P. Orlik and L. Solomon
Nagoya Math. J.
Vol. 109 (1988), 23-45

DISCRIMINANTS IN THE INVARIANT THEORY OF REFLECTION GROUPS

PETER ORLIK AND LOUIS SOLOMON*

§ 1. Introduction

Let V be a complex vector space of dimension l and let $G \subset GL(V)$ be a finite reflection group. Let S be the C-algebra of polynomial functions on V with its usual G-module structure $(gf)(v) = f(g^{-1}v)$. Let R be the subalgebra of G-invariant polynomials. By Chevalley's theorem there exists a set $\mathscr{B} = \{f_1, \cdots, f_l\}$ of homogeneous polynomials such that $R = C[f_1, \cdots, f_l]$. We call $\mathscr{B}$ a set of basic invariants or a *basic set* for G. The degrees $d_i = \deg f_i$ are uniquely determined by G. We agree to number them so that $d_1 \leq \cdots \leq d_l$. The map τ: $V/G \to C^l$ defined by

$$\tau(Gv) = (f_1(v), \cdots, f_l(v)) \tag{1.1}$$

is a bijection. Each reflection in G fixes some hyperplane in V. Let $\mathscr{A} = \mathscr{A}(G)$ be the set of reflecting hyperplanes and let

$$N(G) = \bigcup_{H \in \mathscr{A}} H \tag{1.2}$$

$$M(G) = V - \bigcup_{H \in \mathscr{A}} H. \tag{1.3}$$

If $H \in \mathscr{A}$ let e_H be the order of the (cyclic) subgroup fixing H and let $\alpha_H \in V^*$ be a linear form with kernel H. Since $\prod_{H \in \mathscr{A}} \alpha_H^{e_H} \in R$ we may define a polynomial $\Delta(T_1, \cdots, T_l; \mathscr{B})$ in the indeterminates $T_1, \cdots, T_l$ by

$$\Delta(f_1, \cdots, f_l; \mathscr{B}) = \prod_{H \in \mathscr{A}} \alpha_H^{e_H}. \tag{1.4}$$

We call the polynomial $\Delta(T_1, \cdots, T_l; \mathscr{B})$ the *discriminant* of G relative to $\mathscr{B}$ since it depends on the basic invariants. The hypersurface

$$\tau(N(G)/G) = \{(z_1, \cdots, z_l) \in C^l \mid \Delta(z_1, \cdots, z_l; \mathscr{B}) = 0\} \tag{1.5}$$

Received February 6, 1986.
*) This work was supported in part by the National Science Foundation.

will be called the discriminant locus. E. Bannai [1] computed the fundamental group of the complement $\tau(M(G)/G)$ of the discriminant locus for all irreducible unitary reflection groups with $\dim V = 2$. In the course of this work she computed the discriminants $\Delta(f_1, f_2; \mathscr{B})$.

If $V_R \subset V$ is a real form of V and $W \subset GL(V_R)$ is a finite Coxeter group we may view $W \subset GL(V)$ as a unitary reflection group. In an earlier paper [14] we defined a class of finite irreducible unitary reflection groups, called Shephard groups. In (2.11) we associate to each Shephard group $G \subset GL(V)$ a finite irreducible Coxeter group $W \subset GL(V)$ which is determined up to isomorphism. Both G and W are isomorphic to quotients of the same Artin group. Since W is also a reflection group it has basic invariants, discriminant, etc. When both groups G, W are in question we use notation such as f_i^G, f_i^W, Δ_G, Δ_W, d_i^G, d_i^W, etc. to indicate the dependence on G, W. If a statement involves only one Shephard group we usually suppress this dependence and may apply the statement to G or W. Given a Shephard group G and the corresponding Coxeter group W, inspection of the known values of d_i^G, d_i^W as listed in Table 1 reveals the remarkable fact that

$$d_1^G/d_1^W = \cdots = d_l^G/d_l^W . \tag{1.6}$$

This suggests that there may be connections between the invariant theory of G and W. Corollary (2.26) asserts that with suitable basic sets $\mathscr{B}_G$, $\mathscr{B}_W$ for G, W the discriminant loci $\tau(N(G)/G)$ and $\tau(N(W)/W)$ are defined by the same polynomial:

$$\Delta_G(T_1, \cdots, T_l; \mathscr{B}_G) \approx \Delta_W(T_1, \cdots, T_l; \mathscr{B}_W) . \tag{1.7}$$

Here, and in the rest of this paper it is convenient to write $a \approx b$ if $b \in C^*a$. In (5.1) we use (1.7) and work of Deligne [5] to show that

(1.8) If $G \subset GL(V)$ is a Shephard group then $M(G)$ is a $K(\pi, 1)$ space.

We illustrate (1.6) and (1.7) for the pair (G, W) where $G = G_{25}$ in the Shephard-Todd classification [19] and W is the Coxeter group of type $A_3 = D_3$. The degrees d_i^G are 6, 9, 12 and the degrees d_i^W are 2, 3, 4. Thus $d_i^G/d_i^W = 3$ does not depend on i. To illustrate (1.7) we use polynomials C_6, C_9, C_{12} and $\mathfrak{C}_{12}$ defined by Maschke [9, p. 326]. Shephard and Todd [19, p. 286] remarked that we may choose $\mathscr{B}_G = \{C_6, C_9, C_{12}\}$ and that $\mathfrak{C}_{12} \approx \prod_{H \in \mathscr{A}(G)} \alpha_H$. It follows from Maschke's work [9, p. 326] that $\mathfrak{C}_{12}^3 =$

Table 1.

	G		W	
S & T	$\Gamma(G)$	d_i^G	$\Gamma(W)$	d_i^W
$G(p,1,l)$	$\overset{p}{0}\overset{4}{—}0—\cdots—0$	$p, 2p, \cdots, lp$	$0\overset{4}{—}0—\cdots—0$	$2, 4, \cdots, 2l$
3	$\overset{p}{0}$	p	0	2
4	$\overset{3}{0}—\overset{3}{0}$	4, 6	$0—0$	2, 3
8	$\overset{4}{0}—\overset{4}{0}$	8, 12	$0—0$	2, 3
16	$\overset{5}{0}—\overset{5}{0}$	20, 30	$0—0$	2, 3
5	$\overset{3}{0}\overset{4}{—}\overset{3}{0}$	6, 12	$0\overset{4}{—}0$	2, 4
10	$\overset{4}{0}\overset{4}{—}\overset{3}{0}$	12, 24	$0\overset{4}{—}0$	2, 4
18	$\overset{5}{0}\overset{4}{—}\overset{3}{0}$	30, 60	$0\overset{4}{—}0$	2, 4
20	$\overset{3}{0}\overset{5}{—}\overset{3}{0}$	12, 30	$0\overset{5}{—}0$	2, 5
6	$\overset{3}{0}\overset{6}{—}0$	4, 12	$0\overset{6}{—}0$	2, 6
9	$\overset{4}{0}\overset{6}{—}0$	8, 24	$0\overset{6}{—}0$	2, 6
17	$\overset{5}{0}\overset{6}{—}0$	20, 60	$0\overset{6}{—}0$	2, 6
14	$\overset{3}{0}\overset{8}{—}0$	6, 24	$0\overset{8}{—}0$	2, 8
21	$\overset{3}{0}\overset{10}{—}0$	12, 60	$0\overset{10}{—}0$	2, 10
25	$\overset{3}{0}—\overset{3}{0}—\overset{3}{0}$	6, 9, 12	$0—0—0$	2, 3, 4
26	$0\overset{4}{—}\overset{3}{0}—\overset{3}{0}$	6, 12, 18	$0\overset{4}{—}0—0$	2, 4, 6
32	$\overset{3}{0}—\overset{3}{0}—\overset{3}{0}—\overset{3}{0}$	12, 18, 24, 30	$0—0—0—0$	2, 3, 4, 5

$(432C_9^2 - C_6^3 + 3C_6C_{12})^2 - 4C_{12}^3$ and thus

$$\Delta_G(T_1, T_2, T_3; \mathscr{B}_G) = (432T_2^2 - T_1^3 + 3T_1T_3)^2 - 4T_3^3. \tag{1.9}$$

To compute the discriminant for $W = D_3$ choose a basis x_1, x_2, x_3 for V^* such that $\prod_{H \in \mathscr{A}(W)} \alpha_H = (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)$. In this coordinate system $p_1 = x_1^2 + x_2^2 + x_3^2$, $p_2 = x_1x_2x_3$ and $p_3 = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2$ is a basic set for W. Consider a cubic polynomial with roots x_1^2, x_2^2, x_3^2. The formula for the discriminant of this cubic, expressed in terms of the elementary symmetric functions of the roots, gives the identity

$$(1.10) \qquad \prod_{H\in\mathscr{A}(W)} \alpha_H^{e_H} = (2p_1^3 - 9p_1p_3 + 27p_2^2)^2 - 4(p_1^2 - 3p_3)^3 .$$

Let $f_1 = p_1$, $f_2 = p_2/4$ and $f_3 = p_1^2 - 3p_3$. Then $\mathscr{B}_W = \{f_1, f_2, f_3\}$ is a basic set for W and we have

$$(1.11) \qquad \Delta_W(T_1, T_2, T_3; \mathscr{B}_W) = (432T_2^2 - T_1^3 + 3T_1T_3)^2 - 4T_3^3 .$$

Comparison of (1.9) and (1.11) illustrates (1.7) for this pair of groups (G, W).

