it is clear that $\eta \circ \varphi^*$ is near to $\varphi(x)$ if η is near x.

To fix notations let $\overline{T}_A(M)$ denote the set of A-near points over M and $\overline{T}_A(\varphi)$ denote the map described above from $\overline{T}_A(M)$ to $\overline{T}_A(N)$; $\overline{T}_A(M)$ can be given a canonical manifold structure in such a way that $\overline{T}_A(\varphi)$ is a map between manifolds for any φ, further the correspondences $M \rightsquigarrow \overline{T}_A(M)$ $\varphi \rightsquigarrow \overline{T}_A(\varphi)$ gives a covariant endofunctor $\overline{T}_A$ of $\mathfrak{M}$. Also for any morphism $A \xrightarrow{\lambda} B$ of local algebras we have a canonical natural transformation $\overline{T}_A \xrightarrow{\overline{T}(\lambda)} \overline{T}_B$ given by $\eta \to \lambda \circ \eta$ for any $\eta \in \overline{T}_A(M)$. It will be useful to distinguish with the notation τ the natural isomorphism $\overline{T}_{\boldsymbol{R}} \to \mathrm{Id}$ given by the bijective correspondence between $\boldsymbol{R}$-near points and points on manifolds; here Id denotes the identity functor of $\mathfrak{M}$.

All that was described above in a succinct way is contained in [4] although in another language; we have stressed the functorial aspect in describing these ideas for it is exactly this the spirit of this paper.

In 1969, A. Morimoto in his lecture notes [3] worked on many applications about prolongations of geometrical structures over manifolds by means of the functors $\overline{T}_A$. To be explicit about the origins of the problem treated in this paper we reproduce that part of the introduction to [3] which refers to it:

"The purpose of this series of papers is to give the results of my recent works on the prolongation of geometrical structures on a differentiable manifold M to the tangent bundles of higher order, bundles of p^r-jets or more generally to bundles of infinitely near points of arbitrary kind. Our general method is based on the consideration of the functor $\tilde{T}$ which assigns to M the bundle $\tilde{T}(M)$ of some kind. The properties which we use in most parts of this series are the following (1)~(7):

(1) The covariant functor $\tilde{T}$ is multiplicative in the following sense: $\tilde{T}(M \times N)$ is canonically diffeomorphic to $\tilde{T}(M) \times \tilde{T}(N)$.

(2) For differentiable maps Φ, Ψ we have the following equalities (in the case when they have meanings):

$$\tilde{T}(\Phi \circ \Psi) = \tilde{T}(\Phi) \circ \tilde{T}(\Psi),$$
$$\tilde{T}(\Phi \times \Psi) = \tilde{T}(\Phi) \times \tilde{T}(\Psi),$$
$$\tilde{T}(\Phi, \Psi) = (\tilde{T}(\Phi), \tilde{T}(\Psi))$$

$$\tilde{T}1_M = 1_{\tilde{T}(M)}$$

where 1_M stands for the identity map of M.

(3) If V is a real vector space of finite dimension, then $\tilde{T}V$ is canonically a vector space.

(4) If G is a Lie group (acting on a manifold M) then $\tilde{T}G$ is also a Lie group (acting on the manifold $\tilde{T}M$).

(5) $\tilde{T}GL(n)$ acts on $\tilde{T}\boldsymbol{R}^n$ effectively as a linear transformation group.

(6) If $P(M, \pi, G)$ is a principal fibre bundle then $\tilde{T}P(\tilde{T}M, \tilde{T}\pi, \tilde{T}G)$ is also a principal fibre bundle.

(7) Let $\{x_1, \cdots, x_n\}$ (resp. $\{y_1, \cdots, y_n\}$) be a local coordinate system on a neighborhood U and let J: $U \to GL(n)$ be the Jacobian matrix with respect to the coordinate systems $\{x_i\}$, $\{y_i\}$ and let $\{\tilde{x}_1, \cdots, \tilde{x}_N\}$ (resp. $\{\tilde{y}_1, \cdots, \tilde{y}_N\}$) be the induced coordinate system on $\pi^{-1}(U)$ in some sense, then we have the following equality:

$$j^{(\tilde{T})}(\tilde{T}J) = \tilde{J}$$

where $\tilde{J}$: $\pi^{-1}(U) \to GL(\tilde{T}\boldsymbol{R}^n)$ denotes the Jacobian matrix with respect to $\{\tilde{x}_\nu\}$ and $\{\tilde{y}_\nu\}$ and where $j^{(\tilde{T})}$: $\tilde{T}GL(n) \to GL(\tilde{T}\boldsymbol{R}^n)$ is a homomorphism defined by property (5).

... (He continues describing the distribution of subjects in the work, which doesn't concern us here, and then finalizes with:).

The author conjectures that if a covariant functor $\tilde{T}$ has the properties (1)$\sim$(7), then there exists a local algebra A such that $\tilde{T}$ will be (or will be near to) the functor which assigns the bundle of infinitely near points of A-kind".

The last assertion above was the motivation for the investigations made in this paper.

§ 2. Auxiliary results and remarks

Before stating the approach we had given to the conjecture raised by Morimoto we will make some remarks of auxiliary nature.

R1. If a covariant endofunctor T of $\mathfrak{M}$ preserves products then $T(\mathbf{1})$ is a singleton for any singleton $\mathbf{1}$.

In fact, it suffices to observe that if $\mathbf{1} \xrightarrow{q} \mathbf{1} \times \mathbf{1} \xrightarrow{p} \mathbf{1}$ is the cartesian product then $T(\mathbf{1}) \xleftarrow{T(q)} T(\mathbf{1} \times \mathbf{1}) \xrightarrow{T(p)} T(\mathbf{1})$ must be also a product in $\mathfrak{M}$, that is, $T(\mathbf{1}\times\mathbf{1})\xrightarrow{T(p),T(q)}T(\mathbf{1})\times T(\mathbf{1})$ is a diffeomorphism and observe that $T(p)=T(q)$ (because $p=q$).

Now if $M\xrightarrow{\varphi}N$ is any constant map between manifolds whose unique value is $y\in N$ then $T(\varphi)$ is constant since $T(\varphi)$ factorizes through the singleton $T(\{y\})$ and the value of $T(\varphi)$ is that of the map $T(\{y\})\to T(N)$ obtained from the inclusion $\{y\}\to N$.

R2. If T is a product preserving covariant endofunctor of $\mathfrak{M}$ we can define for any manifold M a map $M\xrightarrow{i_M}T(M)$ such that for any map $M\xrightarrow{\varphi}N$ between manifolds we have the following commutative diagram:

$$\begin{tikzcd} T(M) \arrow[r, "T(\varphi)"] & T(N) \\ M \arrow[u, "i_M"] \arrow[r, "\varphi"] & N \arrow[u, "i_N"'] \end{tikzcd}$$

In fact, let $i_M(x)$ be the value of the constant map $T(\{x\})\to T(M)$ obtained by application of T on the inclusion $\{x\}\to M$; the commutativity of the above diagram follows from the commutativity of:

$$\begin{tikzcd} T(\{x\}) \arrow[r] \arrow[d] & T(M) \arrow[d, "T(\varphi)"] \\ T(\{\varphi(x)\}) \arrow[r] & T(N) \end{tikzcd}$$

Further the smoothness of the maps i_M depends only on the smoothness of $\boldsymbol{R}\xrightarrow{i_{\boldsymbol{R}}}T(\boldsymbol{R})$. To see this observe that i_M is smooth iff $i_M|U$ is smooth for any domain of a chart $U\xrightarrow{\theta}\boldsymbol{R}^n$ onto $\boldsymbol{R}^n$.

