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Nagoya Math. J.
Vol. 108 (1987), 67-76

ITO'S FORMULA AND LÉVY'S LAPLACIAN

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§ 1. Introduction

The class of normal functionals

$$\int \cdots \int_{R^n} f(x_1, \cdots, x_n) : \dot{B}_{x_1}^{p_1} \cdots \dot{B}_{x_n}^{p_n} : dx_1 \cdots dx_n, \qquad f \in L^1(\boldsymbol{R}^n),$$
$$(p_1, \cdots, p_n) \in (\boldsymbol{N} \cup \{0\})^n,$$

is, as is well known, adapted to the domain of Lévy's Laplacian and plays important roles in the works by P. Lévy and T. Hida (cf. [1], [2] and [8]), where $\dot{B}_x$ denotes one-dimensional parameter white noise and $:\dot{B}_{x_1}^{p_1} \cdots \dot{B}_{x_n}^{p_n}:$ denotes the renormalization of $\dot{B}_{x_1}^{p_1} \cdots \dot{B}_{x_n}^{p_n}$.

We are interested in a generalization of this class to that of generalized functionals of two-dimensional parameter white noise $\{W(t, x); (t, x) \in \boldsymbol{R}^2\}$, which is a generalized stochastic process with the characteristic functional

$$C(\xi) = E(\exp\{i\langle W, \xi\rangle\}) = \exp\left\{-\frac{1}{2}\|\xi\|^2\right\}, \qquad \xi \in S(\boldsymbol{R}^2).$$

As in the case [1], we are able to introduce, in Section 2, a space $(L^2)^{(-\alpha)}$ of generalized functionals and the $\mathscr{S}$-transform on $(L^2)^{(\alpha)}$ for every $\alpha > 0$. Then the calculus in terms of the white noise $W(t, x)$ will quickly be discussed.

The main purpose of this paper is to investigate how Lévy's Laplacian appears in Itô's formula for generalized Brownian functionals depending on t. To this end we first discuss a class of generalized Brownian functionals, often without any renormalization, having interest in its own right. For instance, a monomial $B_x(t)^p$ is sometimes more significant rather than the renormalized quantity $:B_x(t)^p: \equiv :\left\{\int_0^t W(r, x)dr\right\}^p:$ which is living in $(L^2)^{(-\alpha)}$. We are therefore led to construct a new space $[\![L^2]\!]^{(-\alpha)}$,

Received April 11, 1986.

in which $B_x(t)^p$ lives, in Section 3. The $\mathscr{S}$-transform and the $W(t, x)$-differentiation can be introduced on $[\![L^2]\!]^{(-\alpha)}$ for every $\alpha > 0$ in a similar manner to those in [6]. The symbol $1/dx$ which has often been used by H.H. Kuo (cf. [7]) is now understood as a shift operator acting on $[\![L^2]\!]^{(-\alpha)}$.

In Section 4, we define $B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n}$ by

$$B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n} = \Big[\!\!\Big[:B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n}:, \frac{p_1(p_1-1)}{2}(\cdot):B_{x_1}(\cdot)^{p_1-2}B_{x_2}(\cdot)^{p_2}\cdots B_{x_n}(\cdot)^{p_n}: + \cdots + \frac{p_n(p_n-1)}{2}(\cdot):B_{x_1}(\cdot)^{p_1}\cdots B_{x_{n-1}}(\cdot)^{p_{n-1}}B_{x_n}(\cdot)^{p_n-2}: \Big]\!\!\Big],$$

for any $n \in \boldsymbol{N}$, $(p_1, \cdots, p_n) \in (\boldsymbol{N} \cup \{0\})^n$, and $x_1, \cdots, x_n \in \boldsymbol{R}$, and we introduce a class $\mathscr{D}_L$ of generalized functionals as follows:

$$\mathscr{D}_L = LS\Big\{\int\cdots\int_{\boldsymbol{R}^n} f(x_1, \cdots, x_n)B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n}dx_1\cdots dx_n;\ f \in L^1(\boldsymbol{R}^n),\ (p_1, \cdots, p_n) \in (\boldsymbol{N} \cup \{0\})^n,\ n = 0, 1, 2, \cdots\Big\},$$

where LS means the linear span. Then it holds that $\mathscr{D}_L$ is contained in $\mathscr{C}([0, \infty) \to [\![L^2]\!]^{(-\alpha)})$ for any $\alpha > 5/6$ and that for $\phi(B(\cdot))$ in $\mathscr{D}_L$, the $W(t, x)$-derivative $\partial_{s,x}\phi(B(t))$ is independent of the choice of s in an open interval $(0, t)$. With these property, Itô's formula for elements in $\mathscr{D}_L$ is proved in Theorem:

If $\phi(B(\cdot))$ is in $\mathscr{D}_L$, then

$$\text{(4.7)} \quad \phi(B(t)) - \phi(B(s)) = \int_{\boldsymbol{R}}\int_s^t \partial_x\phi(B(u))dB_x(u)dx + \frac{1}{2}\cdot\frac{1}{dx}\cdot\int_s^t \Delta_L\phi(B(u))du$$

holds for $0 \leqq s \leqq t$.

Finally, we should like to note that the Lévy's Laplacian Δ_L is involved in the Itô's formula only for generalized Brownian functionals and that Δ_L, in fact, annihilates ordinary Brownian functionals.

§ 2. Preliminaries

1°) Let $S(\boldsymbol{R}^2)$ be the Schwartz space on $\boldsymbol{R}^2$ and $S^*(\boldsymbol{R}^2)$ be the dual space of $S(\boldsymbol{R}^2)$. Let μ be the measure of white noise introduced on $S^*(\boldsymbol{R}^2)$ by the characteristic functional

$$C(\xi) = \exp\Big\{-\frac{1}{2}\|\xi\|^2\Big\}, \qquad \xi \in S(\boldsymbol{R}^2),$$

where $\|\cdot\|$ denotes the $L(\boldsymbol{R}^2)$-norm and set $(L^2) = L^2(S^*(\boldsymbol{R}^2), \mu)$. The Hilbert space (L^2) admits the Wiener-Itô decomposition

$$(L^2) = \sum_{n=0}^{\infty} \oplus \mathscr{H}_n ,$$

where $\mathscr{H}_n$ is the space of n-ple Wiener integrals, i.e.

$$\mathscr{H}_n = \left\{ \int \cdots \int_{\boldsymbol{R}^{2n}} F(t_1, x_1, \cdots, t_n, x_n) W(t_1, x_1) \cdots W(t_n, x_n) dt_1 dx_1 \cdots dt_n dx_n ; \right. \\ \left. F \in \hat{L}^2((\boldsymbol{R}^2)^n) \right\} ,$$

the space $\hat{L}^2((\boldsymbol{R}^2)^n)$ being the totality of symmetric $L^2((\boldsymbol{R}^2)^n)$-functions.

The $\mathscr{S}$-transform of a Brownian functional ϕ in (L^2) is defined by

$$(\mathscr{S}\phi)(\xi) = \int_{S^*(\boldsymbol{R}^2)} \phi(W + \xi) d\mu(W) , \qquad \xi \in S(\boldsymbol{R}^2) .$$

It can be easily checked that

$$\mathscr{S}\mathscr{H}_n = \left\{ \int \cdots \int_{\boldsymbol{R}^{2n}} F(t_1, x_1, \cdots, t_n, x_n) \xi(t_1, x_1) \cdots \xi(t_n, x_n) dt_1 dx_1 \cdots dt_n dx_n ; \right. \\ \left. F \in \hat{L}^2((\boldsymbol{R}^2)^n) \right\} .$$

We denote the space $\mathscr{S}\mathscr{H}_n$ by $\boldsymbol{F}_n$.

2°) We then come to a background in order to introduce a class of normal functionals of $\boldsymbol{R}^2$-parameter. Take a complete orthonormal system (c.o.n.s.) in $L^2(\boldsymbol{R}^2)$ formed by

$$\xi_{(j,k)} = \xi_j \otimes \xi_k , \quad \xi_j(u) = (2^j j! \sqrt{\pi})^{-1/2} \cdot H_j(u) \cdot e^{-u^2/2}, \quad j, k = 0, 1, 2, \cdots ,$$

where H_j denotes the Hermite polynomial of degree j. With this c.o.n.s., we introduce a Hilbertian norm $\|\cdot\|_{\alpha,n}$ by