In (2.18) we define, for each Shephard group G and basic set $\mathscr{B}$, an $l \times l$ matrix $\boldsymbol{\Delta}(T_1, \cdots, T_l; \mathscr{B})$ with entries in $\boldsymbol{C}[T_1, \cdots, T_l]$. We call $\boldsymbol{\Delta}(T_1, \cdots, T_l; \mathscr{B})$ the *discriminant matrix* of G with respect to $\mathscr{B}$. It follows from the definition of the discriminant matrix that

$$(1.12) \qquad \Delta(T_1, \cdots, T_l; \mathscr{B}) \approx \det \boldsymbol{\Delta}(T_1, \cdots, T_l; \mathscr{B}) .$$

Suppose $\bar{\mathscr{B}} = \{\bar{f}_1, \cdots, \bar{f}_l)$ is another basic set. Define polynomials $\varphi_j(T_1, \cdots, T_l)$ by $\bar{f}_j = \varphi_j(f_1, \cdots, f_l)$. Let $\bar{T}_j = \varphi_j(T_1, \cdots, T_l)$. It follows from (1.4) that

$$(1.13) \qquad \Delta(\bar{T}_1, \cdots, \bar{T}_l; \bar{\mathscr{B}}) \approx \Delta(T_1, \cdots, T_l; \mathscr{B}) .$$

The corresponding transformation formula for discriminant matrices is more complicated; see (2.22). Write $\kappa = d_l^G/d_l^W$ for the constant defined in (1.6). The main result (2.25) of this paper implies that if G is a Shephard group and W is the associated Coxeter group then there exist basic sets $\mathscr{B}_G$ and $\mathscr{B}_W$ such that

$$(1.14) \qquad \boldsymbol{\Delta}_G(T_1, \cdots, T_l; \mathscr{B}_G) = \boldsymbol{\Delta}_W(\kappa T_1, \cdots, \kappa T_l; \mathscr{B}_W) .$$

Since (1.14) asserts an equality of matrices, it is a much sharper statement than (1.7), which asserts an equality of polynomials. We derive (1.7) from (1.14) in (2.26). Our proof of (1.14) uses the classification of Shephard groups. For the exceptional groups with dim $V = 3, 4$ we used MACSYMA on a VAX computer to find the discriminant matrix. We comment on the methods in Section 4 and give the results in the Appendix.

The discriminant $\Delta(T_1, \cdots, T_l; \mathscr{B})$ is a weighted homogeneous polynomial. We show in (5.13) that if G is any irreducible unitary reflection group then the weights of $\Delta(T_1, \cdots, T_l; \mathscr{B})$ are unique. We use this to show that (1.7) implies (1.6).

We would like to thank Ben Noble for introducing us to MACSYMA, and the Mathematics Research Center of the University of Wisconsin for providing computer time. We also want to thank Hiroaki Terao for many helpful conversations.

§ 2. The Comparision Theorem

In this section we state the main result of this paper, Theorem (2.25). In the first part of this section we assume that $G \subset GL(V)$ is any finite unitary reflection group and use the notation and the results of our earlier paper [14]. Let Der_S be the S-module of derivations of S and let $\Omega_S = \mathrm{Hom}_S(\mathrm{Der}_S, S)$ be the S-module of differential 1-forms. Let $\{e_i\}$ be a basis for V and let $\{x_i\}$ be the dual basis for V^*, fixed throughout this paper. Then $\{D_i = \partial/\partial x_i\}$ and $\{dx_i\}$ are bases for Der_S and Ω_S as S-modules. These modules are graded as follows. Give $S = \bigoplus_{p\geq 0} S_p$ its usual grading so that $S_1 = V^*$. We call *nonzero* elements of S forms of degree p. Grade Der_S by $\theta \in (\mathrm{Der}_S)_q$ if $\theta S_p \subseteq S_{p+q}$ for all p and grade Ω_S by $\omega \in (\Omega_S)_q$ if $\omega((\mathrm{Der}_S)_p) \subseteq S_{p+q}$ for all p. Thus D_i has degree -1 and dx_i has degree $+1$. Both Der_S and Ω_S are G-modules. The G-action is given by $(g\theta)(f) = g(\theta(g^{-1}f))$ and $(g\omega)(\theta) = g(\omega(g^{-1}\theta))$ where $\theta \in \mathrm{Der}_S$ and $\omega \in \Omega_S$.

Recall from [14] that the R-module Ω_S^G of G-invariant 1-forms is free of rank l with a basis of homogeneous elements $df_1, \cdots, df_l$. The R-module Der_S^G of G-invariant derivations is free of rank l with a basis $\mathscr{D} = \{\theta_1, \cdots, \theta_l\}$ of homogeneous elements. We call $\mathscr{D}$ a set of *basic derivations*. Note that $\deg(df_i) = d_i = m_i + 1$ where the m_i are the *exponents* of G. Similarly $\deg \theta_i = n_i - 1$ where the n_i are the *coexponents* of G as in [12, 14]. We agree to number the n_i in increasing order so that we have

$$m_1 \leq \cdots \leq m_l \quad \text{and} \quad n_1 \leq \cdots \leq n_l\,. \tag{2.1}$$

Define the Jacobian matrix $\mathbf{J} = \mathbf{J}(f_1, \cdots, f_l)$ by $\mathbf{J}_{ij} = D_i f_j$. If $\theta \in \mathrm{Der}_S$ then $\theta = \sum (\theta x_i) D_i$. Define a matrix $\mathbf{Q} = \mathbf{Q}(\theta_1, \cdots, \theta_l)$ by $\mathbf{Q}_{ij} = \theta_j x_i$. It follows from the definition of the G-action that if $f \in R$ and $\theta \in \mathrm{Der}_S^G$ then $\theta(f) \in R$. Since

$$(\mathbf{J}^T\mathbf{Q})_{ij} = \sum (\theta_j x_k)(D_k f_i) = \theta_j f_i \tag{2.2}$$

it follows that $\mathbf{J}^T\mathbf{Q} \in M_l(R)$ where $\mathbf{J}^T$ is the transpose of $\mathbf{J}$. Let $T_1, \cdots, T_l$ be indeterminates. Since $R = \boldsymbol{C}[f_1, \cdots, f_l]$ and the basic invariants $f_1, \cdots, f_l$ are algebraically independent, there exist unique polynomials $\psi_{ij}(T_1, \cdots, T_l) \in \boldsymbol{C}[T_1, \cdots, T_l]$ such that $(\mathbf{J}^T\mathbf{Q})_{ij} = \psi_{ij}(f_1, \cdots, f_l)$.

(2.3) DEFINITION. Let $\mathscr{B} = \{f_1, \cdots, f_l\}$ be a set of basic invariants and let $\mathscr{D} = \{\theta_1, \cdots, \theta_l\}$ be a set of basic derivations. Define the discriminant matrix $\varDelta(T_1, \cdots, T_l; \mathscr{B}, \mathscr{D})$ by $\varDelta(T_1, \cdots, T_l; \mathscr{B}, \mathscr{D})_{ij} = \psi_{ij}(T_1, \cdots, T_l)$. Thus

$$\varDelta(f_1, \cdots, f_l; \mathscr{B}, \mathscr{D}) = \mathbf{J}(f_1, \cdots, f_l)^T \mathbf{Q}(\theta_1, \cdots, \theta_l) . \tag{2.4}$$

Define homogeneous polynomials J and Q by

$$J = \prod_{H \in \mathscr{A}} \alpha_H^{e_H - 1} . \qquad Q = \prod_{H \in \mathscr{A}} \alpha_H . \tag{2.5}$$

The degree of J is the number m of reflections in G. The degree of Q is the number n of reflecting hyperplanes [12, 14]. It is proved in [19] that $J \approx \det \mathbf{J}(f_1, \cdots, f_l)$ and in [14, 2.28] that $Q \approx \det \mathbf{Q}(\theta_1, \cdots, \theta_l)$. It follows from (1.4), (2.4) and (2.5) that

$$\varDelta(f_1, \cdots, f_l; \mathscr{B}) \approx \det \varDelta(f_1, \cdots, f_l; \mathscr{B}, \mathscr{D}) \approx JQ . \tag{2.6}$$

Thus $\varDelta(T_1, \cdots, T_l; \mathscr{B}, \mathscr{D})$ depends on $\mathscr{B}$ and $\mathscr{D}$ but its determinant depends only on $\mathscr{B}$. For Shephard groups there is a natural choice of $\mathscr{D}$ in terms of $\mathscr{B}$. We repeat their definition [14]. Shephard [17, 18] introduced the notion of a regular complex polytope $\mathscr{P}$ and showed that its symmetry group $G = \mathrm{Aut}(\mathscr{P})$ is an irreducible unitary reflection group. A regular convex polytope in $\boldsymbol{R}^l$ defines a regular complex polytope in $V = \boldsymbol{C}^l$ by scalar extension. If $\mathscr{P}$ arises in this way we say that $\mathscr{P}$ has a real form. In this case G is a finite irreducible Coxeter group. Not all finite irreducible Coxeter groups arise in this way.

(2.7) DEFINITION. A *Shephard group* is the symmetry group of a regular complex polytope.

Coxeter [4, pps. 94–5, 147–9] showed that for each Shephard group G there exist generating reflections $s_1, \cdots, s_l$ and integers $p_1, \cdots, p_l$ and integers $q_1, \cdots, q_{l-1}$ such that G has a presentation with defining relations:

$$s_j^{p_j} = 1 \tag{2.8}$$

$$s_j s_k = s_k s_j \qquad \text{if } |j - k| \geq 2 \tag{2.9}$$

$$s_{j+1} s_j s_{j+1} \cdots = s_j s_{j+1} s_j \cdots \qquad \text{if } 1 \leq j \leq l-1 \tag{2.10}$$

where there are q_j terms on each side of (2.10). Coxeter associated the symbol $p_1[q_1]p_2 \cdots p_{l-1}[q_{l-1}]p_l$ to the group G. It follows from the classification of Shephard groups that the symbol is uniquely determined by G up to replacement of $p_1[q_1]p_2\cdots p_{l-1}[q_{l-1}]p_l$ by its reversal $p_l[q_{l-1}]p_{l-1}\cdots p_2[q_1]p_1$. Thus we may make the following definition.