Now $i_M|U=i_M\circ\nu$ where $U\xrightarrow{\nu}M$ is the inclusion and so the following commutative diagram gives the conclusion for $n>0$:

$$\begin{tikzcd} & \boldsymbol{R}^n \arrow[dl, "(i_{\boldsymbol{R}})^n"'] \arrow[d, "i_{\boldsymbol{R}^n}"] & U \arrow[l, "\theta"'] \arrow[r, "\nu"] \arrow[d, "i_U"] & M \arrow[d, "i_M"] \\ T(\boldsymbol{R})^n \arrow[r] & T(\boldsymbol{R}^n) & T(U) \arrow[l, "T(\theta)"'] \arrow[r, "T(\nu)"] & T(M) \end{tikzcd}$$

Note that the assertion of the smoothness of $i_M|U$ is trivial for $n=0$ since then U is a singleton.

Now if the maps i_M are smooth they furnish a natural transformation

$\mathrm{Id} \xrightarrow{i} T$, where Id denotes the identity functor of $\mathfrak{M}$.

R3. If a product preserving covariant endofunctor T of $\mathfrak{M}$ admits a natural transformation $\mathrm{Id} \xrightarrow{i} T$, then it is unique and is given by the maps described at R2. In fact, given such i we look at the following diagram, with $x \in M$:

$$\begin{tikzcd} T(\{x\}) \arrow[r] & T(M) \\ \{x\} \arrow[u, "i_{\{x\}}"] \arrow[r] & M \arrow[u, "i_M"'] \end{tikzcd}$$

where the base arrow is the inclusion and the top arrow T applied to it.

R4. Let $\boldsymbol{R} \times \boldsymbol{R} \overset{m}{\underset{a}{\rightrightarrows}} \boldsymbol{R}$ be the ring operations on $\boldsymbol{R}$, that is, m and a are respectively the multiplication and addition; applying T to m and a and using the canonical diffeomorphism $T(\boldsymbol{R} \times \boldsymbol{R}) \to T(\boldsymbol{R}) \times T(\boldsymbol{R})$ (note that we are always considering T covariant product preserving endofunctor of $\mathfrak{M}$) we obtain two smooth maps $T(\boldsymbol{R}) \times T(\boldsymbol{R}) \overset{\mu}{\underset{\alpha}{\rightrightarrows}} T(\boldsymbol{R})$. Since commutativity and associativity of m and a can be expressed by means of commutative diagrams these are also properties of μ and α as a consequence of functoriality; by a similar argument we have that μ is distributive with respect to α. Now the question of neutral elements for both μ and α can be treated as follows. We will exemplify only with α since it is almost the same with μ. Let $\boldsymbol{R} \xrightarrow{Z} \boldsymbol{R}$ be the constant map whose value is zero; we have $a \circ (\mathrm{id}_{\boldsymbol{R}}, Z) = \mathrm{id}_{\boldsymbol{R}} = a \circ (Z, \mathrm{id}_{\boldsymbol{R}})$ and so $\alpha \circ (\mathrm{id}_{T(\boldsymbol{R})}, T(Z)) = \mathrm{id}_{T(\boldsymbol{R})} = \alpha \circ (T(Z), \mathrm{id}_{T(\boldsymbol{R})})$ and as was observed at R1 the map $T(Z)$ is constant and has as its unique value $i_{\boldsymbol{R}}(0)$ which in that manner is the neutral element of α. In this way we have a ring with unity ($i_{\boldsymbol{R}}(1)$) structure on $T(\boldsymbol{R})$ given by α and μ.

Observe now that "set-theoretical naturality" of the maps i_M remarked at R2 reduces the question about the smoothness of $i_{\boldsymbol{R}}$ to that of continuity since we have now $i_{\boldsymbol{R}}$ as a ring morphism and it is sufficient to consider the (abelian) Lie group structure of $\boldsymbol{R}$ and $T(\boldsymbol{R})$. The author has strong reasons to belief that it is the case that $i_{\boldsymbol{R}}$ is continuous.

Now since $\boldsymbol{R}$ is a field we find that $i_{\boldsymbol{R}}$ is injective or constant; in the last case $i_{\boldsymbol{R}}(0) = i_{\boldsymbol{R}}(1)$ and these being the neutral elements of $T(\boldsymbol{R})$ we have that $T(\boldsymbol{R})$ is trivial: it reduces to $\{i_{\boldsymbol{R}}(0)\}$. This together with an examination of the last diagram of R2 shows that i_M is locally constant and so constant in the connected components of M. Since we are primarily interested in prolongations of manifolds we will work in the sequel only with these T for which i_R is injective and continuous. This will have as a consequence that i_M is always an embedding (see the end of Section 3).

EXAMPLE. With respect to the injectivity of i_M we can look at the following situation: let C be the connected component functor which associates to any manifold M the discrete manifold $C(M)$ —set of connected components of M— and for any map $M \xrightarrow{\varphi} N$ the map $C(M) \xrightarrow{C(\varphi)} C(N)$ sending $S \in C(M)$ to the element of $C(N)$ which contains $\varphi(S)$. In this case the map $M \xrightarrow{i_M} C(M)$ associates to any $x \in M$ the connected component of x in M, and obviously is not injective, unless M is discrete.

2.1. THEOREM. *Let T be a covariant product preserving endofunctor of $\mathfrak{M}$ such that $\boldsymbol{R} \xrightarrow{i_R} T(\boldsymbol{R})$ is continuous and injective. Then the $\boldsymbol{R}$-algebra structure given by i_R and the operations μ and α (described at* R4) *splits as an $\boldsymbol{R}$-algebra into a finite sum of local algebras, and these components are uniquely determined.*

Proof. $T(\boldsymbol{R})$ is a finite dimensional $\boldsymbol{R}$-algebra by Lemma 1, Appendix; to get the conclusion by applying Lemma 3, it is necessary to show that $1 + x^2$ is invertible in $T(\boldsymbol{R})$ for any $x \in T(\boldsymbol{R})$. Now the consideration of the map $\boldsymbol{R} \to \boldsymbol{R}$, $t \mapsto 1 + t^2$, which is smooth and multiplicatively invertible, gives the desired result, by the definitions of the ring operations on $T(\boldsymbol{R})$ and functoriality.

§ 3. Generalized Weil's functors and generalized prolongation functors

3.1. DEFINITION. $\boldsymbol{SL}$ will denote the category that has as objects the $\boldsymbol{R}$-algebras which splits as $\boldsymbol{R}$-algebras in a finite sum of local algebras and as morphisms the $\boldsymbol{R}$-algebra morphisms.

See Lemma 3 in the Appendix another characterization of the objects of $\boldsymbol{SL}$.

3.2. DEFINITION. $\boldsymbol{GP}$ will denote the category that has as objects the product preserving covariant endofunctors $\mathfrak{M} \xrightarrow{T} \mathfrak{M}$ for which $\boldsymbol{R} \xrightarrow{i_R}$

$T(\boldsymbol{R})$ is injective and continuous (see R2, 3, 4 of §1), and as morphisms $T \xrightarrow{\Lambda} S$ the natural transformations.

Let A be an object of $\boldsymbol{SL}$ and $A_1 \oplus \cdots \oplus A_s$ an splitting of it as a finite sum of local algebras. There is an obvious bijection between $\hom_{\boldsymbol{R}\text{-alg}}(C^\infty(M), A)$ and $\overline{T}_{A_1}(M) \times \cdots \times \overline{T}_{A_s}(M)$, for any manifold M, and by transport we have a canonical smooth manifold structure on $\hom_{\boldsymbol{R}\text{-alg}}(C^\infty(M), A)$, which by Lemma 3, Appendix, depends only on A; let's still use the notation $\overline{T}_A(M)$ for this manifold. We will call $\overline{T}_A$ a generalized Weil's functor.

Now it is clear that for any smooth map $M \xrightarrow{\varphi} N$ we can define, like in the case of Weil's functors, a smooth map $\overline{T}_A M \xrightarrow{\overline{T}_A(\varphi)} \overline{T}_A(N)$ in such a way that $M \rightsquigarrow \overline{T}_A(M)$ $\varphi \rightsquigarrow \overline{T}_A(\varphi)$ is a covariant endofunctor $\overline{T}_A$ of $\mathfrak{M}$ and that it is product preserving. It is also clear how to define natural transformations $\overline{T}_A \xrightarrow{\overline{T}(\lambda)} \overline{T}_B$ for any morphism $A \xrightarrow{\lambda} B$ of $\boldsymbol{SL}$ in such a way that $\overline{T}(\lambda \circ \mu) = \overline{T}(\lambda) \circ \overline{T}(\mu)$ if $C \xrightarrow{\mu} A$ is another morphism of $\boldsymbol{SL}$; in particular we have $\mathrm{Id} \to \overline{T}_A$ corresponding to $\boldsymbol{R} \to A$ for any A—it is clear that $\boldsymbol{R} \xrightarrow{i_{\boldsymbol{R}}} \overline{T}_A(\boldsymbol{R})$ is injective and continuous.