$$\|f\|_{\alpha,n}^2 = \sum_{j_1,k_1,\cdots,j_n,k_n=0}^{\infty} \left\{ \prod_{\nu=1}^{n} (2j_\nu + 1)(2k_\nu + 1) \right\}^{\alpha} \cdot (f, \xi_{(j_1,k_1)} \otimes \cdots \otimes \xi_{(j_n,k_n)})^2 , \\ f \in L^2((\boldsymbol{R}^2)^n) , \qquad \alpha > 0 ,$$

where $(\cdot, \cdot)$ denotes the $L^2((\boldsymbol{R}^2)^n)$-inner product. For $\alpha > 0$ we form Hilbert spaces

$$S_\alpha((\boldsymbol{R}^2)^n) = \{ f \in L^2((\boldsymbol{R}^2)^n);\ \|f\|_{\alpha,n} < \infty \} , \\ \hat{S}_\alpha((\boldsymbol{R}^2)^n) = \{ f \in S_\alpha((\boldsymbol{R}^2)^n);\ f \text{ is symmetric} \} , \quad \alpha > 0 .$$

Let $\hat{S}_{-\alpha}((\boldsymbol{R}^2)^n)$ be the dual space of $\hat{S}_\alpha((\boldsymbol{R}^2)^n)$ for $\alpha > 0$. The space $\boldsymbol{F}_n^{(\alpha)}$ of U-functionals is introduced in the same manner as in [2],

$$F_n^{(\alpha)} = \left\{\int\cdots\int_{R^{2n}} F(t_1, x_1, \cdots, t_n, x_n)\xi(t_1, x_1)\cdots\xi(t_n, x_n)dt_1dx_1\cdots dt_ndx_n\ ;\right.$$
$$\left. F \in \hat{S}_\alpha((R^2)^n)\right\}, \qquad \alpha > 0 .$$

With the help of the $\mathcal{S}$-transform, we can define a subspace $\mathcal{H}_n^{(\alpha)}$ by

$$\mathcal{H}_n^{(\alpha)} = \mathcal{S}^{-1}F_n^{(\alpha)} .$$

For U_i in $F_n^{(\alpha)}$ with kernel F_i, $i = 1, 2$, we have

$$(U_1, U_2)_{F_n^{(\alpha)}} = n!(F_1, F_2)_{S_\alpha((R^2)^n)} .$$

This is rephrased in the form

$$(\phi_1, \phi_2)_{\mathcal{H}_n^{(\alpha)}} = (\mathcal{S}\phi_1, \mathcal{S}\phi_2)_{F_n^{(\alpha)}}, \qquad \phi_1, \phi_2 \in \mathcal{H}_n^{(\alpha)} .$$

Let $\mathcal{H}_n^{(-\alpha)}$, $\alpha > 0$, be the dual space of $\mathcal{H}_n^{(\alpha)}$, and define the spaces $(L^2)^{(\alpha)} = \sum_{n=0}^\infty \oplus \mathcal{H}_n^{(\alpha)}$ and $(L^2)^{(-\alpha)} = \sum_{n=0}^\infty \oplus \mathcal{H}_n^{(-\alpha)}$ to obtain a Gel'fand triple:

$$(L^2)^{(\alpha)} \subset (L^2) \subset (L^2)^{(-\alpha)} .$$

The $\mathcal{S}$-transform can be extended to the space $(L^2)^{(-\alpha)}$ to have

$$\mathcal{S}\mathcal{H}_n^{(-\alpha)} = \{\langle F, \xi^{\otimes n}\rangle;\ F \in \hat{S}_{-\alpha}((R^2)^n)\},$$

which is denoted by $F_n^{(-\alpha)}$.

3°) The $W(t, x)$-derivative $\partial_{t,x}\phi \equiv \partial\phi/\partial W(t, x)$ of a generalized Brownian functional ϕ is defined by

$$\partial_{t,x}\phi = \mathcal{S}^{-1}\frac{\delta}{\delta\xi(t, x)}\mathcal{S}\phi, \qquad (t, x) \in R^2 ,$$

where $(\delta/\delta\xi(t, x))\mathcal{S}\phi$ denotes the functional derivative of $\mathcal{S}\phi$. If the second variation of the $\mathcal{S}$-transform $\mathcal{S}\phi$ of ϕ in $(L^2)^{(-\alpha)}$ is given by a following form

$$(\delta^2\mathcal{S}\phi)_\xi(\eta, \zeta) = \iint_{R^2} U_1''(\xi;\, t, x)\eta(t, x)\zeta(t, x)dtdx$$
$$+ \iiiint_{R^4} U_2''(\xi;\, t, x, s, y)\eta(t, x)\zeta(s, y)dtdxdsdy, \qquad \xi, \eta, \zeta \in S(R^2),$$

then the Lévy's Laplacian Δ_L is defined by

$$\Delta_L\phi = \mathcal{S}^{-1}\left\{\iint_{R^2} U_1''(\xi;\, t, x)dtdx\right\} \qquad \text{(see [2], [7] and [8]).}$$