(2.11) DEFINITION. If $G \subset GL(V)$ is a Shephard group with symbol $p_1[q_1]p_2\cdots p_{l-1}[q_{l-1}]p_l$, let $W \subset GL(V)$ be the Coxeter group with symbol $2[q_1]2\cdots 2[q_{l-1}]2$. We call W the Coxeter group associated to G.

The group W is uniquely determined by G up to conjugacy in $GL(V)$. If $\mathscr{P}$ has a real form then $W = G$.

(2.12) DEFINITION. Let $G \subset GL(V)$ be a Shephard group with symbol $p_1[q_1]p_2\cdots p_{l-1}[q_{l-1}]p_l$. Associate to G a graph $\Gamma(G)$ which has vertices $s_1, \cdots, s_l$ with labels $p_1, \cdots, p_l$ and edges between vertices s_j and s_{j+1} which are labeled q_j.

These graphs were introduced by Coxeter. If W is a Coxeter group, let $\Gamma(W)$ be its Coxeter graph. In Table 1 we list the pairs (G, W) in (2.11) together with their graphs. We omit the labels $p_j = 2$ and $q_j = 3$. Table 1 also contains the invariant degrees d_i^G, d_i^W. In [2] Coxeter noted in case $l = 2$ and $q = q_1$ that

$$d_1^G = 2h/q \quad \text{and} \quad d_2^G = h \tag{2.13}$$

where h is the order of $s_1 s_2$. Since $d_1^W = 2$ and $d_2^W = q$ this implies (1.6) in case $l = 2$. In fact if G is any Shephard group and h^G is the order of $s_1 s_2 \cdots s_l$ then a case by case check shows that $d_i^G/d_i^W = h^G/h^W$ where h^W is the Coxeter number of W. Since the order of the center $Z(G)$ is the greatest common divisor of $d_1^G, \cdots, d_l^G$ [20, Cor. 3.3] we get a stronger form of (1.6):

$$\kappa = d_i^G/d_i^W = |Z(G)|/|Z(W)| = h^G/h^W . \tag{2.14}$$

Now we return to the invariant theory and the discriminant for a Shephard group. Let $G \subset GL(V)$ be a Shephard group and let f_1 be an invariant form of minimal positive degree. The main result of [14] is that the R-linear map $\mathrm{Hess}(f_1)\colon \mathrm{Der}_S^G \to \Omega_S^G$ defined by

$$\mathrm{Hess}(f_1)\colon \theta \to \sum \theta(D_i f_1) dx_i \qquad \theta \in \mathrm{Der}_S^G \tag{2.15}$$

is an isomorphism of graded R-modules. In particular it follows that for given basic invariants $f_1, \cdots, f_l$ there exist unique basic derivations $\theta_1, \cdots, \theta_l$ with

(2.16) $$\mathrm{Hess}(f_1)\theta_i = m_1 df_i \qquad i = 1, \cdots, l.$$

The matrix of $\mathrm{Hess}(f_1)$ relative to the pair of bases $\{D_i\}$ and $\{dx_i\}$ is the Hessian matrix $\mathbf{H}(f_1)$ defined by $\mathbf{H}(f_1)_{ij} = D_i D_j f_1$. The formula (2.16) is equivalent to the matrix equation

(2.17) $$\mathbf{H}(f_1)\mathbf{Q}(\theta_1, \cdots, \theta_l) = m_1 \mathbf{J}(f_1, \cdots, f_l).$$

(2.18) DEFINITION. Let G be a Shephard group and let $\mathscr{B} = \{f_1, \cdots, f_l\}$ be a set of basic invariants. If $\mathscr{D} = \{\theta_1, \cdots, \theta_l\}$ satisfies (2.17) we call $\mathscr{D}$ the set of basic derivations associated to $\mathscr{B}$ and define

$$\varDelta(T_1, \cdots, T_l; \mathscr{B}) = \varDelta(T_1, \cdots, T_l; \mathscr{B}, \mathscr{D}).$$

We call $\varDelta(T_1, \cdots, T_l; \mathscr{B})$ the discriminant matrix of G with respect to $\mathscr{B}$.

(2.19) LEMMA. (i) $\varDelta(f_1, \cdots, f_l; \mathscr{B}) = m_1 \mathbf{J}^T \mathbf{H}(f_1)^{-1} \mathbf{J}$;

(ii) $\varDelta(T_1, \cdots, T_l; \mathscr{B})$ *is a symmetric matrix*;

(iii) $\theta_i f_j = \theta_j f_i$ *for* $1 \leq i, j \leq l$.

Proof. Formula (i) follows from (2.4) and (2.17); (ii) follows from (i); (iii) follows from (i) and (2.2). □

We showed in [14, 2.23] that $\theta_1 = \sum x_i D_i$ is the Euler derivation. It follows from (2.19. iii) that $\theta_1 f_j = \theta_j f_1 = d_j f_j$. It follows from (2.19. ii) that

$$\varDelta(T_1, \cdots, T_l; \mathscr{B}) = \begin{bmatrix} d_1 T_1 & d_2 T_2 & \cdots & d_l T_l \\ d_2 T_2 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ d_l T_l & * & & * \end{bmatrix}.$$

For convenience we define a matrix $\mathbf{D}(T_1, \cdots, T_l; \mathscr{B})$ by

(2.20) $$\mathbf{D}(T_1, \cdots, T_l; \mathscr{B}) = \varDelta(T_1/d_1, \cdots, T_l/d_l; \mathscr{B}).$$

We also call $\mathbf{D}(T_1, \cdots, T_l; \mathscr{B})$ the discriminant matrix of G with respect to $\mathscr{B}$. Note that

(2.21) $$\mathbf{D}(d_1 f_1, \cdots, d_l f_l; \mathscr{B}) = \varDelta(f_1, \cdots, f_l; \mathscr{B}).$$

In the next lemma we compare $\mathbf{D}(T_1, \cdots, T_l; \mathscr{B})$ and $\mathbf{D}(T_1, \cdots, T_l, \bar{\mathscr{B}})$ for basic sets $\mathscr{B}$, $\bar{\mathscr{B}}$. In the argument we use the fact proved in [14, 5.4] that if G is a Shephard group, then an invariant form of minimal positive degree is unique up to a constant multiple.

(2.22) LEMMA. *Let $\mathscr{B} = \{f_1, \cdots, f_l\}$ and $\bar{\mathscr{B}} = \{\bar{f}_1, \cdots, \bar{f}_l\}$ be basic sets for the Shephard group G. Define $c \in C^*$ by $\bar{f}_1 = cf_1$. Let $\mathbf{E}$ be the $l \times l$ diagonal matrix $\mathbf{E} = \operatorname{diag}(d_1, \cdots, d_l)$. Let p_j be the polynomial defined by $d_j\bar{f}_j = p_j(d_1f_1, \cdots, d_lf_l)$. Let $\bar{T}_j = p_j(T_1, \cdots, T_l)$. Let $\mathbf{M}$ be the $l \times l$ matrix with (k, j) entry $\mathbf{M}_{kj} = \partial\bar{T}_j/\partial T_k$. Let $\mathbf{N} = \mathbf{E}\mathbf{M}\mathbf{E}^{-1}$. Then*

$$\mathbf{D}(\bar{T}_1, \cdots, \bar{T}_l; \bar{\mathscr{B}}) = c^{-1}\mathbf{N}^T\mathbf{D}(T_1, \cdots, T_l; \mathscr{B})\mathbf{N}. \tag{2.23}$$

Proof. Write $\mathbf{J}(\mathscr{B}) = \mathbf{J}(f_1, \cdots, f_l)$. A calculation using the chain rule gives $\mathbf{J}(\bar{\mathscr{B}}) = \mathbf{J}(\mathscr{B})\mathbf{N}$. Thus we get $\mathbf{D}(d_1\bar{f}_1, \cdots, d_l\bar{f}_l; \bar{\mathscr{B}}) = m_1\mathbf{J}(\bar{\mathscr{B}})^T\mathbf{H}(\bar{f}_1)^{-1}\mathbf{J}(\bar{\mathscr{B}}) = m_1\mathbf{N}^T\mathbf{J}(\mathscr{B})^T\mathbf{H}(\bar{f}_1)^{-1}\mathbf{J}(\mathscr{B})\mathbf{N}$. Since $\bar{f}_1 = cf_1$ we have $\mathbf{H}(\bar{f}_1) = c\mathbf{H}(f_1)$ so $\mathbf{D}(d_1\bar{f}_1, \cdots, d_l\bar{f}_l; \bar{\mathscr{B}}) = c^{-1}\mathbf{N}^T\mathbf{D}(d_1f_1, \cdots, d_lf_l; \mathscr{B})\mathbf{N}$. □

Now we are in position to compare the discriminant matrices $\mathbf{D}_G(T_1, \cdots, T_l; \mathscr{B})$ of a Shephard group G and $\mathbf{D}_W(T_1, \cdots, T_l; \mathscr{B}_W)$ of the associated Coxeter group W.

(2.24) NOTATION. Let $G \subset GL(V)$ be a Shephard group and let W be the corresponding Coxeter group. Given basic sets $\mathscr{B}_G$ and $\mathscr{B}_W$ we sometimes write $\mathscr{B}_G \sim \mathscr{B}_W$ if $\mathbf{D}_G(T_1, \cdots, T_l; \mathscr{B}_G) = \mathbf{D}_W(T_1, \cdots, T_l; \mathscr{B}_W)$.

The main result of this paper, proved in Sections 3 and 4, is the following comparison theorem for discriminant matrices of G and W. In view of (2.20) this implies (1.14).