3.3. Proposition. *$\overline{T}$: $\boldsymbol{SL} \to \boldsymbol{GP}$ given by $A \rightsquigarrow \overline{T}_A$ and $(A \xrightarrow{\lambda} B) \rightsquigarrow \overline{T}_A \xrightarrow{\overline{T}(\lambda)} \overline{T}_B$ is a covariant functor.*

Proof. Immediate.

On the basis of 2.1 we know that any object of $\boldsymbol{GP}$ gives origin to an object $T(\boldsymbol{R})$ of $\boldsymbol{SL}$. As a consequence of the naturality we see easily that $T(\boldsymbol{R}) \xrightarrow{\Lambda_{\boldsymbol{R}}} S(\boldsymbol{R})$ is a morphism of $\boldsymbol{SL}$ if $T \xrightarrow{\Lambda} S$ is a morphism of $\boldsymbol{GL}$ (of course together the observation that if $\mathrm{Id} \xrightarrow{i} T$ and $\mathrm{Id} \xrightarrow{j} S$ are the unique morphisms from Id to T and S respectively (see R2, 3, of §1) then $\Lambda \circ i = j$ since $\mathrm{Id} \xrightarrow{\Lambda \circ i} S$ is also a morphism of $\boldsymbol{GP}$).

3.4. Proposition. *The correspondences $T \rightsquigarrow T(\boldsymbol{R})$, $\Lambda \rightsquigarrow \Lambda_{\boldsymbol{R}}$ define a covariant functor $\mathcal{A}$: $\boldsymbol{GP} \to \boldsymbol{SL}$.*

We have now two endofunctors $\mathcal{A} \circ \overline{T}$ of $\boldsymbol{SL}$, and $\overline{T} \circ \mathcal{A}$ of $\boldsymbol{GP}$. It is immediate to define, like in [4], a natural isomorphism $\mathcal{A} \circ \overline{T} \xrightarrow{\omega} 1_{\boldsymbol{SL}}$ by $\omega_A(\eta) = \eta(\mathrm{id}_{\boldsymbol{R}})$ for any object A of $\boldsymbol{SL}$ and any η in $\mathcal{A}(\overline{T}_A) = \overline{T}_A(\boldsymbol{R})$.

Also for any object T of $\boldsymbol{GP}$ and for any manifold M by the definition of the $\boldsymbol{R}$-algebra structure on $\mathcal{A}(T)$ we see that the map $C^\infty(M) \to \mathcal{A}(T)$,

$f \mapsto T(f)(z)$ is an element of $T_{A(T)}(M)$ for any $z \in T(M)$, and thus we have a map $T(M) \xrightarrow{(\Xi_T)_M} \overline{T}_{A(T)}(M)$; the verification that for any T we have a morphism $T \xrightarrow{\Xi_T} \overline{T}_{A(T)}$ given by these maps is a straight forward one, as is the verification that $1_{GL} \xrightarrow{\Xi} \overline{T} \circ A$ given by these Ξ_T is a natural transformation.

To see that Ξ is not an isomorphism it is sufficient to look at the functors $T \times C$ of $\boldsymbol{GP}$, where T is an object of $\boldsymbol{GP}$ and C is the functor of the example after R4, § 1.

Although Ξ_T in general is not an isomorphism there is a simple case in which it is and has a nice consequence. In fact we can see directly from the definition of Ξ_T and of ω_A that $\Xi_{\overline{T}_A}$ is just $\overline{T}(\omega_A^{-1}) = \overline{T}(\omega_A)^{-1}$ and so $\Xi_{\overline{T}_A}$: $\overline{T}_A \to \overline{T}_{A(\overline{T}_A)}$ is an isomorphism and then:

3.5. PROPOSITION. *Every natural transformation $\overline{T}_A \xrightarrow{\Lambda} \overline{T}_B$ is determined by the commutative diagram*:

$$\begin{tikzcd} \overline{T}_A \arrow[r, "\Lambda"] \arrow[d, "\Xi_{\overline{T}_A}"'] & \overline{T}_B \arrow[d, "\Xi_{\overline{T}_B}"] \\ \overline{T}_{A(\overline{T}_A)} \arrow[r, "\overline{T}(\Lambda_R)"] & \overline{T}_{A(\overline{T}_B)} \end{tikzcd}$$

that is any such Λ is given by $\overline{T}(\omega_B \circ \Lambda_R \circ \omega_A^{-1})$, in particular there are no other morphisms in $\boldsymbol{GP}$, $\overline{T}_A \xrightarrow{\Lambda} \overline{T}_B$ besides those given by the functor $\overline{T}$; Λ is an isomorphism if and only if Λ_R is so.

Proof. Immediate from what has been said above.

3.6. PROPOSITION. *For any object T of $\boldsymbol{GP}$ there exist canonical bijections between the following sets*:

(i) *the morphisms of $\boldsymbol{GP}$ from T to* Id;

(ii) *the morphisms of $\boldsymbol{SP}$ from $A(T)$ to $\boldsymbol{R}$*;

(iii) *the maximal ideals of $A(T)$.*

Moreover if $A_1 \oplus \cdots \oplus A_s$ is a splitting of $A(T)$ into a sum of local algebras there are exactly s morphisms $A(T) \xrightarrow{p_i} \boldsymbol{R}$, $1 \leq i \leq s$, determined by $p_i(e_j) = \delta_{ij}$, where e_i is the unity of A_i.

Proof. (i) $\leftrightarrow$ (ii). Given $T \xrightarrow{\pi}$ Id let $A(\pi) = \pi_R$ associated to it; if $A(T) \xrightarrow{p} \boldsymbol{R}$ is given we take $\pi = \tau \circ \overline{T}(p) \circ \Xi_T$ where $\overline{T}_R \xrightarrow{\tau}$ Id is the canonical isomorphism. It is esay to see that these are inverses one of the other.

(ii) $\leftrightarrow$ (iii). Given $A(T) \xrightarrow{p} \boldsymbol{R}$ we have that $\operatorname{Ker}(p)$ is a maximal ideal; given a maximal ideal I of $A(T)$ we know that $A(T) = \boldsymbol{R} \oplus I$, by Lemma 2, Appendix, since as was shown at the proof of 2.1 we have $1 + a^2$ invertible for any $a \in A(T)$; take then for p the composition $A(T) \to A(T)/I \cong \boldsymbol{R}$.

Now, since for any morphism $A(T) \xrightarrow{p} \boldsymbol{R}$ we have $p(e_i)^2 = p(e_i)$ and $p(e_i)p(e_j) = 0$ if $i \neq j$, it is obvious that there is one and only one $i \in \{1, \cdots, s\}$ such that $p(e_j) = \delta_{ij}$, $1 \leq j \leq s$; since e_i is the unity of A_i and each A_i is a local algebra p is determined by the values $p(e_i)$; on the other side the composition $A \to A_i \to \boldsymbol{R}$ gives a morphism $A \xrightarrow{p_i} \boldsymbol{R}$ such that $p_i(e_j) = \delta_{ij}$, $1 \leq j \leq s$. ■

Let now T be an object of $\boldsymbol{GP}$, $\mathrm{Id} \xrightarrow{i} T$ the unique morphism from Id to T and $T \xrightarrow{\pi} \mathrm{Id}$ any one of the morphisms described in the last proposition. Since we have that both $\pi \circ i$ and 1 are morphisms $\mathrm{Id} \to \mathrm{Id}$, by R2, Section 1, we have $\pi \circ i = 1$, that is, for any manifold M the map i_M is a section of π_M.

