Si Si
Nagoya Math. J.
Vol. 108 (1987), 121-130

A NOTE ON LÉVY'S BROWNIAN MOTION

SI SI

Dedicated to Professor Takeyuki Hida on the occasion of his sixtieth birthday

§ 1. Introduction

The Lévy Brownian motion with multidimensional parameter was introduced and discussed in his book [1] and it is known as the most important random field. Many approaches have been made to the investigation of the Lévy Brownian motion by H.P. McKean [7], Yu. A. Rozanov and others, by using various techniques.

We wish to find out the way of dependency of Lévy's Brownian motion $X(A)$ as the time parameter A runs through a certain domain of the parameter space R^n. The conditional expectation, given the values of the Brownian motion over a manifold in R^n, serves as one of the most significant quantity that shows the dependency. We shall, as a first step, take a one-dimensional manifold, that is a curve C, then $\{X(A); A \in C\}$ determines a continuous Gaussian process with a linear parameter, denoted by $X(t)$, which we are going to discuss in this note.

The well-known theory of the canonical representation of Gaussian processes, introduced by P. Lévy (see, e.g. T. Hida [5]), is now ready to be applied to the Brownian motion when the parameter A is restricted to a plane curve C in R^n. In fact, the canonical representation of such a process plays an essential role in our study. Actually an explicit form of the conditional expectation $E[X(P)/X(A), A \in C]$ can be given by using the canonical representation, as is prescribed by Theorem 2 and Theorem 3. With that expression, we can speak how $X(P)$ is related to $X(A)$ when A runs through the curve C.

It deserves to mention that the canonical representation of $X(t)$ is not only useful for obtaining the conditional expectation, but also it provides an interesting example of the theory of representation of Gaussian processes.

Received July 4, 1986.

§ 2. Canonical representation

This section is devoted to the investigation of the canonical representation of Lévy's Brownian motion on a smooth one dimensional manifold C in R^n. Let us assume in what follows that C is a simple curve of C^3-class being originated from the origin and denote the Brownian motion on C by $X(t)$, where t is taken as the arc length. We have a Gaussian process $X(t)$ with $X(0) = 0$.

Before discussing the canonical representation of $X(t)$ for a general C, we consider the particular case where C is a circle for which the process is denoted by $X(\theta)$, $0 \leq \theta \leq 2\pi$. The representation of $X(\theta)$ is not only interesting for itself but also serving as a step to that for general C.

The covariance function of $X(\theta)$ is

$$\Gamma(\theta, \theta') = \sin(\theta/2)(1 - \cos(\theta'/2)) + (1 + \cos(\theta/2)) \sin(\theta'/2), \qquad \theta \geq \theta'. \tag{2.1}$$

Change the variable θ to t by $t = \tan(\theta/4)$, and multiply by $(1 + t^2)/2$ to have a Gaussian process $Y(t)$. Then the covariance function of $Y(t)$ is

$$\gamma(t, s) = ts^2 + s, \qquad t \geq s. \tag{2.2}$$

We are now in search of the canonical representation of $Y(t)$. If it exists, then the kernel has to be a Goursat kernel of order 2 of the form

$$G(t, u) = tg_1(u) + g_2(u) \tag{2.3}$$

and the covariance function of $Y(t)$ is given by

$$\gamma(t, s) = \int_0^s (tsg_1^2(u) + (t + s)g_1(u)g_2(u) + g_2^2(u))du, \qquad t \geq s. \tag{2.4}$$

Equations (2.2), (2.4) and the fact

$$|G(t, t)|^2 = \lim_{\delta t \to 0} \frac{E(\delta Y(t))^2}{\delta t} = t^2 + 1,$$

give us Riccati's differential equation

$$g_1'(s) + \frac{1}{\sqrt{s^2 + 1}} g_1^2(s) + \frac{s}{s^2 + 1} g_1(s) + \frac{2}{\sqrt{s^2 + 1}} = 0. \tag{2.5}$$

A special solution of which is $\frac{\sqrt{s^2 + 1}}{s}$, therefore we have the solution

$$g_1(s) = \frac{\sqrt{s^2+1}}{s} - \frac{1}{s\sqrt{s^2+1}(1+s\tan^{-1}s)}\,, \tag{2.6}$$

which we are looking for. It follows that

$$g_2(s) = \{\sqrt{s^2+1}(1+s\tan^{-1}s)\}^{-1} \tag{2.7}$$

and then a representation of $Y(t)$ is

$$Y(t) = \int_0^t (tg_1(u) + g_2(u))d\tilde{B}(u)\,, \tag{2.8}$$

where $g_1(u)$ and $g_2(u)$ are given by (2.6) and (2.7) respectively.