(2.25) THEOREM. *Let $G \subset GL(V)$ be a Shephard group and let W be the corresponding Coxeter group.* (i) *If $\mathscr{B}_W$ is a basic set for W then there exists a basic set $\mathscr{B}_G$ for G such that $\mathscr{B}_G \sim \mathscr{B}_W$.* (ii) *If $\mathscr{B}_G$ is a basic set for G then there exists a basic set $\mathscr{B}_W$ for W such that $\mathscr{B}_G \sim \mathscr{B}_W$.*

(2.26) COROLLARY. *Let $G \subset GL(V)$ be a Shephard group and let W be the corresponding Coxeter group. Then there exist basic sets $\mathscr{B}_G$, $\mathscr{B}_W$ such that*

$$\Delta_G(T_1, \cdots, T_l; \mathscr{B}_G) \approx \Delta_W(T_1, \cdots, T_l; \mathscr{B}_W).$$

Proof. By (2.25) there exist basic sets $\bar{\mathscr{B}}_G = \{f_1^G, \cdots, f_l^G\}$ and $\bar{\mathscr{B}}_W = \{f_1^W, \cdots, f_l^W\}$ such that $\bar{\mathscr{B}}_G \sim \bar{\mathscr{B}}_W$. Thus $\mathbf{D}_G(T_1, \cdots, T_l; \bar{\mathscr{B}}_G) = \mathbf{D}_W(T_1, \cdots, T_l, \bar{\mathscr{B}}_W)$. It follows from (2.20) that

$$\Delta_G(T_1/d_1^G, \cdots, T_l/d_l^G; \bar{\mathscr{B}}_G) = \Delta_W(T_1/d_1^W, \cdots, T_l/d_l^W; \bar{\mathscr{B}}_W).$$

Let $\mathscr{B}_G = \{d_1^Gf_1^G, \cdots, d_l^Gf_l^G\}$ and let $\mathscr{B}_W = \{d_1^Wf_1^W, \cdots, d_l^Wf_l^W\}$. The assertion follows from (1.13) by taking determinants. □

The next proposition shows that it suffices to prove (2.25) for one particular basic set $\mathscr{B}_W$ in (i) and one particular basic set $\mathscr{B}_G$ in (ii).

(2.27) PROPOSITION. *Suppose there exist basic sets $\mathscr{B}_W$, $\mathscr{B}_G$ for W, G such that $\mathscr{B}_G \sim \mathscr{B}_W$. (i) If $\bar{\mathscr{B}}_W$ is any basic set for W then there exists a basic set $\bar{\mathscr{B}}_G$ for G such that $\bar{\mathscr{B}}_G \sim \bar{\mathscr{B}}_W$. (ii) If $\bar{\mathscr{B}}_G$ is any basic set for G then there exists a basic set $\bar{\mathscr{B}}_W$ for W such that $\bar{\mathscr{B}}_G \sim \bar{\mathscr{B}}_W$.*

Proof. (i) We apply (2.22) to both G and W. To indicate the dependence on G and W we write c_G, $\mathbf{E}_G$, $\mathbf{M}_G$, $\mathbf{N}_G$ and c_W, $\mathbf{E}_W$, $\mathbf{M}_W$, $\mathbf{N}_W$. Define polynomials p_j by $d_j^W \bar{f}_j^W = p_j(d_1^W f_1^W, \cdots, d_l^W f_l^W)$. Define $\bar{f}_j^G$ by $d_j^G \bar{f}_j^G = p_j(d_1^G f_1^G, \cdots, d_l^G f_l^G)$. Then $c_G = c_W$ and $\mathbf{M}_G = \mathbf{M}_W$. It follows from (1.6) that $\mathbf{E}_G = k \cdot \mathbf{E}_W$ for some $k \in C^*$. Thus $\mathbf{N}_G = \mathbf{N}_W$. Assertion (i) follows from (2.23) applied to both G and W and the assumption $\mathbf{D}_G(T_1, \cdots, T_l; \mathscr{B}_G) = \mathbf{D}_W(T_1, \cdots, T_l; \mathscr{B}_W)$. Assertion (ii) is proved in the same way. □

§ 3. The case dim $V = 2$

(3.1) THEOREM. *Suppose* $\dim V = 2$ *and* $G \subset GL(V)$ *is a Shephard group with symbol* $p_1[q]p_2$. *Then there exists a basic set* $\mathscr{B} = \{f_1, f_2\}$ *such that*

$$\mathbf{D}(T_1, T_2; \mathscr{B}) = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1^{q-1} \end{bmatrix}.$$

Theorem (3.1) is a consequence of Lemmas (3.4)–(3.9). We introduce some notation. If $p_1, p_2 \in S$ we write $J(p_1, p_2) = \det \mathbf{J}(p_1, p_2)$ for the Jacobian determinant, a notation used throughout this section for various polynomials p_1, p_2. It follows from (2.5) that for any basic set $\mathscr{B} = \{f_1, f_2\}$ we have $J(f_1, f_2) \approx J$. Let θ_1, θ_2 be the basic derivations associated to $\mathscr{B}$ as in (2.18). Since $\theta_1 = x_1 D_1 + x_2 D_2$ is the Euler derivation, (2.4) and (2.19. iii) give

$$\Delta(f_1, f_2; \mathscr{B}) = \begin{bmatrix} d_1 f_1 & d_2 f_2 \\ d_2 f_2 & \theta_2 f_2 \end{bmatrix}. \tag{3.2}$$

Let $\psi(T_1, T_2)$ be the polynomial defined by $\psi(d_1 f_1, d_2 f_2) = \theta_2 f_2$. Thus

$$\mathbf{D}(T_1, T_2; \mathscr{B}) = \begin{bmatrix} T_1 & T_2 \\ T_2 & \psi(T_1, T_2) \end{bmatrix}.$$

To prove (3.1) it remains to show that

$$\theta_2 f_2 = (d_1 f_1)^{q-1} \tag{3.3}$$

for suitable choice of f_1, f_2. Then $\psi(T_1, T_2) = T_1^{q-1}$.

(3.4) LEMMA. $\theta_2 f_2 \neq 0$.

Proof. Suppose $\theta_2 f_2 = 0$. It follows from (2.6) and (3.2) that $JQ \approx f_2^2$. Since Q is square-free, Q divides f_2, and thus f_2 divides J. For $g \in G$ let $\delta(g) = \det g$. It is known [21, p. 85] that J is a semi-invariant of G of minimal degree with character δ. Since J/f_2 is also a semi-invariant of G with character δ we have a contradiction. □

We have already remarked preceding (2.22) that, since G is a Shephard group, the invariant f_1 is unique up to a constant multiple. We showed in [14] that $\{f_1, J(Q, f_1)\}$ is a basic set. Suppose $\{f_1, f_2\}$ is any basic set. Then $f_2 \in \sum \boldsymbol{C} f_1^\alpha J(Q, f_1)^\beta$ where the sum is over all nonnegative integers α, β such that $\alpha d_1 + \beta d_2 = d_2$. Thus either $(\alpha, \beta) = (0, 1)$ or $\beta = 0$, in which case d_1 divides d_2. It follows from (2.13) that $2d_2 = qd_1$ so $\beta = 0$ can only occur if q is even. Thus there are two possibilities:

(3.5.i) $\quad f_2 = aJ(Q, f_1) \quad$ if q is odd,

(3.5.ii) $\quad f_2 = aJ(Q, f_1) + bf_1^r \quad$ if $q = 2r$ is even,

where $a \in \boldsymbol{C}^*$ and $b \in \boldsymbol{C}$. In the next lemma we compute the basic derivations θ_1, θ_2 associated to f_1, f_2 where f_2 is given by (3.5). Let

$$\eta = -(D_2 Q)D_1 + (D_1 Q)D_2 . \tag{3.6}$$

Note that $\eta\varphi = J(Q, \varphi)$ for any $\varphi \in S$.

(3.7) LEMMA. *Let G be a Shephard group. Let f_1 be an invariant form of degree d_1 and let f_2 be defined by* (3.5). *Then the basic derivations θ_1, θ_2 associated to the basic invariants f_1, f_2 are $\theta_1 = x_1 D_1 + x_2 D_2$ and*

(i) $\theta_2 = d_2 a\eta$ *if q is odd,*

(ii) $\theta_2 = d_2 a\eta + brf_1^{r-1}\theta_1$ *if $q = 2r$ is even.*

Proof. Recall from (2.5) that $n = \deg Q$. We showed in [14] that $\mathbf{H}(f_1)\mathbf{Q}(\theta_1, \eta) = m_1 \mathbf{J}(f_1, \psi)$ where $(n + d_1 - 2)\psi = J(Q, f_1)$. Note that $n + d_1 - 2 = \deg J(Q, f_1) = \deg \psi = d_2$. In view of the equivalence of (2.16) and (2.17) we have $\operatorname{Hess}(f_1)\theta_1 = m_1 df_1$ and $\operatorname{Hess}(f_1)\eta = m_1 d\psi$. Now R-linearity of $\operatorname{Hess}(f_1)$ shows that $\operatorname{Hess}(f_1)\theta_2 = m_1 df_2$. □

(3.8) LEMMA. *The form* $J(Q, J(Q, f_1))$ *is a nonzero invariant of degree* $2d_2 - d_1 = (q-1)d_1$.