3.7. Proposition. *For any object T of $\boldsymbol{GP}$ we have that the maps i_M given by the unique morphism $\mathrm{Id} \xrightarrow{i} T$ are embeddings, being sections of π_M for any morphism $T \xrightarrow{\pi} \mathrm{Id}$ of $\boldsymbol{GP}$.*

This justifies to call the objects of $\boldsymbol{GP}$ generalized prolongation functors.

There is also the following remark:

If an object T of $\boldsymbol{GP}$ is isomorphic to some $\overline{T}_A$, with A an object of $\boldsymbol{SL}$ through Σ then Ξ_T is an isomorphism and $\Sigma = \overline{T}(\sigma) \circ \Xi_T$ where $A(T) \xrightarrow{\sigma} A$ is an isomorphism of $\boldsymbol{SL}$.

In fact applying A to Σ gives the isomorphism $A(T) \xrightarrow{A(\Sigma)} A(\overline{T}_A)$. On the other side $A(\overline{T}_A) \xrightarrow{\omega_A} A$ is an isomorphism of $\boldsymbol{SL}$, and the commutative diagram below, together $\overline{T}(\omega_A^{-1}) = \Xi_{\overline{T}_A}$ as was remarked earlier, gives the result with $\sigma = \omega_A \circ A(\Sigma)$.

$$\begin{array}{ccc} T & \xrightarrow{\Sigma} & T_A \\ {\scriptstyle \Xi_T}\downarrow & & \downarrow{\scriptstyle \Xi_{\overline{T}_A}} \\ \overline{T}_{A(T)} & \xrightarrow[\overline{T}_{(A(\Sigma))}]{} & \overline{T}_{A(\overline{T}_A)} \end{array}$$

§ 4. A full subcategory of *GP* and an answer to Morimoto's conjecture

4.1. DEFINITION. ***L*** will denote the full subcategory of ***SL*** that has as objects the local algebras.

4.2. DEFINITION. ***IP*** will denote the full subcategory of ***GP*** that has as objects these T of ***GP*** satisfying: ***AL*** (Algebraic locality), $A(T)$ is a local algebra, ***LD*** (Local determination). For any manifold M and open submanifold U of M the diagram below is a pull-back:

$$\begin{array}{ccc} T(U) & \xrightarrow{T(\nu)} & T(M) \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ U & \xrightarrow{\nu} & M \end{array}$$

where $U \xrightarrow{\nu} M$ is the inclusion and $T \xrightarrow{\pi} \mathrm{Id}$ is the unique morphism from T to Id (see 3.6).

Remark. Since for $W = \pi_M^{-1}(U)$ we have a pull back diagram

$$\begin{array}{ccc} W & \longrightarrow & T(M) \\ {\scriptstyle \pi_M | W}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ U & \xrightarrow{\nu} & M \end{array}$$

where the top arrow is the inclusion, the requirement ***LD*** above is equivalent to: $T(\nu)$ is a diffeomorphism onto $\pi_M^{-1}(U)$.

It is trivial that the restrictions of A to ***IP*** and $\overline{T}$ to ***L*** gives respectively functors ***IP*** $\to$ ***L*** and ***L*** $\to$ ***IP***, which will be still denoted by A and $\overline{T}$.

4.3. THEOREM. *($\overline{T}$, A) is an equivalence between the categories **IP** and **L**.*

Proof. Since, as we have seen after 3.4, $A \circ \overline{T} \xrightarrow{\omega} 1_{SL}$ is a natural isomorphism, with the old A and $\overline{T}$, restricting to ***L*** gives the first part; also by restricting the natural transformation $1_{GL} \xrightarrow{\Xi} \overline{T} \circ A$ (with the old A and $\overline{T}$) to ***IP*** and still denoting it by Ξ, we will show that it is an isomorphism. Since by 3.6 we have unique natural transformations $T \xrightarrow{\pi} \mathrm{Id}$ and $\overline{T}_{A(T)} \xrightarrow{\chi} \mathrm{Id}$ we have $\chi \cdot \Xi_T = \pi$; then to show that for any T in ***IP*** and any manifold M we have a diffeomorphism given by $(\Xi_T)_M$ it is sufficient to show $(\Xi_T)_M | \pi_M^{-1}(U)$ is a diffeomorphism onto $\chi_M^{-1}(U)$, where

U ranges through an open covering of M.

Since $T(\nu)$ is a diffeomorphism onto $\pi_M^{-1}(U)$ and $T_{A(T)}(\nu)$ a diffeomorphism onto $\chi_M^{-1}(U)$, where $U \xrightarrow{\nu} M$ is the inclusion, the diagram below shows that it is sufficient to verify that $(\Xi_T)_U$ is a diffeomorphism when U ranges through an open cover of M.

$$\begin{tikzcd} T(U) \arrow[r, "T(\nu)"] \arrow[d, "(\Xi_T)_U"'] & T(M) \arrow[d, "(\Xi_T)_M"] \\ \overline{T}_{A(T)}(U) \arrow[r, "\overline{T}_{A(T)}(\nu)"] & \overline{T}_{A(T)}(M) \end{tikzcd}$$

Consider the open cover of M consisting of those U for which there exists a chart $U \xrightarrow{\theta} \boldsymbol{R}^n$ onto $\boldsymbol{R}^n$. Then we have the commutative diagram:

$$\begin{tikzcd} T(U) \arrow[r, "T(\theta)"] \arrow[d, "(\Xi_T)_U"'] & T(\boldsymbol{R}^n) \arrow[r, "\sim"] \arrow[d, "(\Xi_T)_{\boldsymbol{R}^n}"] & T(\boldsymbol{R})^n \arrow[d, "(\Xi_T)_{\boldsymbol{R}}^n"] \\ \overline{T}_{A(T)}(U) \arrow[r, "\overline{T}_{A(T)}(\theta)"'] & \overline{T}_{A(T)}(\boldsymbol{R}^n) \arrow[r, "\sim"'] & \overline{T}_{A(T)}(\boldsymbol{R})^n \end{tikzcd}$$

It is trivial to verify that the map $(\Xi_T)_{\boldsymbol{R}}$ is the inverse of $\omega_{A(T)}$; together with the fact that horizontal arrows are diffeomorphisms the conclusion follows. ∎

The theorem above gives an answer to Morimoto's conjecture. It is to be noticed that from the seven conditions stated in [3] we used only (1) and local determination ***LD*** of 4.2, which is very natural; besides this the injectivity of $\boldsymbol{R} \xrightarrow{i_{\boldsymbol{R}}} \overline{T}(\boldsymbol{R})$ is also assumed (and is justified by commentaries in the end of R 4, § 1) giving as a consequence the fact that $M \xrightarrow{i_M} T(M)$ are embeddings (3.7). It is also nice that the natural transformation given by the maps $T(M) \xrightarrow{\pi_M} M$ is an information contained in T; besides this by the theorem above they are locally trivial fibrations and also they are the unique natural family of maps $T(M) \xrightarrow{\pi_M} M$.

We give below another description of the objects in ***IP*** without making the algebraic hypothesis *AL* of 4.2.