The kernel function of the above representation satisfies, as is easily seen, the criterion for *a proper canonical representation,* given by T. Hida [5].

By changing back the parameter t to θ, we obtain the canonical representation of our original Brownian motion:

$$\begin{aligned} X(\theta) = \int_0^\theta \Big\{ &\sin(\theta/2)\Big(\operatorname{cosec}(\theta'/2) - \frac{\cot(\theta'/4)}{2}h(\theta')\Big) \\ &+ \cos^2(\theta/4)h(\theta')\Big\} dB(\theta')\,, \end{aligned} \tag{2.9}$$

where

$$h(\theta) = \{1 + (\theta/4)\tan(\theta/4)\}^{-1}\,.$$

This result tells us that $X(\theta)$ is a double Markov Gaussian process.

Hence we have proved the following theorem.

THEOREM 1. *The Brownian motion on a circle is a double Markov Gaussian process and has the canonical representation given by* (2.9).

Remark. There is a double Markov Gaussian process, in the restricted sense, expressed in the form

$$\begin{aligned} Z(\theta) = \int_0^\theta \Big\{ &\Big(\operatorname{cosec}(\theta'/2) - \frac{\cot(\theta'/4)}{2}h(\theta')\Big) \\ &- \Big(\frac{1}{2}\tan(\theta/4) + \frac{\theta}{8}\sec^2(\theta/4)\Big)h(\theta')\Big\} dB(\theta') \end{aligned} \tag{2.10}$$

such that

$$X(\theta) = -4\frac{d}{d\theta}(\cos^2(\theta/4)Z(\theta)). \tag{2.11}$$

We have

$$dB(\theta) = -\frac{4}{h(\theta)} d\Big(\cos^2(\theta/4)h(\theta)\frac{d}{d\theta}Z(\theta)\Big), \tag{2.12}$$

and by noting

$$Z(\theta) = -\frac{1}{4}\sec^2(\theta/4)\int_0^\theta X(\theta')d\theta',$$

we can conclude that $dB(\theta)$, in the expression (2.9), is obtained as a function of $X(\theta')$, $\theta' \leq \theta + \delta\theta$.

Since the canonical representation of the Brownian motion on a circle has been obtained, we can now think of the representation of a Brownian motion on a curve C, in a class $\mathbf{C}$, where

$$\mathbf{C} = \{\text{simple plane curve, } C^3\text{-manifold}\}. \tag{2.13}$$

The curve C can be expressed in terms of a parameter t, the arc length, as

$$C = \{A(t);\ 0 \leq t \leq T,\ A(0) = 0\},$$

where T may be finite or infinite.

The curvature of the curve C is locally bounded, so we can approximate a part of C (within $A(t)$ and $A(t+\delta t)$) by an arc of a circle S (If the curvature of C is zero at $A(t)$, then S is taken to be a straight line). We denote the Brownian motion on the part of the circle S by $X_1(t)$ and have an evaluation

$$E[X(t+\delta t) - X_1(t+\delta t)]^2 = O(\delta t)^3. \tag{2.14}$$

So we have

$$dX(t) = dX_1(t) + c_t\xi_t(dt)^k; \qquad k \geq 3/2, \tag{2.15}$$

where c_t is bounded in t and ξ_t is a standard Gaussian random variable, and hence it can be seen that the stochastic integral $\int f(u)dX(u)$ is well defined for any continuous function f and that the multiplicity of $X(t)$ is one.