Proof. Let f, φ, ψ be any binary forms of degrees m, n, p at least 2. Let $H(f) = \det \mathbf{H}(f)$ be the Hessian determinant of f and let

$$\tau_2(f, \varphi) = (D_1^2 f)(D_2^2\varphi) - 2(D_1D_2f)(D_1D_2\varphi) + (D_2^2 f)(D_1^2\varphi)$$

be their second transvectant [21, p. 57]. Define $(f, \varphi)^2$ by

$$\tau_2(f, \varphi) = m(m-1)n(n-1)(f, \varphi)^2 .$$

We use a known formula for the Jacobian of a Jacobian [23, p. 223]:

$$J(J(f, \varphi), \psi) \approx \frac{m-n}{m+n-2}(f, \varphi)^2\psi + (f, \psi)^2\varphi - (\varphi, \psi)^2 f .$$

Set $f = f_1$ and $\varphi = \psi = Q$. We proved in [14] that $\tau_2(f, Q) = 0$. Thus

$$J(Q, J(Q, f_1)) \approx \tau_2(Q, Q)f_1 \approx H(Q)f_1 .$$

Since Q is not a power of a linear form $H(Q) \neq 0$ [6, p. 235] and thus $J(Q, J(Q, f_1)) \neq 0$. In general, if G is a subgroup of $GL(V)$ and f, φ are semi-invariants of G with characters λ, μ then $J(f, \varphi)$ is a semi-invariant of G with character $\delta\lambda\mu$ [21, p. 97] where $\delta(g) = \det g$. Since Q is a semi-invariant with character δ^{-1} [14, 2.27] and f_1 is an invariant form it follows that $J(Q, J(Q, f_1))$ is an invariant of degree $2n + d_1 - 4$. We remarked in the proof of (3.7) that $n + d_1 - 2 = d_2$. Since $2d_2 = qd_1$ the proof is complete. □

(3.9) LEMMA. (i) *If* q *is odd then there exists* $c \in \boldsymbol{C}^*$ *with* $J(Q, J(Q, f_1)) = cf_1^{q-1}$. (ii) *If* $q = 2r$ *is even then there exist* $c_1, c_2 \in \boldsymbol{C}$ *not both zero with* $J(Q, J(Q, f_1)) = c_1 f_1^{r-1} J(Q, f_1) + c_2 f_1^{q-1}$.

Proof. It follows from (3.8) that $J(Q, J(Q, f_1)) \in \sum \boldsymbol{C} f_1^\alpha f_2^\beta$ where the sum is over all nonnegative integers α, β such that $\alpha d_1 + \beta d_2 = 2d_2 - d_1$. Thus $\beta = 0, 1$. If $\beta = 0$ then $\alpha d_1 = 2d_2 - d_1$. Since $2d_2 = qd_1$ it follows that $\alpha = q - 1$. If $\beta = 1$ then $(\alpha + 1)d_1 = d_2$ so d_1 divides d_2 and q must be even. Since $J(Q, J(Q, f_1)) \neq 0$ this completes the proof. □

To prove (3.3) it remains to show that constants $a \in \boldsymbol{C}^*$ and $b \in \boldsymbol{C}$ may be chosen so that $\theta_2 f_2 = (d_1 f_1)^{q-1}$. Suppose first that q is odd. Then by (3.5)–(3.9) we have

$$\theta_2 f_2 = d_2 a^2 \eta J(Q, f_1) = d_2 a^2 J(Q, J(Q, f_1)) = d_2 a^2 c f_1^{q-1}.$$

Since $c \neq 0$ we may choose $a \in \boldsymbol{C}^*$ so that $d_2 a^2 c = d_1^{q-1}$. Now suppose $q = 2r$ is even. A degree argument as in (3.9) shows that $\theta_2 f_2 = k_1 f_1^{r-1} J(Q, f_1) + k_2 f_1^{q-1}$ where $k_1, k_2 \in \boldsymbol{C}$. It follows from (3.3) that k_1, k_2 are not both zero. We must show that there exist $a \in \boldsymbol{C}^*$, $b \in \boldsymbol{C}$ such that $k_1 = 0$ and $k_2 = d_1^{q-1}$. Direct calculation gives

$$\begin{aligned}\theta_2 f_2 &= d_2 a^2 \eta J(Q, f_1) + d_2 ab\eta f_1^r + brf_1^{r-1}\theta_1 f_2 \\ &= d_2 a^2 J(Q, J(Q, f_1)) + 2d_2 abrf_1^{r-1} J(Q, f_1) + d_2 b^2 r f_1^{q-1}.\end{aligned}$$

From (3.9) we get $k_1 = d_2 a^2 c_1 + 2d_2 abr$ and $k_2 = d_2 a^2 c_2 + d_2 b^2 r$. Choose $b \in \boldsymbol{C}$ so that $ac_1 + 2br = 0$. Then $k_1 = 0$ and $k_2 = a^2(d_2 c_2 + d_2 c_1^2/4r)$. Since $k_1 = 0$ we have $k_2 \neq 0$ and thus we may choose $a \in \boldsymbol{C}^*$ so that $k_2 = d_1^{q-1}$. This proves (3.3) and hence completes the argument for (3.1). □

(3.10) *Remark.* It follows from the preceding computation that for given f_1, an invariant f_2 which satisfies (3.1) is determined uniquely up to sign.

To complete the proof of Theorem (2.25) in case $\dim V = 2$, we apply (3.1) to both G and W, where W has symbol $2[q]2$, to conclude that there exist bases $\mathscr{B}_G$, $\mathscr{B}_W$ such that

$$\mathbf{D}_G(T_1, T_2; \mathscr{B}_G) = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1^{q-1} \end{bmatrix} = \mathbf{D}_W(T_1, T_2; \mathscr{B}_W). \quad \square$$

Table 2.

S & T	Coxeter	f_1	f_2
$G(q, q; 2)$	$2[q]2$	$x_1 x_2$	$(2^{q/2-1}/q)(x_1^q + x_2^q)$
$G(p, 1; 2)$	$p[4]2$	$x_1^p + x_2^p$	$(p/2)(x_1^{2p} - 6x_1^p x_2^p + x_2^{2p})$
4	$3[3]3$	Φ	$8(-1/3)^{1/4} t$
8	$4[3]4$	W	$(4/3)2^{1/2}\chi$
16	$5[3]5$	H	$(4/3)(-5)^{1/2} T$
5	$3[4]3$	t	$(1/6)(-3)^{1/2}\chi$
10	$4[4]3$	χ	$6(2W^3 - \chi^2)$
18	$5[4]3$	T	$15(T^2 + 2H^3)$
20	$3[5]3$	f	$(2/5)T$
6	$3[6]2$	Φ	$(16/3)(\Phi^3 - 2\Psi^3)$
9	$4[6]2$	W	$(64/3)(W^3 - 2\chi^2)$
17	$5[6]2$	H	$(400/3)(2T^2 + H^3)$
14	$3[8]2$	t	$(1/2)(\chi^2 + W^3)$
21	$3[10]2$	f	$(12/5)(H^3 - T^2)$

In Table 2 we list the Shephard groups for $l = 2$ together with a basic set $\mathscr{B} = \{f_1, f_2\}$ which satisfies Theorem (3.1). The invariant f_1 is chosen as in [19, 4.14] to be one of Klein's polynomials Φ, Ψ, t, W, χ, f, H, T, [7]. The invariant f_2, determined up to sign, is also expressed in terms of Klein's polynomials.

§ 4. The case dim $V \geq 3$

If dim $V \geq 3$ the Shephard groups which are not Coxeter groups are the monomial groups $G(p, 1, l)$ and the groups G_{25}, G_{26}, G_{32} in the classification of Shephard and Todd. We use (2.27) which tells us that it suffices to exhibit basic sets $\mathscr{B}_G$, $\mathscr{B}_W$ such that $\mathbf{D}_G(T_1, \cdots, T_l; \mathscr{B}_G) = \mathbf{D}_W(T_1, \cdots, T_l; \mathscr{B}_W)$.

First we consider the monomial group $G = G(p, 1, l)$. In this case the corresponding Coxeter group $W = G(2, 1, l)$ is of type B_l. It suffices to show that G has a basic set $\mathscr{B}$ such that the corresponding discriminant matrix $\mathbf{D}(T_1, \cdots, T_l; \mathscr{B})$ is the same for all p. For $k = 1, 2, 3, \cdots$ let $s_k = x_1^k + \cdots + x_l^k$. Define polynomials $\varphi_k = \varphi_k(T_1, \cdots, T_l)$ by

$$s_k = \varphi_k(s_1, \cdots, s_l) \qquad k = 1, 2, 3, \cdots . \tag{4.1}$$

Thus $\varphi_k = T_k$ for $k = 1, \cdots, l$. Let $\mathscr{B} = \{f_1, \cdots, f_l\}$ be the basic set for $G(p, 1, l)$ defined by

$$kpf_k = s_{kp} \qquad 1 \leq k \leq l . \tag{4.2}$$

It follows from (2.17) that the associated basic derivations are

$$\theta_k = \sum_{j=1}^{l} x_j^{(k-1)p+1} D_j \qquad 1 \leq k \leq l . \tag{4.3}$$

Thus

$$\mathbf{J}(f_1, \cdots, f_l)^T \mathbf{Q}(\theta_1, \cdots, \theta_l) = \begin{bmatrix} s_p & s_{2p} & \cdots & s_{lp} \\ s_{2p} & s_{3p} & \cdots & s_{(l+1)p} \\ \vdots & \vdots & \vdots & \vdots \\ s_{lp} & s_{(l+1)p} & \cdots & s_{(2l-1)p} \end{bmatrix} .$$

Since $s_{kp} = \varphi_k(s_p, \cdots, s_{lp})$, the discriminant matrix

$$\mathbf{D}(T_1, \cdots, T_l; \mathscr{B}) = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_l \\ \varphi_2 & \varphi_3 & \cdots & \varphi_{l+1} \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_l & \varphi_{l+1} & \cdots & \varphi_{2l-1} \end{bmatrix} .$$

is the same for all p. This proves (2.25) for $G = G(p, 1, l)$ and $W = B_l$.