4.4. Proposition. *An object T of **GP** is an object of **IP** iff there exists a morphism $T \xrightarrow{\pi} \mathrm{Id}$ of **GP** such that condition **LD** of* 4.2 *is satisfied for π.*

Proof. We show the non trivial half. Let $T \xrightarrow{\pi} \mathrm{Id}$ satisfy ***LD*** of 4.2. If we show that $\mathrm{Ker}(\pi_{\boldsymbol{R}}) = \pi_{\boldsymbol{R}}^{-1}(0)$ is the unique maximal ideal of $T(\boldsymbol{R})$ we are done. Consider the inclusion $\boldsymbol{R}_* \xrightarrow{\nu} \boldsymbol{R}$ of the set of non-zero (inveritible) real numbers and the diffeomorphism $T(\nu)$ of $T(\boldsymbol{R}_*)$ onto $\pi_{\boldsymbol{R}}^{-1}(\boldsymbol{R}*) = T(\boldsymbol{R})\backslash \pi_{\boldsymbol{R}}^{-1}(0)$; the invertibility of the elements of $T(\boldsymbol{R})\backslash\pi_{\boldsymbol{R}}^{-1}(0)$ is a consequence of the equation $\overline{m}\circ(\mathrm{id}_{\boldsymbol{R}_*,\sigma}) = u = \overline{m}\circ(\sigma, \mathrm{id}_{\boldsymbol{R}_*})$, where $\boldsymbol{R}_* \xrightarrow{u} \boldsymbol{R}_*$ is the constant map $t\mapsto 1$, $\boldsymbol{R}_* \xrightarrow{\sigma} \boldsymbol{R}_*$ is the map $t\mapsto t^{-1}$ and $\boldsymbol{R}_*\times\boldsymbol{R}_* \xrightarrow{\overline{m}} \boldsymbol{R}_*$, $(t, s) \mapsto t.s$, together with the fact that $\nu\circ\overline{m} = m\circ(\nu\times\nu)$ where $\boldsymbol{R}\times\boldsymbol{R}\xrightarrow{m}\boldsymbol{R}$ is the multiplication of $\boldsymbol{R}$. ∎

Remark. We also note that the requirement on T, to be an element of $\boldsymbol{IP}$, can be weakened to the condition that T is covariant product preserving, together with the existence of π satisfying $\boldsymbol{LD}$ of 4.2 and the continuity of $\boldsymbol{R}\xrightarrow{i_R} T(\boldsymbol{R})$ since the injectivity of i_R is a consequence of $\pi_R\circ i_R = \mathrm{id}_R$, this last equality being a consequence of the commutativity of the diagram below:

$$\begin{tikzcd}
\boldsymbol{R} \arrow[r, "i_R"] & T(\boldsymbol{R}) \arrow[r, "\pi_R"] & \boldsymbol{R} \\
\{t\} \arrow[u] \arrow[r, "\{s\}"'] & T(\{t\}) \arrow[u] \arrow[r, "\pi_{\{t\}}"'] & \{t\} \arrow[u]
\end{tikzcd}$$ ∎

4.5. Proposition. *If T is an object of $\boldsymbol{IP}$ the natural isomorphism $T\xrightarrow{\Xi_T}\overline{T}_{A(T)}$ is unique up to natural automorphisms of $\overline{T}_{A(T)}$ of the type $\overline{T}(\sigma)$ where $A(T)\xrightarrow{\sigma}A(T)$ is an automorphism.*

Proof. If $T\xrightarrow{\Sigma}\overline{T}_{A(T)}$ is another isomorphism look at $\Lambda = \Sigma\circ(\Xi_T)^{-1}$ and use 3.5. ∎

We will make now a general discussion which applies to all objects of $\boldsymbol{GP}$.

Let E be a finite dimensional real vector space with operation $\boldsymbol{R}\times E\to E$ (scalar multiplication) and $E\times E\to E$ (addition).

Considering E with its structure of manifold the maps above are smooth and we can obviously obtain from them $A(T)\times T(E)\to T(E)$ and $T(E)\times T(E)\to T(E)$ which furnish a structure of $A(T)$-module, as can be shown with arguments in the line of R4, §1. Moreover it is very simple to show that if $\{e_i | i\in\Gamma\}$ and $\{\xi_i | i\in\Gamma\}$ are dual basis of E and E^* then the map $T(E)\to A(T)\otimes E$ given by $z\mapsto\sum_{i\in\Gamma} T(\xi_i)(z)\otimes e_i$ doesn't depend on the basis and is an $A(T)$-linear isomorphism. If in addition we are given a structure of $\boldsymbol{R}$-algebra by a map $E \times E \xrightarrow{m} E$, $T(E)$ will be equipped with a structure of $A(T)$-algebra, where multiplication is deduced from $T(m)$; further the above isomorphism of $A(T)$-module is also a $A(T)$-algebra isomorphism.

Consider now another object S of $\boldsymbol{GP}$ and let $S \circ T$ be the composed functor. First of all it is obviously a covariant product preserving functor of $\mathfrak{M}$; on the other side if $\mathrm{Id} \xrightarrow{i} T$ and $\mathrm{Id} \xrightarrow{j} S$ are the unique morphisms from Id to T and S then the maps $M \xrightarrow{S(i_M) \circ j_M} S(T(M))$ and also $M \xrightarrow{j_{T(M)} \circ i_M} S(T(M))$ furnish natural transformations $\mathrm{Id} \xrightarrow{k} S \circ T$ which coincide by R2, §2; now take any $T \xrightarrow{\pi} \mathrm{Id}$, $S \xrightarrow{\chi} \mathrm{Id}$, by 3.7 we have $\pi_M \circ \chi_{T(M)} \circ k_M = \mathrm{id}_M$ and so k_M is injective. Then $S \circ T$ is an object of $\boldsymbol{GP}$ and since $A(S \circ T) = S(T(\boldsymbol{R}))$ by what has been observed above we have a canonical isomorphism $A(S \circ T) \rightleftarrows A(S) \times A(T)$.

4.6. PROPOSITION. *Let T be an object of $\boldsymbol{GP}$ and E a finite dimensional real vector space. Then the $A(T)$-module structure on $T(E)$ obtained by means of T from the operations on E is canonically isomorphic to $A(T) \otimes E$. If moreover E has an $\boldsymbol{R}$-algebra structure the $A(T)$-algebra structure obtained in a similar way is also isomorphic to $A(T) \otimes E$ by the above map $T(E) \to A(T) \otimes E$; in particular we have a canonical isomorphism $A(S \circ T) \rightleftarrows A(S) \otimes A(T)$.*

Suppose in particular that S and T are objects of $\boldsymbol{IP}$. Then $A(S \circ T)$ is a local algebra isomorphic to $A(S) \otimes A(T)$. To see that $S \circ T$ is an object of $\boldsymbol{IP}$ it remains to check condition $\boldsymbol{LD}$ of 4.2. Let M be a manifold, $U \xrightarrow{\nu} M$ then inclusion of an open submanifold U of M, $T \xrightarrow{\pi} \mathrm{Id}$, $S \xrightarrow{\chi} \mathrm{Id}$ (the unique) natural transformations from T and S to Id, $W = \pi_M^{-1}(U)$, $W \xrightarrow{\mu} T(M)$ the inclusion and $T(U) \xrightarrow{\bar{\nu}} W$ the diffeomorphism given by $T(\nu)$. Since $\mu \circ \bar{\nu} = T(\nu)$ and $S(\mu)$ is a diffeomorphism onto $\chi_{T(M)}^{-1}(W) \chi_{T(M)}^{-1}(\pi_M^{-1}(U)) = (\pi_M \circ \chi_{T(M)})(U)$ the commutative diagram below together with the fact that the map $\pi_M \circ \chi_{T(M)}$ gives a natural transformation $S \circ T \to \mathrm{Id}$ lead to the desired conclusion:

$$\begin{tikzcd}
S(T(U)) \arrow[r, "S(T(\nu))"] \arrow[d, "\chi_{T(U)}"'] & S(T(M)) \arrow[d, "\chi_{M(U)}"] \\
T(U) \arrow[r, "T(\nu)"] \arrow[d, "\pi_U"'] & T(M) \arrow[d, "\pi_M"] \\
U \arrow[r, "\nu"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
S(T(U)) \arrow[r, "S(\bar{\nu})"] \arrow[d, "\chi_{T(U)}"'] & S(W) \arrow[r, "S(\nu)"] \arrow[d, "\chi_W"] & S(T(M)) \arrow[d, "\chi_{T(M)}"] \\
T(U) \arrow[r, "\bar{\nu}"] & W \arrow[r, "\nu"] & T(M)
\end{tikzcd}$$

We must notice that the map $S(\pi_M) \circ \chi_M$ also furnishes a natural transformation $S \circ T \to \mathrm{Id}$ and so coincides with $\pi_M \circ \chi_{T(M)}$.