It is known that the conditional expectation $E[dX(t)/X(s),\ s \leq t]$ can be expressed as a stochastic integral with respect to $dX(u)$ and $dX(t)$ itself is a sum

$$dX(t) = W_t + dt\int_0^t g(t, u)dX(u)\,,$$

where W_t is independent of $\{X(u);\ u \leq t\}$. From (2.15), one can see that W_t is of order $\sqrt{dt}$, and hence we may write

$$dX(t) = dB(t) + dt\int_0^t g(t, u)dX(u) + o(dt) \tag{2.16}$$

(see P. Lévy [2]). The kernel function $g(t, u)$ can be determined by the following proposition.

PROPOSITION. *The function $g(t, u)$, satisfying* (2.16), *is the solution of the Fredholm integral equation*

$$\tilde{\gamma}_t(s) = \int_0^t \tilde{g}_t(u)\gamma(s, u)du - \tilde{g}_t(s)\,, \tag{2.17}$$

in which $\tilde{g}_t(u)$ and $\tilde{\gamma}_t(s)$ denote $g(t, u)$ and $\gamma(t, s)$ respectively for fixed t and $\gamma(s, u)$ is a symmetric continuous kernel, determined by

$$\frac{\partial^2}{\partial u \partial s}\Gamma(u, s) = \gamma(u, s) - \delta(u - s)\,, \tag{2.18}$$

where $\Gamma(u, s)$ is the covariance of $X(u)$ and $X(s)$ and where δ is the delta function.

The main part of the proof is to establish (2.18) which actually comes from

$$\Gamma(u, s) = \frac{1}{2}(\rho(0, u) + \rho(0, s) - \rho(u, s))\,, \tag{2.19}$$

$\rho(u, s)$ being the distance between $A(u)$ and $A(s)$, and from

$$\rho(s + h, s) = |h| + o(h)\,, \tag{2.20}$$

which shows that $\rho(u, s)$ has a singularity at $u = s$.

From (2.16) we obtain, by the iterative substitution, the equation

$$\begin{aligned} dX(t) = dB(t) &+ dt\int_0^t g(t, u)dB(u) \\ &+ dt\int_0^t dB(u)\int_u^t du_1 g(t, u_1)g(u_1, u) \\ &+ dt\int_0^t dB(u)\int_u^t du_1 g(t, u_1)\int_u^{u_1} du_2 g(u_1, u)g(u_1, u_2) \\ &+ \cdots \end{aligned} \tag{2.21}$$

and then the representation of $X(\tau)$ is obtained as

$$X(\tau) = \int_0^\tau F(\tau, u) dB(u) \tag{2.22}$$

with

$$\begin{aligned} F(\tau, u) = 1 &+ \int_u^\tau g(t, u)dt + \int_u^\tau dt \int_u^t du_1 g(t, u_1) g(u_1, u) \\ &+ \int_u^\tau dt \int_u^t du_1 g(t, u_1) \int_u^{u_1} du_2 g(u_1, u) g(u_1, u_2) + \cdots, \end{aligned} \tag{2.23}$$

satisfying $|F(\tau, u)| \leq \exp(k(\tau - u))$, where k is the bound of $g(t, u)$ and $F(\tau, \cdot) \in L^2$ for every τ. This kernel F is proper canonical, as is easily seen from (2.16) and (2.22).

Summing up, the following theorem is obtained.

THEOREM 2. *The Brownian motion on a simple plane curve of C^3-class has the canonical representation and the kernel of which is given by* (2.23).

§ 3. Conditional expectation

In this section, we discuss the conditional expectation of the Lévy Brownian motion to show the way of its dependency. When the Brownian motion on a curve is given, one may ask if the conditional expectation could be expressed in the following form:

$$E[X(P)/X(A),\ A \in C] = \int_{A \in C} f(P, A) X(A) dA\,, \tag{3.1}$$

in which P is a point which does not lie on a given curve C and dA is a line element of C.

In general, the conditional expectation may not be expressed in the form (3.1) by using an ordinary function f. Here are some illustrative examples (see, in particular, examples 1(b), 1(c)).

EXAMPLE 1(a). Let us assume that the Brownian motion on the entire line is given and P be a point which is not on the line. If $|OP| = t$ and the inclination of OP from the given line is θ,

$$E[X(P)/X(x),\ -\infty < x < \infty;\ X(0) = 0] = \int_{-\infty}^{+\infty} f(P, x) X(x) dx \tag{3.2}$$

such that

$$f(P, x) = \frac{t^2 \sin^2\theta}{2\rho(x, t, \theta)^3}, \tag{3.3}$$

where

$$\rho(x, t, \theta) = (x^2 + t^2 - 2|x| t \cos\theta)^{1/2}. \tag{3.4}$$

(This notation ρ will be often used in what follows.)