To complete the proof of the main theorem we have to find $\mathscr{B}_G$, $\mathscr{B}_W$ such that $\mathscr{B}_G \sim \mathscr{B}_W$ for the three remaining pairs (G, W) which are (G_{25}, D_3), (G_{26}, B_3) and (G_{32}, A_4). There are done by explicit calculation, in part using machine computation in MACSYMA. We choose a basic set $\mathscr{B}_W = \{f_1^W, \cdots, f_l^W\}$. Since f_1^W is a quadratic form, the Hessian $\mathbf{H}(f_1^W)$ is a constant matrix, so it is easy to compute the matrix $\mathbf{Q}(\theta_1^W, \cdots, \theta_l^W) = m_1^W \mathbf{H}(f_1^W)^{-1}\mathbf{J}(f_1^W, \cdots, f_l^W)$ which satisfies (2.17). Note that $m_1^W = 1$. Next we compute $\mathbf{J}(f_1^W, \cdots, f_l^W)^T\mathbf{Q}(\theta_1^W, \cdots, \theta_l^W)$. This is a matrix of W-invariants. We express each matrix entry as a polynomial in the variables $d_1^W f_1^W, \cdots, d_l^W f_l^W$ to obtain the matrix $\mathbf{D}_W(T_1, \cdots, T_l; \mathscr{B}_W)$. Basic invariants for the corresponding groups G were determined by Shephard and Todd using work of Maschke [9]. Let $\mathscr{B} = \{f_1, \cdots, f_l\}$ be their basic set for G. Any basic invariant f_i^G of degree d_i^G has the form $f_i^G = \varphi_i(f_1, \cdots, f_l)$ where the φ_i are polynomials. Since the G-invariant form of minimal degree is unique up to a constant we may choose $f_1^G = f_1$. Degree considerations restrict the polynomials φ_i so that there are only a few free parameters in each case. The matrix $\mathbf{J}(f_1^G, \cdots, f_l^G)$ contains these parameters. The matrix $\mathbf{H}(f_1^G)$ has polynomial entries. In G_{32} the entries of the 4×4 matrix $\mathbf{H}(f_1^G)$ are polynomials of degree 10 in 4 variables. Thus it is not easy to compute the matrix $\mathbf{Q}(\theta_1^G, \cdots, \theta_l^G) = m_1^G \mathbf{H}(f_1^G)^{-1}\mathbf{J}(f_1^G, \cdots, f_l^G)$ which satisfies (2.17). As in the case of W we compute $\mathbf{J}(f_1^G, \cdots, f_l^G)^T\mathbf{Q}(\theta_1^G, \cdots, \theta_l^G)$ which still contains the free parameters. This is a matrix of G-invariants. We express each matrix entry as a polynomial in the variables $d_1^G f_1^G, \cdots, d_l^G f_l^G$ and force the parameters to satisfy $\mathbf{D}_G(T_1, \cdots, T_l; \mathscr{B}_G) = \mathbf{D}_W(T_1, \cdots, T_l; \mathscr{B}_W)$. The results of the calculation are given in the Appendix where we also exhibit the unique basic derivations $\theta_1, \cdots, \theta_l$ associated to $\mathscr{B}_G$ which satisfy (2.17).

§ 5. Related results

The comparison theorem (2.25) allows us to deduce further properties of a Shephard group G from the corresponding properties of the associated Coxeter group W.

(5.1) THEOREM. *If $G \subset GL(V)$ is a Shephard group then $M(G) = V - \bigcup_{H \in \mathscr{A}(G)} H$ is a $K(\pi, 1)$ space.*

Proof. Let W be the Coxeter group associated to G. Deligne [5] showed that $M(W)$ is a $K(\pi, 1)$ space and hence so is $M(W)/W$. It follows from (1.7) that for suitable choice of basic invariants the discriminant loci for G and W are the zero sets of the same polynomial. Thus $M(G)/G = M(W)/W$, so $M(G)/G$ and hence $M(G)$ is a $K(\pi, 1)$ space. □

If $l = 2$ then the complement of *any* finite set of hyperplanes containing the origin is a $K(\pi, 1)$ space. We proved in [13] that $M(G)$ is a $K(\pi, 1)$ space for the Shephard groups $G = G(p, 1, l)$. Nakamura [11] showed that $M(G)$ is a $K(\pi, 1)$ space for all imprimitive unitary reflection groups.

(5.2) THEOREM. *Let $G \subset GL(V)$ be a Shephard group and let $\mathscr{B}_G = \{f_1^G, \cdots, f_l^G\}$ be a basic set. Then*

$$\Delta_G(f_1^G, \cdots, f_l^G) \equiv (f_l^G)^l \qquad \mathrm{mod}\,(f_1^G, \cdots, f_{l-1}^G)\,.$$

Proof. Let W be the associated Coxeter group. By (2.25) there exists a basic set $\mathscr{B}_W = \{f_1^W, \cdots, f_l^W\}$ such that $\mathscr{B}_G \sim \mathscr{B}_W$. Saito [15] proved that $\Delta_W(f_1^W, \cdots, f_l^W; \mathscr{B}_W) \equiv (f_l^W)^l \,\mathrm{mod}\,(f_1^W, \cdots, f_{l-1}^W)$. Thus $\Delta_W(T_1, \cdots, T_l; \mathscr{B}_W) \equiv T_l^l \,\mathrm{mod}\,(T_1, \cdots, T_{l-1})$. The assertion follows from (2.26). □

Suppose $W \subset GL(V)$ is a finite irreducible Coxeter group. Let $\mathscr{B} = \{f_1, \cdots, f_l\}$ be a basic set for W and write $\mathbf{J} = \mathbf{J}(f_1, \cdots, f_l)$. Saito, Sekiguchi and Yano [15, 16, 24] have used the matrix $\mathbf{J}^T\mathbf{J}$. We may choose coordinates $x_1, \cdots, x_l$ so that $f_1 = \sum x_i^2$ and choose $\theta_j = \frac{1}{2}\sum (D_i f_j) D_i$. Then $\mathbf{J} = 2\mathbf{Q}(\theta_1, \cdots, \theta_l)$ and (2.17) is satisfied. Thus $\mathbf{J}^T\mathbf{J} \approx \mathbf{\Delta}(f_1, \cdots, f_l; \mathscr{B})$ in the notation of this paper. Since $d_{l-1} < d_l$ the operator $\partial/\partial f_l\colon R \to R$ is uniquely determined up to constant. Saito, Sekiguchi and Yano [15, 16] proved that there exists a basic set $\mathscr{B}$ for W, which they call a *flat basic set*, such that $(\partial/\partial f_l)(\mathbf{J}^T\mathbf{J}) \in M_l(\boldsymbol{R})$.

If $G \subset GL(V)$ is a Shephard group, it follows from (1.6) that $d_{l-1}^G < d_l^G$ and thus the operator $\partial/\partial f_l\colon R \to R$ is again uniquely determined up to constant.

(5.3) DEFINITION. Let G be a Shephard group. We call a basic set $\mathscr{B}$ a *flat basic set* if $(\partial/\partial f_l)\mathbf{\Delta}(f_1, \cdots, f_l; \mathscr{B}) \in M_l(\boldsymbol{C})$.

(5.4) THEOREM. *Let $G \subset GL(V)$ be a Shephard group. Then G has a flat basic set.*

Proof. In view of (2.20) a basic set $\mathscr{B}$ is flat if and only if

$(\partial/\partial T_l)\mathbf{D}(T_1, \cdots, T_l; \mathscr{B}) \in M_l(C)$. The result follows from (2.25) and the existence of flat basic sets for finite irreducible Coxeter groups. □

(5.5) *Remark.* If $\dim V = 2$ then

$$\frac{\partial}{\partial T_2}\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1^{q-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus the basic sets constructed in Theorem (3.1) are flat.

We conclude this section with some results about the discriminant which are true for *any* irreducible unitary reflection group $G \subset GL(V)$. Let $\mathscr{B} = \{f_1, \cdots, f_l\}$ be a basic set and let $\mathscr{D} = \{\theta_1, \cdots, \theta_l\}$ be a set of basic derivations. As before let $R = S^G = C[f_1, \cdots, f_l]$. Let Der_R be the R-module of derivations of R. Then Der_R has an R-basis $\{D_{f_i} = \partial/\partial f_i\}$ where $D_{f_i}(f_j) = \delta_{ij}$. If $\theta \in \mathrm{Der}_S$ and $a \in S$ then $g(a\theta) = (ga)(g\theta)$ for all $g \in G$. Thus if $\theta \in \mathrm{Der}_S^G$ then $\theta R \subseteq R$. Let $\bar{\theta}$ be the restriction of θ to R. Then

$$\bar{\theta} = \sum_{i=1}^{l} \theta(f_i) D_{f_i}. \tag{5.6}$$

As in the introduction let $\mathscr{A}$ be the set of reflecting hyperplanes and let $\Delta = \Delta(f_1, \cdots, f_l; \mathscr{B})$ be the discriminant. The following proposition is due to H. Terao; see [22, Thm. D] for the analytic version.

(5.7) PROPOSITION. *Let* $D_R(\Delta) = \{\eta \in \mathrm{Der}_R \mid \eta\Delta \in R\Delta\}$. *If* $\theta_1, \cdots, \theta_l$ *is an R-basis for* Der_S^G *then* $D_R(\Delta)$ *is a free R-module with R-basis* $\bar{\theta}_1, \cdots, \bar{\theta}_l$.

(5.8) *Remark.* In view of (5.6) and (2.2) the columns of the matrix $\boldsymbol{\Delta}(f_1, \cdots, f_l; \mathscr{B}, \mathscr{D})$ defined in (2.4) are the coefficients of the derivations $\bar{\theta}_1, \cdots, \bar{\theta}_l$ when written as R-linear combinations of $D_{f_1}, \cdots, D_{f_l}$.