4.7. Corollary. (Transitivity of Prolongations [4]). *If S and T are objects of **IP** so is $S \circ T$ and there is a canonical natural isomorphism $S \circ T \rightrightarrows \overline{T}_{A(S) \otimes A(T)}$.*

Proof. As was observed $A(S \circ T)$ is isomorphic to $A(S) \otimes A(T)$ by a canonically given ζ; by 4.3 $\Xi_{S \circ T}$ is a natural isomorphism, since $S \circ T$ is an object of ***IP***, so it is sufficient to consider $\overline{T}(\zeta) \circ \Xi_{S \circ T}$. ■

§ 5. Pseudo infinitesimal prolongations and a projection functor

In this final paragraph we will look at a full subcategory ***PIP*** of ***GP*** which includes ***IP*** and has as endofunctor R with the property $R \circ R = R$ and having ***IP*** as its image. Besides this there is a natural transformation $1_{PIP} \xrightarrow{r} R$ such that the diagram below commutes:

$$\begin{tikzcd}
 & 1_{PIP} \arrow[d, "r"] \\
R \arrow[ur] \arrow[r, "1"] & R
\end{tikzcd}$$

where the oblique arrow is a natural transformation given by inclusions. In concrete terms, for any object T of ***PIP*** and any manifold M, $R(T) = {}^R T$ is an object of ***IP***, ${}^R T(M)$ is a closed regular submanifold of $T(M)$ and ${}^R T(M) \xrightarrow{(r_T)_M} {}^R T(M)$ is a retraction onto ${}^R T(M)$. Details can be seen in the discussion after the following definition.

5.1. Definition. ***PIP*** will denote the full subcategory of ***GP*** that has as objects those T of ***GP*** for which $A(T)$ is a local algebra.

Let T be an object of ***PIP***, M a manifold and U an open submanifold of M such that a chart $U \xrightarrow{\theta} \boldsymbol{R}^n$ onto $\boldsymbol{R}^n$ exists; we have the commutative diagram below:

$$\begin{tikzcd} T(U) \arrow[r, "T(\theta)"] \arrow[d, "(\Xi_T)_U"'] & T(\boldsymbol{R}^m) \arrow[r, "\sim"] \arrow[d, "(\Xi_T)_{\boldsymbol{R}^m}"] & T(\boldsymbol{R})^m \arrow[d, "(\Xi_T)_{\boldsymbol{R}}^m"] \\ \overline{T}_{A(T)}(U) \arrow[r, "\overline{T}_{A(T)}(\theta)"'] & \overline{T}_{A(T)}(\boldsymbol{R}^m) \arrow[r, "\sim"] & T_{A(T)}(\boldsymbol{R})^m \end{tikzcd}$$

and horizontal arrows are diffeomorphisms and also $(\Xi_T)_{\boldsymbol{R}}$, which is the inverse of $\omega_{A(T)}$, so that the vertical arrows are diffeomorphisms, in particular $(\Xi_T)_U$. We have also another commutative diagram:

$$\begin{tikzcd} T(U) \arrow[r, "T(\nu)"] \arrow[d, "(\Xi_T)_U"'] & T(M) \arrow[d, "(\Xi_T)_M"] \\ \overline{T}_{A(T)}(U) \arrow[r, "\overline{T}_{A(T)}(\nu)"'] & \overline{T}_{A(T)}(M) \end{tikzcd}$$

where $U \xrightarrow{\nu} M$ is the inclusion and the base arrow is a diffeomorphism onto $\chi_M^{-1}(U)$, where $\overline{T}_{A(T)} \xrightarrow{\chi} \mathrm{Id}$ is the unique morphism from $\overline{T}_{A(T)}$ to Id, since $\overline{T}_{A(T)}$ is an object of $\boldsymbol{IP}$. From this it follows, by means of the above diagram, that there exists a smooth map $\pi_M^{-1}(U) \xrightarrow{\rho_U^M} T(U)$, where $T \xrightarrow{\pi}$ Id is the unique morphism from T to Id, in such a way that $\rho_U^M \circ T(\nu)$ is the identity of $T(U)$, so that $T(\nu)$ is an embedding and $r_U^M = T(\nu) \circ \rho_U^M$: $\pi_M^{-1}(U) \to \pi_M^{-1}(U)$ is a retraction whose image is that of $T(\nu)$.

Consider now an open submanifold W of U for which there exists a chart $W \to \boldsymbol{R}^m$ onto $\boldsymbol{R}^m$; we have a similar commutative diagram:

$$\begin{tikzcd} T(W) \arrow[r, "T(\sigma)"] \arrow[d, "(\Xi_T)_W"'] & T(U) \arrow[d, "(\Xi_T)_U"] \\ \overline{T}_{A(T)}(W) \arrow[r, "\overline{T}_{A(T)}(\sigma)"'] & \overline{T}_{A(T)}(U) \end{tikzcd}$$

where $W \xrightarrow{\sigma} U$ is the inclusion. But in this case the two vertical arrows are diffeomorphisms and so the image of $T(\sigma)$ is exactly $\pi_U^{-1}(W)$, that is $T(\sigma)$ and ρ_W^U are inverses of each other. Examining the last two diagrams, taking account of the observation made just now and the definition of the maps ρ's and r's we see that $T(\sigma) \circ \rho_W^M = \rho_U^M | \pi_M^{-1}(W)$ and from this $r_U^M | \pi_M^{-1}(W) = r_W^M$. Finally if U and V are open submanifolds of M for which there exists charts $U \to \boldsymbol{R}^m$, $V \to \boldsymbol{R}^m$ onto $\boldsymbol{R}^m$ and $U \cap V$ is non empty, let x be a point of $U \cap V$ and W an open submanifold such that $x \in W \subset U \cap V$ and there exists a chart $W \to \boldsymbol{R}^m$ onto $\boldsymbol{R}^m$. By the argument given above we have $r_U^M|\pi_M^{-1}(W) = r_W^M = r_V^M|\pi_M^{-1}(V)$, and so this being the case for any W we have $r_U^M|\pi_M^{-1}(U \cap V) = r_V^M|\pi_M^{-1}(U \cap V)$. This gives a well defined smooth map $T(M) \xrightarrow{r_M} T(M)$, since $\pi_M^{-1}(U)$ cover $T(M)$ when U ranges through the open submanifolds considered above and the maps $\pi_M^{-1}(U) \xrightarrow{r_U^M} \pi_M^{-1}(U)$ are compatible; we have $r_M|\pi_M^{-1}(U) = r_U^M$ and from $r_U^M \circ r_U^M = r_U^M$ we conclude $r_M \circ r_M = r_M$ and also that the image of r_M is a regular submanifold of $T(M)$ since $(\operatorname{Im} r_M) \cap \pi_M^{-1}(U) = \operatorname{Im}(r_U^M) = \operatorname{Im}(T(\nu))$ and $T(\nu)$ is an embedding; also $\operatorname{Im}(r_M)$ is closed since it is the fixed point set of r_M.

Through up to now we have worked with a single T, in what follows all objects of ***GP*** will enter the discussion so we will denote the maps r_M above by $(r_T)_M$ and r_U^M by $(r_T)_U^M$. Let $T \xrightarrow{\Lambda} S$ be a morphism of ***PIP***. We have the commutative diagram

$$\begin{tikzcd}
\bar{T}_{A(T)}(M) \arrow[rrr, "\bar{T}(\Lambda_R)_M"] & & & \bar{T}_{A(S)}(M) \\
& T(M) \arrow[ul, "(\Xi_T)_M"'] \arrow[r, "\Lambda_M"] & S(M) \arrow[ur, "(\Xi_S)_M"] & \\
& T(U) \arrow[u, "T(\nu)"] \arrow[r, "\Lambda_U"] \arrow[dl, "(\Xi_T)_U"] & S(U) \arrow[u, "S(\nu)"'] \arrow[dr, "(\Xi_S)_U"'] & \\
\bar{T}_{A(T)}(U) \arrow[uuu, "\bar{T}_{A(T)}(\nu)"] \arrow[rrr, "\bar{T}(\Lambda_R)_U"'] & & & \bar{T}_{A(S)}(U) \arrow[uuu, "\bar{T}_{A(S)}(\nu)"']
\end{tikzcd}$$

which gives $(r_S)_U^M \circ (\Lambda_M|\pi_M^{-1}(U)) = \Lambda_M \circ (r_T)_U^M$ for any U, so $(r_S) \circ \Lambda_M = \Lambda_M \circ (r_T)_M$ for any M, in other words $r_S \circ \Lambda = \Lambda \circ r_T$.