§ 3. The spaces of generalized functionals

In this section, we construct the various spaces of generalized functionals, on which the $W(t, x)$-differentiation, the operator $1/dx$ and other related notations are introduced.

We introduce the spaces $(\tilde{L}^2)^{(\alpha)}$ and $\tilde{F}^{(\alpha)}$ for every $\alpha \in \boldsymbol{R}$:

$$(\tilde{L}^2)^{(\alpha)} = \left\{\phi = (\phi_1, \phi_2, \cdots, \phi_n, \cdots);\ \phi_j \in (L^2)^{(\alpha)},\ j = 1, 2, \cdots, n, \cdots,\ \sum_{j=1}^{\infty} \|\phi_j\|^2_{(L^2)^{(\alpha)}} < \infty\right\},$$

$$\tilde{\boldsymbol{F}}^{(\alpha)} = \left\{f = (f_1, f_2, \cdots, f_n, \cdots);\ f_j \in \boldsymbol{F}^{(\alpha)},\ j = 1, 2, \cdots, n, \cdots,\ \sum_{j=1}^{\infty} \|f_j\|^2_{\boldsymbol{F}^{(\alpha)}} < \infty\right\},$$

$$\boldsymbol{F}^{(\alpha)} = \sum_{n=0}^{\infty} \oplus \boldsymbol{F}_n^{(\alpha)}.$$

The spaces $(\tilde{L}^2)^{(\alpha)}$ and $\tilde{\boldsymbol{F}}^{(\alpha)}$ are Hilbert spaces with the inner products

$$(\phi, \psi)_{(\tilde{L}^2)^{(\alpha)}} = \sum_{j=1}^{\infty} (\phi_j, \psi_j)_{(L^2)^{(\alpha)}}, \quad \phi = (\phi_1, \phi_2, \cdots), \quad \psi = (\psi_1, \psi_2, \cdots) \in (\tilde{L}^2)^{(\alpha)}$$

and

$$(f, g)_{\tilde{\boldsymbol{F}}^{(\alpha)}} = \sum_{j=1}^{\infty} (f_j, g_j)_{\boldsymbol{F}^{(\alpha)}}, \quad f = (f_1, f_2, \cdots), \quad g = (g_1, g_2, \cdots) \in \tilde{\boldsymbol{F}}^{(\alpha)}$$

respectively. We define the spaces $(\tilde{L}^2)_*^{(\alpha)}$ and $\tilde{\boldsymbol{F}}_*^{(\alpha)}$ for every $\alpha \in \boldsymbol{R}$ as follows:

$$(\tilde{L}^2)_*^{(\alpha)} = \{\phi = (\phi_1, \phi_2, \cdots) \in (\tilde{L}^2)^{(\alpha)};\ \phi_1 = \phi_2 = 0\},$$
$$\tilde{\boldsymbol{F}}_*^{(\alpha)} = \{f = (f_1, f_2, \cdots) \in \tilde{\boldsymbol{F}}^{(\alpha)};\ f_1 = f_2 = 0\}.$$

The spaces $(\tilde{L}^2)_*^{(\alpha)}$ and $\tilde{\boldsymbol{F}}_*^{(\alpha)}$ are closed subspaces of $(\tilde{L}^2)^{(\alpha)}$ and $\tilde{\boldsymbol{F}}^{(\alpha)}$ respectively. Set $[\![L^2]\!]^{(\alpha)} = (\tilde{L}^2)^{(\alpha)}/(\tilde{L}^2)_*^{(\alpha)}$ and set $[\![\boldsymbol{F}]\!]^{(\alpha)} = \tilde{\boldsymbol{F}}^{(\alpha)}/\tilde{\boldsymbol{F}}_*^{(\alpha)}$. Both $[\![L^2]\!]^{(\alpha)}$ and $[\![\boldsymbol{F}]\!]^{(\alpha)}$ are Hilbert spaces with the norms