We have used (1.6) in the proof of the comparison theorem and hence in the proof of (1.7). We show in (5.15) that conversely (1.7) implies (1.6). To do this we must consider gradings of $R = C[f_1, \cdots, f_l]$. There is a natural grading $R = \oplus R_p$ inherited from S in which $\deg f_i = d_i$ and $R_p = R \cap S_p$. If $\alpha = (a_1, \cdots, a_l)$ is any l-tuple of positive integers we may also grade R by letting $\deg f_i = a_i$. Let R_p^α denote the p-th homogeneous component in this grading. Thus $R = \oplus R_p^\alpha$. If $f \in R_p^\alpha$ we say that f is $(a_1, \cdots, a_l)$-homogeneous of degree p. If $f \in R_p^\alpha$ then the Euler formula says

$$\sum_{i=1}^{l} a_i f_i D_{f_i}(f) = pf. \tag{5.9}$$

Let $\theta \in \mathrm{Der}_R$. We say that θ is $(a_1, \cdots, a_l)$-homogeneous of degree r if $\theta(R_p^a) \subseteq R_{p+r}^a$. For example D_{f_i} is $(a_1, \cdots, a_l)$-homogeneous of degree $-a_i$. It follows from (2.5) and (2.6) that in the natural grading the discriminant Δ is $(d_1, \cdots, d_l)$-homogeneous of degree $m + n$.

(5.10) PROPOSITION. *Let $G \subset GL(V)$ be any finite irreducible unitary reflection group. Let $(a_1, \cdots, a_l)$ be an l-tuple of positive integers. Suppose Δ is $(a_1, \cdots, a_l)$-homogeneous of degree p. Then there exists k such that $p = k(m + n)$ and $a_i = kd_i$ for $1 \le i \le l$.*

Proof. Let $\theta_1, \cdots, \theta_l$ be a set of basic derivations of degrees $n_1 - 1, \cdots, n_l - 1$. It follows from (5.7) that $\bar{\theta}_1, \cdots, \bar{\theta}_l$ is an R-basis for $D_R(\Delta)$. Note that $\bar{\theta}_i$ is $(d_1, \cdots, d_l)$-homogeneous of degree $n_i - 1$. We proved in [12] that $1 = n_1 < n_2$ for any irreducible unitary reflection group. Thus

$$\bar{\theta}_1 = d_1 f_1 D_{f_1} + \cdots + d_l f_l D_{f_l} \tag{5.11}$$

is, up to constant, the unique $(d_1, \cdots, d_l)$-homogeneous element of degree 0 in $D_R(\Delta)$. Let $(a_1, \cdots, a_l)$ be any l-tuple of positive integers such that Δ is $(a_1, \cdots, a_l)$-homogeneous of degree p. Define

$$\eta = a_1 f_1 D_{f_1} + \cdots + a_l f_l D_{f_l}\,. \tag{5.12}$$

By the Euler formula (5.9) we have $\eta\Delta = p\Delta$ so $\eta \in D_R(\Delta)$. Clearly η is $(d_1, \cdots, d_l)$-homogeneous of degree 0. Thus there exists $k \neq 0$ such that $\eta = k\bar{\theta}_1$. The conclusion follows. □

Proposition (5.10) may be restated using Milnor's notion [10, p. 75] of weighted homogeneous polynomials. In the $(d_1, \cdots, d_l)$-grading of $C[T_1, \cdots, T_l]$ the polynomial $\Delta = \Delta(T_1, \cdots, T_l; \mathscr{B})$ is homogeneous of degree $(m + n)$ and hence it is weighted homogeneous with weights $((m + n)/d_1, \cdots, (m + n)/d_l)$.

(5.13) COROLLARY. *Let $G \subset GL(V)$ be any irreducible unitary reflection group. The discriminant $\Delta(T_1, \cdots, T_l; \mathscr{B})$ has uniquely determined weights. These weights are also independent of $\mathscr{B}$.* □

(5.14) *Remark.* The assertions in (5.10) and (5.13) need not hold for reducible groups. For example if G is of type $A_1 \times A_1$ acting naturally on C^2 then $Q = x_1 x_2$. If we choose $\mathscr{B} = \{x_1^2, x_2^2\}$ then $\Delta(T_1, T_2; \mathscr{B}) = T_1 T_2$, which is (a_1, a_2)-homogeneous of degree $a_1 + a_2$ for any positive integers a_1, a_2. The weights are $((a_1 + a_2)/a_1, (a_1 + a_2)/a_2)$.

(5.15) COROLLARY. *Let $G \subset GL(V)$ be a Shephard group and let W be the corresponding Coxeter group. Suppose there exist basic sets $\mathscr{B}_G$, $\mathscr{B}_W$ such that $\Delta_G(T_1, \cdots, T_l; \mathscr{B}_G) = \Delta_W(T_1, \cdots, T_l; \mathscr{B}_W)$. Then $d_1^G/d_1^W = \cdots = d_l^G/d_l^W$.*

Proof. Apply (5.13) to both $\Delta_G(T_1, \cdots, T_l; \mathscr{B}_G)$ and $\Delta_W(T_1, \cdots, T_l; \mathscr{B}_W)$, This shows $(m^G + n^G)/d_i^G = (m^W + n^W)/d_i^W$ for $1 \leq i \leq l$. □

Appendix

(A.1) *The pairs* (G, W). In Table 1 we list the pairs (G, W) where G is a Shephard group, W is the corresponding Coxeter group and $G \neq W$. Given the graph $\Gamma(G)$ we obtain $\Gamma(W)$ by omitting the labels on the nodes of $\Gamma(G)$.

(A.2) *Flat basic sets for* $l = 2$. In Table 2 we list basic sets $\mathscr{B} = \{f_1, f_2\}$ for Shephard groups G with $l = 2$ which satisfy (3.1). It follows from (5.5) that these basic sets are flat. The invariants are either given explicitly or in terms of the polynomials Φ, Ψ, t, W, χ, f, H, T of Klein [7].

(A.3) *Conventions.* For the Shephard groups G_{25}, G_{26}, G_{32} we use work of Maschke [9]. The computations were done using MACSYMA. In order to be able to use the polynomials C_6, C_9, C_{12}, C_{18} defined in [9, p. 326] and F_{12}, F_{18}, F_{24}, F_{30} defined in [9, p. 337] we agree to let the basis of V^* be z_1, z_2, z_3 for $l = 3$ and z_0, z_1, z_2, z_3 for $l = 4$. This allows us to use Maschke's convention that in formulas where the subscripts i, $i+1$, $i+2$ appear, they represent the integers 1, 2, 3 in cyclic permutation.

In the description of the basic derivations associated to $\mathscr{B}_G$ for $G = G_{32}$ we need additional polynomials defined below:

$$C_6(0) = C_6, \quad C_9(0) = C_9, \quad C_{12}(0) = C_{12}$$

and for $i = 1, 2, 3$

$$\begin{aligned} C_6(i) &= z_0^6 + z_{i+1}^6 + z_{i+2}^6 + 10(z_{i+1}^3 z_{i+2}^3 - z_0^3 z_{i+1}^3 + z_0^3 z_{i+2}^3) \\ C_9(i) &= z_{i+1}^3 z_{i+2}^6 + z_{i+1}^6 z_{i+2}^3 + z_0^3 z_{i+1}^6 - z_0^6 z_{i+1}^3 - z_0^3 z_{i+2}^6 - z_0^6 z_{i+2}^3 \\ C_{12}(i) &= z_0^{12} + z_{i+1}^{12} + z_{i+2}^{12} - 4(z_{i+1}^3 z_{i+2}^9 + z_{i+1}^9 z_{i+2}^3 + z_0^3 z_{i+2}^9 + z_0^9 z_{i+2}^3 \\ &\quad - z_0^3 z_{i+1}^9 - z_0^9 z_{i+1}^3) + 6(z_0^6 z_{i+1}^6 + z_0^6 z_{i+2}^6 + z_{i+1}^6 z_{i+2}^6) \\ &\quad - 228(z_0^6 z_{i+1}^3 z_{i+2}^3 + z_0^3 z_{i+1}^6 z_{i+2}^3 - z_0^3 z_{i+1}^3 z_{i+2}^6)\,. \end{aligned}$$

For the Coxeter groups D_3, B_3, A_4 we found flat basic sets $\mathscr{B}_W = \{f_1^W, \cdots, f_l^W\}$

using free parameters as described in Section 4. We checked that these basic sets agree, up to multiplication of each f_i^W by a constant c_i, with the flat basic sets found by Saito, Sekiguchi and Yano in [16].