Let now $M \xrightarrow{\varphi} N$ be a map between manifolds. We can cover M by open submanifolds U such that there exists for any such U an open submanifold V of N and $\varphi(U) \subseteq V$ as well there are charts $U \to \boldsymbol{R}^m$, $V \to \boldsymbol{R}^n$, onto $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ respectively; we will use the notation ψ to the map $U \xrightarrow{\varphi|U} V$ in what follows.

The following commutative diagram and the definition of the maps $(r_T)_U^M$ and $(r_T)_V^N$ show that $(r_T)_V^N \circ (T(\varphi)\,|\,\pi_M^{-1}(U)) = T(\varphi) \circ (r_T)_U^M$, and since this is true for any such pair we have $(r_T)_V^N \circ (T(\varphi)) = T(\varphi) \circ (r_T)_M$, in particular $T(\varphi)(\operatorname{Im} r_T)_M) \subseteq \operatorname{Im}(r_T)_N$

$$\begin{tikzcd}
\overline{T}_{A(T)}(M) \arrow[rrr, "\overline{T}_{A(T)}(\varphi)"] & & & \overline{T}_{A(T)}(N) \\
& T(M) \arrow[ul, "(\Xi_T)_M"'] \arrow[r, "T(\varphi)"] & T(N) \arrow[ur, "(\Xi_T)_M"] & \\
& T(U) \arrow[u, "T(\nu)"] \arrow[r, "T(\psi)"] \arrow[dl, "(\Xi_T)_U"] & T(V) \arrow[u, "T(\mu)"'] \arrow[dr, "(\Xi_T)_V"'] & \\
\overline{T}_{A(T)}(U) \arrow[uuu, "\overline{T}_{A(T)}(\nu)"] \arrow[rrr, "\overline{T}(\Lambda_R)_U"'] & & & \overline{T}_{A(T)}(V) \arrow[uuu, "\overline{T}_{A(T)}(\mu)"']
\end{tikzcd}$$

where $U \xrightarrow{\nu} M$, $V \xrightarrow{\mu} N$ are inclusions.

Summing up we can say that we have for any object T in ***PIP*** a covariant endofunctor ${}^RT$ of $\mathfrak{M}$ given by $M \to \operatorname{Im}(r_T)_M$ $\varphi \to T(\varphi)|\operatorname{Im}(r_T)_M$.

If P and Q are manifolds and $P \times Q \xrightarrow{p} P$, $P \times Q \xrightarrow{q} Q$ are the projections onto the factor of the product we have a commutative diagram

$$\begin{tikzcd}
T(P \times Q) \arrow[r, "(T(p),\, T(q))"] \arrow[d, "(r_T)_{P\times Q}"'] & T(P) \times T(Q) \arrow[d, "(r_T)_P \times (r_T)_Q"] \\
{}^RT(P \times Q) \arrow[r, "({}^RT(p),\, {}^RT(q))"] & {}^RT(P) \times {}^RT(Q)
\end{tikzcd}$$

so that the fact that the top arrow is a diffeomorphism and so is the arrow which is the restriction of it to the submanifold ${}^RT(P \times Q)$ gives the conclusion that ${}^RT$ is product preserving.

Now it is obvious that $A({}^RT) = A(T)$ and since $\Lambda \circ r_T = r_S \circ \Lambda$ for any morphism $T \xrightarrow{\Lambda} S$ we have in particular $i = r_T \circ i$, where $\mathrm{Id} \xrightarrow{i} T$ is the unique morphism from Id to T (since $r_{\mathrm{Id}} = 1$ obviously), so that the image of i_M is contained in ${}^RT(M)$ and the morphism $\mathrm{Id} \to {}^RT$ is just i; moreover $R \xrightarrow{i_R} {}^RT(R)$ is injective. All this gives that ${}^RT$ is an object of ***PIP*** and also gives the functor $\boldsymbol{PIP} \xrightarrow{R} \boldsymbol{PIP}$, $T \rightsquigarrow {}^RT$, $\Lambda \rightsquigarrow R(\Lambda)$ where $R(\Lambda)_M = \Lambda_M|{}^RT(M)$ for any manifold M.

It is clear that if T is an object of ***IP*** then ${}^RT = T$. Now we see just by the definition of ${}^RT$ that it is isomorphic with $\overline{T}_{A(T)}$ by means of the restriction of Ξ_T to it, and so by this reason ${}^RT$ is an object of ***IP*** and so ${}^R({}^RT) = {}^RT$, which gives $R \circ R = R$; in this way the image of R is ***IP***.

Since $\overline{T}_{A(T)}$ is an object of ***IP*** we have ${}^R\overline{T}_{A(T)} = \overline{T}_{A(T)}$ and $r_{\overline{T}_{A(T)}} = 1$ so that $\Xi_{{}^RT}$ is the restriction of Ξ_T to ${}^RT$ and

$$\begin{tikzcd} T \arrow[r, "\Xi_T"] \arrow[dr, "r_T"'] & \overline{T}_{A(T)} \\ & {}^R T \arrow[u, "\Xi_{RT}"'] \end{tikzcd}$$

commute; here we are considering the maps $(r_T)_M$ as maps $T(M) \to T(M)$ which give a morphism $T \to {}^R T$, still denoted by r_T. Considering r_T as above we have the commutative diagram

$$\begin{tikzcd} & T \arrow[d, "r_T"] \\ {}^R T \arrow[ur] \arrow[r, "1"] & {}^R T \end{tikzcd}$$

where the oblique arrow is given by inclusions. Since for any $T \xrightarrow{A} S$, morphism of ***PIP***, we have $R(A) \circ r_T = r_S \circ R(A)$ we have a natural transformation $1_{PIP} \xrightarrow{r} R$ given by the r_T's.

5.1. PROPOSITION. *There exists a covariant functor* ***PIP*** $\xrightarrow{R}$ ***PIP*** *and a natural transformation* $1_{PIP} \xrightarrow{r} R$ *such that the image of* R *is* ***IP*** *and* $R \circ R = R$, *also for any object* T *of* ***PIP*** *and any manifold* M *we have that* $(r_T)_M$ *is a retraction from* $T(M)$ *to the closed regular submanifold* ${}^R T(M)$ *of* $T(M)$.

From $(\Xi_T)_M \circ (r_T)_M = (\Xi_T)_M$, $(r_T)_M^2 = (r_T)_M$ and the fact that ${}^R T(M)$ is diffeomorphic to $\overline{T}_{A(T)}(M)$ through $(\Xi_T)_M$ for any M, it follows that $(\Xi_T)_M$ is a diffeomorphism iff $(r_T)_M = \mathrm{id}_{T(M)}$ iff ${}^R T(M) = T(M)$ iff $(r_T)_M$ is injective iff $(\Xi_T)_M$ is injective.

A construction of retractions $T(M) \xrightarrow{r_M} T(M)$ works for objects of ***GP*** and connected M with reasonings very similar to those made in this paragraph, the only difference being that in the place of the unique $T \to \mathrm{Id}$ for objects of ***PIP*** we use the morphism $T \xrightarrow{\pi_1, \cdots, \pi_k} \mathrm{Id} \times \cdots \times \mathrm{Id}$ where $\{T \xrightarrow{\pi_i} \mathrm{Id} \mid 1 \leq i \leq k\}$ is the set of morphisms $T \to \mathrm{Id}$ (see 3.6); one needs to remember that for connected M and any k there exists for any $(x_1, \cdots, x_k) \in M^k$ a chart $U \to \boldsymbol{R}^m$ onto $\boldsymbol{R}^m$ such that $(x_1, \cdots, x_k) \in U^k$. The result is:

5.2. PROPOSITION. *For objects* T *of* ***GP*** *and connected* M *there are smooth maps* $T(M) \xrightarrow{(r_T)_M} T(M)$, *natural in* T *and* M, *such that* $(r_T)_M^2 = (r_T)_M$

and the image of $(r_T)_M$ *is a closed regular submanifold of* $T(M)$ *which is diffeomorphic to* $\overline{T}_{A(T)}(M)$ *through* $(\Xi_T)_M$*; moreover* $(\Xi_T)_M \circ (r_T)_M = (\Xi_T)_M$.