If the whole line is replaced by the entire half line, the kernel function $f(P, x)$ does not change. However, if we take a line segment, the kernel function would involve the delta functions sitting at the boundary points (different from the origin).

EXAMPLE 1(b). Suppose the Brownian motion on a finite interval $[a, b]$ is given. To fix the idea, we assume that $0 < a < b$. Then we obtain

$$f(P, x) = \frac{t^2 \sin^2\theta}{2\rho(x, t, \theta)^3} + \alpha\delta_a(x) + \beta\delta_b(x), \tag{3.5}$$

where

$$\alpha = \frac{t}{2a}\left(1 - \frac{t - a\cos\theta}{\rho(a, t, \theta)}\right), \qquad \beta = \frac{1}{2}\left(1 - \frac{b - t\cos\theta}{\rho(b, t, \theta)}\right).$$

EXAMPLE 1(c). When the Brownian motion is given on $J_n = \sum_{i=1}^n I_i$, $I_i = [a_i, b_i]$ being disjoint intervals with $0 < a_i < b_i$, then

$$E[X(P)/X(x);\ x \in J_n] = \sum_{i=1}^n \int_{a_i}^{b_i} \frac{t^2 \sin^2\theta}{2\rho(x, t, \theta)^3} X(x)dx + \sum_{i=1}^n (\alpha_i X(a_i) + \beta_i X(b_i)), \tag{3.6}$$

where

$$\alpha_i = \frac{1}{2(a_i - b_{i-1})}\left(\rho(b_{i-1}, t, \theta) - \frac{g(a_i, b_{i-1}, t, \theta)}{\rho(a_i, t, \theta)}\right); \qquad 1 \le i \le n,$$

$$\beta_i = \frac{1}{2(a_{i+1} - b_i)}\left(\rho(a_{i+1}, t, \theta) - \frac{g(a_{i+1}, b_i, t, \theta)}{\rho(b_i, t, \theta)}\right); \qquad 1 \le i \le n-1,$$

$$\beta_n = \frac{1}{2}\left(1 - \frac{b_n - t\cos\theta}{\rho(b_n, t, \theta)}\right),$$

in which $b_0 = 0$ and $g(a, b, t, \theta) = t^2 - (a + b)t\cos\theta + ab$.

EXAMPLE 2. A special interest may be found in the case where C

is a circle. The conditional expectation of $X(P)$, P being an interior point of C, can be expressed in the form (3.1):

$$E[X(P)/X(\theta);\ 0 \leq \theta \leq 2\pi,\ X(0) = 0] = \int_0^{2\pi} f(P, \theta)X(\theta)d\theta, \tag{3.7}$$

where

$$f(P, \theta) = \frac{(t^2 - x^2)^2}{8t\rho(x, t, \theta - \beta)^3} + \frac{1}{2\pi}\left(1 - \frac{t + x}{2t}E\left(\frac{\pi}{2}, \frac{2\sqrt{tx}}{t + x}\right)\right). \tag{3.8}$$

In the above expression, E is the elliptic function t is the radius of C, $x = |MP|$ and $\beta = \widehat{OMP}$, where M is the centre of C and O is the point on the circle for $\theta = 0$.

By using (2.9), the conditional expectation can be expressed, as is expected, in the form

$$E[X(P)/X(\theta);\ 0 \leq \theta \leq 2\pi,\ X(0) = 0] = \int_0^{2\pi} g_C(P, \theta)dB(\theta), \tag{3.9}$$

where

$$g_C(P, \theta) = \int_\theta^{2\pi} f(P, \alpha)F(\alpha, \theta)d\alpha. \tag{3.10}$$

Also for an exterior point, we can obtain the kernel functions for the conditional expectations (3.7) and (3.9). They are similar to those for an interior point.

Remark. If we take a circle-segment instead of a circle, the kernel function for the conditional expectation involves the delta functions at the boundary points. It also holds for a general curve with boundary points.