(A.4) *The pair* (G_{25}, D_3). For $W = D_3$ we choose the flat basic set $\mathscr{B}_W$:

$$f_1^W = z_1^2 + z_2^2 + z_3^2 ,$$
$$f_2^W = 8z_1z_2z_3 ,$$
$$f_3^W = 6(z_1^2z_2^2 + z_2^2z_3^2 + z_3^2z_1^2) - (z_1^4 + z_2^4 + z_3^4) .$$

This gives the discriminant matrix

$$\varDelta_W(T_1, T_2, T_3; \mathscr{B}_W) = \begin{bmatrix} 2T_1 & 3T_2 & 4T_3 \\ 3T_2 & 4T_1^2 + 4T_3 & 10T_1T_2 \\ 4T_3 & 10T_1T_2 & 8T_1^3 + 6T_2^2 \end{bmatrix}.$$

For $G = G_{25}$ a basic set $\mathscr{B}_G \sim \mathscr{B}_W$ is given by:

$$f_1^G = C_6 ,$$
$$f_2^G = 32\sqrt{3}\, C_9 .$$
$$f_3^G = 5C_6^2 - 8C_{12} .$$

The basic derivations (2.18) associated to $\mathscr{B}_G$ are:

$$\theta_1 = \sum_{i=1}^{3} z_i D_i .$$
$$\theta_2 = 8\sqrt{3} \sum_{i=1}^{3} (z_{i+2}^3 - z_{i+1}^3) z_i D_i ,$$
$$\theta_3 = 6 \sum_{i=1}^{3} (-z_i^6 + 7(z_{i+1}^6 + z_{i+2}^6) - 14z_i^3(z_{i+1}^3 + z_{i+2}^3) + 42z_{i+1}^3 z_{i+2}^3) z_i D_i .$$

(A.5) *The pair* (G_{26}, B_3). For $W = B_3$ we choose the flat basic set $\mathscr{B}_W$:

$$f_1^W = z_1^2 + z_2^2 + z_3^2 ,$$
$$f_2^W = 3(z_1^4 + z_2^4 + z_3^4) - 6(z_1^2z_2^2 + z_2^2z_3^2 + z_3^2z_1^2) ,$$
$$f_3^W = \frac{14}{3}(z_1^6 + z_2^6 + z_3^6) - 10(z_1^4z_2^2 + z_2^4z_3^2 + z_3^4z_1^2 + z_1^2z_2^4 + z_2^2z_3^4 + z_3^2z_1^4) + 100z_1^2z_2^2z_3^2 .$$

This gives the the discriminant matrix

$$\Delta_W(T_1, T_2, T_3; \mathscr{B}_W) = \begin{bmatrix} 2T_1 & 4T_2 & 6T_3 \\ 4T_2 & 8T_1^3 + 12T_1T_2 + 6T_3 & 32T_1^2T_2 + 8T_2^2 \\ 6T_3 & 32T_1^2T_2 + 8T_2^2 & 32T_1^5 + 40T_1T_2^2 \end{bmatrix}.$$

For $G = G_{26}$ a basic set $\mathscr{B}_G \sim \mathscr{B}_W$ is given by:

$$f_1^G = C_6,$$
$$f_2^G = 12C_{12} - 3C_6^2,$$
$$f_3^G = 96C_{18} + 18C_6^3 - 72C_6C_{12}.$$

The basic derivations (2.18) associated to $\mathscr{B}_G$ are:

$$\theta_1 = \sum_{i=1}^{3} z_i D_i.$$

$$\theta_2 = 18\sum_{i=1}^{3} (z_i^6 - 3(z_{i+1}^6 + z_{i+2}^6) + 2z_i^3(z_{i+1}^3 + z_{i+2}^3) - 26z_{i+1}^3 z_{i+2}^3) z_i D_i.$$

$$\begin{aligned}\theta_3 = 18\sum_{i=1}^{3} (&7z_i^{12} + 28z_i^9(z_{i+1}^3 + z_{i+2}^3) + 162z_i^6(z_{i+1}^6 + z_{i+2}^6) \\ &- 1236z_i^6 z_{i+1}^3 z_{i+2}^3 + 2580z_i^3(z_{i+1}^6 z_{i+2}^3 + z_{i+1}^3 z_{i+2}^6) - 308z_i^3(z_{i+1}^9 + z_{i+2}^9) \\ &- 17(z_{i+1}^{12} + z_{i+2}^{12}) + 676(z_{i+1}^9 z_{i+2}^3 + z_{i+1}^3 z_{i+2}^9) + 90z_{i+1}^6 z_{i+2}^6) z_i D_i.\end{aligned}$$

(A.6) *The pair* (G_{32}, A_4). For $W = A_4$ use the notation

$$s_k = z_0^k + z_1^k + z_2^k + z_3^k + (-1)^k(z_0 + z_1 + z_2 + z_3)^k.$$

We choose the flat basic set $\mathscr{B}_W$:

$$f_1^W = \frac{1}{2}s_2,$$
$$f_2^W = \frac{\sqrt{10}}{3}s_3,$$
$$f_3^W = \frac{5}{2}s_4 - \frac{3}{4}s_2^2.$$
$$f_4^W = 2\sqrt{10}\left(s_5 - \frac{2}{3}s_2s_3\right).$$

This gives the discriminant matrix

$$\Delta_W(T_1, T_2, T_3, T_4; \mathscr{B}_W)$$
$$= \begin{pmatrix} 2T_1 & 3T_2 & 4T_3 & 5T_4 \\ 3T_2 & 4T_3 + 4T_1^2 & 5T_4 + 10T_1T_2 & 12T_1T_3 + 6T_2^2 \\ 4T_3 & 5T_4 + 10T_1T_2 & 12T_1T_3 + 12T_2^2 + 8T_1^3 & 14T_2T_3 + 28T_1^2T_2 \\ 5T_4 & 12T_1T_3 + 6T_2^2 & 14T_2T_3 + 28T_1^2T_2 & 8T_3^2 + 32T_1T_2^2 + 16T_1^4 \end{pmatrix}$$

For $G = G_{32}$ a basic set $\mathscr{B}_G \sim \mathscr{B}_W$ is given by

$$f_1^G = F_{12},$$

$$f_2^G = \frac{4}{3} F_{18},$$

$$f_3^G = 21F_{12}^2 - 25F_{24},$$

$$f_4^G = \frac{8}{5}(11F_{12}F_{18} - 25F_{30}).$$

The basic derivations (2.18) associated to $\mathscr{B}_G$ are:

$$\theta_1 = \sum_{i=0}^{3} z_i D_i,$$

$$\theta_2 = 6 \sum_{i=0}^{3} (-3z_i^6 + 7C_6(i)) z_i D_i,$$

$$\theta_3 = 36 \sum_{i=0}^{3} \left(7z_i^{12} - 26z_i^6 C_6(i) + 2080 z_i^3 C_9(i) + \frac{26}{3} C_6(i)^2 - \frac{65}{3} C_{12}(i)\right) z_i D_i,$$

$$\theta_4 = 216 \sum_{i=0}^{3} \Big(-11z_i^{18} + 57 z_i^{12} C_6(i) - 3040 z_i^9 C_9(i) + 722 z_i^6 C_6(i)^2$$
$$- 1235 z_i^6 C_{12}(i) + 4560 z_i^3 C_6(i) C_9(i) - \frac{38}{3} C_6(i)^3$$
$$+ \frac{95}{3} C_6(i) C_{12}(i) + 7600 C_9(i)^2\Big) z_i D_i.$$

References

[1] Etsuko Bannai, Fundamental groups of the spaces of regular orbits of the finite unitary reflection groups of dimension 2, J. Math. Soc. Japan, **28** (1976), 447–454.

[2] H. S. M. Coxeter, The symmetry groups of the regular complex polygons, Arch., Math., **13** (1962), 86–97.

[3] ——, Finite groups generated by unitary reflections, Abhandlungen Math. Sem. Univ. Hamburg, **31** (1967), 127–135.

[4] ——, Regular Complex Polytopes, Cambridge Univ. Press, 1974.

[5] P. Deligne, Les immeubles des groupes de tresses généralisés, Invent. math., **17** (1972), 273–302.

[6] J. H. Grace and A. Young, The Algebra of Invariants, Cambridge Univ. Press, 1903.

[7] F. Klein, Vorlesungen Über das Ikosaeder, Teubner, Leipzig, 1884.

[8] D. W. Koster, Complex Reflection Groups, Thesis, Univ. of Wisconsin, Madison, 1975.

[9] H. Maschke, Aufstellung des vollen Formensystems einer quaternären Gruppe von 51840 linearen substitutionen, Math. Ann., **33** (1888), 317–344.

[10] J. Milnor, Singular points of complex hypersurfaces, Ann. of Math. Studies 61, Princeton Univ. Press, 1968.

[11] T. Nakamura, A Note on the $K(\pi, 1)$ property of the orbit space of the unitary reflection group $G(m, l, n)$, Sci. Papers College of Arts and Sciences, Univ. Tokyo, **33** (1983), 1–6.
[12] P. Orlik and L. Solomon, Unitary reflection groups and cohomology, Invent. math., **59** (1980), 77–94.
[13] ——, Complexes for reflection groups, in Proc. Midwest Algebraic Geometry Conf., Lecture Notes in Math., No. 862, Springer Verlag (1981), 193–207.
[14] ——, The Hessian map in the invariant theory of reflection groups, Nagoya Math. J., **109** (1988), 1–21.
[15] K. Saito, On a Linear Structure of a Quotient Variety by a Finite Reflexion Group, RIMS Report 288, Kyoto 1979.
[16] K. Saito, T. Yano and J. Sekiguchi, On a certain generator system of the ring of invariants of a finite reflection group, Comm. Algebra, **8** (1980), 373–408.
[17] G. C. Shephard, Regular complex polytopes, Proc. London Math. Soc., (3) **2** (1952), 82–97.
[18] ——, Unitary groups generated by reflections, Canad. J. Math., **5** (1953), 364–383.
[19] G. C. Shephard and J. A. Todd, Finite unitary reflection groups, Canad. J. Math., **6** (1954), 274–304.
[20] T. A. Springer, Regular elements of finite reflection groups, Invent. math., **25** (1974), 159–198.
[21] ——, Invariant Theory, Lecture Notes in Math., No. **585**, Springer Verlag, 1977.
[22] H. Terao, Discriminant of a holomorphic map and logarithmic vector fields, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **30** (1983), 379–391.
[23] H. W. Turnbull, The Theory of Determinants, Matrices and Invariants, 3rd Ed., Dover 1960.
[24] T. Yano and J. Sekiguchi, The microlocal structure of weighted homogeneous polynomials associated with Coxeter systems I, II, Tokyo J. Math., **2** (1979), 193–219; **4** (1981), 1–34.

University of Wisconsin
Madison, WI 53706, U.S.A.