It is an easy matter to verify that the functors $\overline{T}_A$, with A in ***SL*** have a local determination property analogous to ***LD*** of 4.2, that is, for any manifold M and open submanifold U of M the diagram below is a pull-back:

$$\begin{array}{ccc} \overline{T}_A(U) & \xrightarrow{\overline{T}_A(\nu)} & \overline{T}_A(M) \\ {\scriptstyle (\pi_1,\cdots,\pi_k)_U}\downarrow & & \downarrow{\scriptstyle (\pi_1,\cdots,\pi_k)_M} \\ U\times\cdots\times U & \xrightarrow{\nu\times\cdots\times\nu} & M\times\cdots\times M \end{array}$$

where $U \xrightarrow{\nu} M$ is the inclusion and $\{\overline{T}_A \xrightarrow{\pi_i} \mathrm{Id} \mid 1 \leq i \leq k\}$ is the set of morphisms $\overline{T}_A \to \mathrm{Id}$ according to 3.7.

As in the discussion of the objects of ***PIP*** we obtain by 5.2 for each object T of ***GP*** a covariant product preserving endofunctor ${}^RT$ of $\mathfrak{M}$, the full subcategory of $\mathfrak{M}$ whose objects are connected manifolds by the same procedure, $M \rightsquigarrow \mathrm{Im}\,(r_T)_M$, $\varphi \mapsto T(\varphi)|\mathrm{Im}\,(r_T)_M$; since $(\Xi_T)_M$ gives a diffeomorphism natural in T and M, by 5.2, of ${}^RT(M)$ onto $\overline{T}_{A(T)}(M)$, we find that ${}^RT$ has also the local determination property described above; further $T \rightsquigarrow {}^RT$, $\Lambda \rightsquigarrow {}^R\Lambda$, where $({}^R\Lambda)_M = \Lambda_M|{}^RT(M)$, M connected, is a covariant functor between obvious categories.

From $(\Xi_T)_M \circ (r_T)_M = (\Xi_T)_M$, $(r_T)^2_M = (r_T)_M$ and the fact that ${}^RT(M)$ is diffeomorphic to $\overline{T}_{A(T)}(M)$, for any connected M, we see that $(\Xi_T)_M$ is a diffeomorphism iff $(r_T)_M = \mathrm{id}_{T(M)}$ iff $(r_T)_M$ is injective iff ${}^RT(M) = T(M)$ iff $(\Xi_T)_M$ is injective.

We observe also that if T satisfies the local determination property cited above we have ${}^RT(M) = T(M)$, for M connected and then $(\Xi_T)_M$ is a diffeomorphism. We don't know what happens for non connected M even if T has the local determination property.

Appendix

Let E be a manifold equipped with smooth operations $\boldsymbol{R} \times E \xrightarrow{s} E$ and $E \times E \xrightarrow{a} E$ that furnishes a real vector space structure. Let $\boldsymbol{D}$ denote the algebra of dual numbers, that is, a 2-dimensional local algebra linearly generated by 1 and d with $d^2 = 0$; then $\overline{T}_{\boldsymbol{D}}$ is (by 4.3) "the tangent bundle," which has an additional structure of vector bundle.

LEMMA 1. *The map* $E \to \overline{T}_D E$, $e \to \overline{T}_D(\gamma_e)(d)$, *where* $\boldsymbol{R} \xrightarrow{\gamma_e} E$, $t \to t.e$, *for any* $e \in E$, *and* $\overline{T}_D(\boldsymbol{R}) = A(\overline{T}_D)$ *had been identified with* $\boldsymbol{D}$ *by means of* ω_D, *is a diffeomorphism onto the fiber of zero in* $\overline{T}_D E$ *and is also linear. In particular* E *has finite dimension over* $\boldsymbol{R}$ *equal to the dimension as a manifold.*

Proof. Trivial consequence of the basic theory of manifolds and Lie groups.

Now we give two algebraic results for the sake of completeness; probably they would be easily deduced from general structure theories. The algebras in question are always commutative, associative with unity $1 \neq 0$.

LEMMA 2. *Let* A *be a* $\boldsymbol{R}$*-algebra and* I *a maximal ideal of finite codimension. If for any* $a \in A$ *we have* $1 + a^2$ *invertible in* A *then* $\boldsymbol{R} \oplus I = A$.

Proof. Trivial.

LEMMA 3. *Let* A *be a finite dimensional* $\boldsymbol{R}$*-algebra. Then the following properties are equivalent*:

(a) *A splits as an algebra into a sum of local algebras.*

(b) *For any* $a \in A$ *the element* $1 + a^2$ *is invertible.*

Moreover if (a) is true (equiv. (b)) for A then the local algebras which make part of the splitting are uniquely determined.

Proof. (a) $\to$ (b). Suppose $A = A_1 \oplus \cdots \oplus A_k$ as an algebra where A_i are local algebras. It is clear that to verify (b) it is sufficient to examine the case $k = 1$, that is when A is a local algebra. Let $A = \boldsymbol{R} \oplus I$ where I is the unique (and nilpotent) maximal ideal; if $a \in A$ then the projection of $1 + a^2$ on $\boldsymbol{R}$ is $1 + t^2$ where t is the projection of a in $\boldsymbol{R}$ so $1 + a^2$ is invertible.

(b) $\to$ (a). We will do induction on $n = \dim_{\boldsymbol{R}} A$. The case $n = 1$ being obvious, let $n > 1$ and suppose the implication valid for algebras having dimension less than n. It is clear that for $a \in A$ we have $aA \subset N$ (N denoting the nil ideal of A) iff $a \in N$, and also that $a \in N$ iff $a^2 \in N$. Let $e_0 \in A$ such that $\dim_{\boldsymbol{R}} e_0 A$ is minimum between $\dim_{\boldsymbol{R}} eA$, e ranging $A \backslash N$. Since $e_0^2 A \subseteq e_0 A$ and $e_0^2 \notin N$ we have $\dim_{\boldsymbol{R}}(e_0^2 A) = \dim_{\boldsymbol{R}}(e_0 A)$ and so $e_0^2 A = e_0 A$; then there exists $a_0 \in A$ such that $e_0 = e_0^2 a_0$, and this implies $(e_0 a_0)^2 = e_0 a_0$ and since $e_0(e_0 a) = e_0$ we have $eA = e_0 A$ where $e = e_0 a_0$; eA is a non trivial ($e \neq 0$) finite dimensional $\boldsymbol{R}$-algebra with e as unity; we will show that eA is a local algebra. Let I be a maximal ideal of eA, $x \in I$ and consider $xA = xeA \subseteq eA$; if $xA = eA$ we have that x is invertible in eA, which is not the case since I is an ideal (proper obviously), so $\dim_R(xA) < \dim_R(eA)$ and then $x \in N$; by Lemma 2 any z in eA is of the form $te + x$ where $x \in N$ and so any z in eA not in I is invertible: I is the unique maximal ideal of eA. Now we have $A = eA \oplus (1-e)A$; if $eA \subsetneq A$, that is $1 \neq e$, $0 < \dim_R((1-e)A) < \dim_R(A)$ since $(1-e)A$ is the quotient of A by the proper ideal eA it has also the required property. So, we are done.

Knowing that A splits as an algebra into a sum of local algebras it is an easy matter to verify that the components of this splitting are exactly the minimal members of the set $\{aA \mid a \notin N\}$ (ordered by inclusion); this gives the last assertion.

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