The conditional expectation of $X(P)$, given the Brownian motion $X(A)$ on a given curve C, is a linear functional of $X(A)$'s, but the kernel function $f(P, A)$ in (3.1) may not be a usual function as we have seen so far. While, if we use $dB(t)$, appeared in the canonical representation of $X(t)$, the conditional expectation can always be expressed as the Wiener integral, the kernel function of which is an L^2-function. The conditional expectation with that expression is obtained by the following theorem.

THEOREM 3. *There exists a kernel function $g_C(P, s)$ such that the conditional expectation of a Brownian motion at a point P, which is not* *on C, is expressed in the form*

$$E[X(P)/X(s);\ 0 \leq s \leq T] = \int_0^T g_C(P, s)dB(s)\,, \tag{3.11}$$

where B(s) is the Brownian motion obtained in Theorem 2. The kernel function g_C *is obtained as*

$$g_C(P, \tau) = \left((I - g)\frac{d}{d\tau}\right)\varphi_P(\tau)\,, \tag{3.12}$$

where I is the identity operator and g is the integral operator defined by the function obtained from the proposition in Section 2.

Proof. Since the conditional expectation should be a linear functional of $\dot{B}(s);\ s \leq T$, it can be expressed in the form (3.11). We can find g_C from the results obtained in Section 2.

According to the property of conditional expectation, we have

$$E[X(P)X(\tau)] = \int_0^T g_C(P, s)E[X(\tau)dB(s)]\,, \qquad 0 \leq \tau \leq T,$$

and then equation (2.22) gives us,

$$E[X(P)X(\tau)] = \int_0^\tau g_C(P, s)F(\tau, s)ds\,.$$

Denoting $E[X(P)X(\tau)]$ and $g_C(P, s)$ by $\varphi_P(\tau)$ and $g_P(s)$ respectively, the above equation becomes

$$\varphi_P(\tau) = \int_0^\tau F(\tau, s)g_P(s)ds\,.$$

Since the kernel F is canonical, which means that F is viewed as a causally invertible operator, we obtain

$$g_C(P, \tau) = F^{-1}\varphi_P(\tau) \tag{3.13}$$

from which the assertion (3.12) follows by using (2.22) and (2.23).

Before closing this paper, we would like to make a concluding remark. As we can see in (3.11), the kernel function $g_C(P, s)$ of the conditional expectation may be viewed as a functional of the curve C. We expect that the functional will tell us a kind of dependency of the Lévy Brownian motion. We are therefore interested in the variation of g_C when C changes, but not yet in a position to discuss it. However, Example 2 gives us an aspect of this question. The kernel function is a sum of two functions f_C^1 and f_C^2 such that f_C^1 is the part proportional to ρ^{-3} while f_C^2 is expressed in terms of the elliptic function, independent of β and θ. They vary as the circle C rotates around the origin O. Hence, there arises a variational question which will be discussed in a separate paper.

ACKNOWLEDGEMENT. The author expresses her thanks to the referee, whose comment has improved the present paper.

REFERENCES

[1] P. Lévy, Processus stochastiques et Mouvment Brownein, Gautier Villars, Paris, (1948).
[2] ——, Random functions: General theory with special reference to Laplacian random functions, Univ. of California Publications in Statistics, **1** (1953), 331–390.
[3] ——, A special problem of Brownian motion, and a general theory of Gaussian Random functions, Proceeding of the Third Berkeley Symposium on Math. Stat. and Prob., **2** (1956), 133–175.
[4] ——, Random functions: A Laplacian random function depending on a point of Hilbert space, Univ. of California Publications in Statistics, **2**, No. 10 (1956), 195–206.
[5] T. Hida, Canonical representation of Gaussian processes and their applications, Mem. Coll. Sci. Univ. Kyoto, **33** (1960), 258–351.
[6] T. Hida and M. Hitsuda, Gaussian Processes, Kinokuniya Pub. Co. 1976 (in Japanese).
[7] H. P. McKean, Jr., Brownian motion with a several-dimensional time, Theory Probab. Appl., **8** (1963), 335–354.

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464
Japan

and

Department of Mathematics
Rangoon University
Rangoon, Burma