$$\|\phi + (\tilde{L}^2)_*^{(\alpha)}\|_{[[L^2]]^{(\alpha)}} = \inf\{\|\psi\|_{(\tilde{L}^2)^{(\alpha)}};\ \psi \in \phi + (\tilde{L}^2)_*^{(\alpha)}\}, \quad \phi \in (\tilde{L}^2)^{(\alpha)},$$

and

$$\|f + \tilde{\boldsymbol{F}}_*^{(\alpha)}\|_{[[\boldsymbol{F}]]^{(\alpha)}} = \inf\{\|g\|_{\tilde{\boldsymbol{F}}^{(\alpha)}};\ g \in f + \tilde{\boldsymbol{F}}_*^{(\alpha)}\}, \quad f \in \tilde{\boldsymbol{F}}^{(\alpha)},$$

respectively. The spaces $[\![L^2]\!]^{(\alpha)}$ and $[\![L^2]\!]^{(-\alpha)}$ are mutually dual by the canonical bilinear form

$$\langle \Phi + (\tilde{L}^2)_*^{(-\alpha)}, \phi + (\tilde{L}^2)_*^{(\alpha)} \rangle_{[[L^2]]^{(-\alpha)}, [[L^2]]^{(\alpha)}} = \langle \Phi_1, \phi_1 \rangle + \langle \Phi_2, \phi_2 \rangle,$$
$$\Phi = (\Phi_1, \Phi_2, \cdots) \in (\tilde{L}^2)^{(-\alpha)}, \quad \phi = (\phi_1, \phi_2, \cdots) \in (\tilde{L}^2)^{(\alpha)},$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical bilinear form connecting $(L^2)^{(-\alpha)}$ and $(L^2)^{(\alpha)}$. Any element $\phi + (\tilde{L}^2)^{(\alpha)}_*$, with $\phi = (\phi_1, \phi_2, \cdots) \in (\tilde{L}^2)^{(\alpha)}$, may be represented as $[\![\phi_1, \phi_2]\!]$. For any $\alpha > 0$, the spaces $[\![L^2]\!]^{(-\alpha)}$ and $[\![L^2]\!]^{(\alpha)}$ are viewed as the space of generalized functionals and the space of testing functionals respectively.

The $\mathscr{S}$-transform on $[\![L^2]\!]^{(-\alpha)}$, $\alpha > 0$, is given by

$$\mathscr{S}[\![\phi_1, \phi_2]\!] = [\![\mathscr{S}\phi_1, \mathscr{S}\phi_2]\!] , \qquad [\![\phi_1, \phi_2]\!] \in [\![L^2]\!]^{(-\alpha)} . \tag{3.1}$$

The $\mathscr{S}$-transform gives an isomorphism $[\![L^2]\!]^{(-\alpha)} \simeq [\![F]\!]^{(-\alpha)}$. The $W(t, x)$-differentiation $\partial_{t,x} \equiv \partial/\partial W(t, x)$ in $[\![L^2]\!]^{(-\alpha)}$, $\alpha > 0$, is naturally defined by

$$\partial_{t,x}[\![\phi_1, \phi_2]\!] = [\![\partial_{t,x}\phi_1, \partial_{t,x}\phi_2]\!] \tag{3.2}$$

for every differentiable element $[\![\phi_1, \phi_2]\!]$ in $[\![L^2]\!]^{(-\alpha)}$. We now introduce the shift $1/dx$ on $[\![L^2]\!]^{(-\alpha)}$ by the formula

$$\frac{1}{dx}[\![\phi_1, \phi_2]\!] = [\![0, \phi_1]\!] , \qquad [\![\phi_1, \phi_2]\!] \in [\![L^2]\!]^{(-\alpha)} . \tag{3.3}$$

For $\phi(B(t)) = [\![\phi_1(B(t)), \phi_2 B(t))]\!]$ in $[\![L^2]\!]^{(-\alpha)}$ for some $\alpha > 0$, we understand the integral $\int_s^t \phi(B(u))du$ as

$$\int_s^t \phi(B(u))du = \left[\!\!\left[\int_s^t \phi_1(B(u))du, \int_s^t \phi_2(B(u))du\right]\!\!\right] . \tag{3.4}$$

Similarly, we can define the stochastic integral $\int_s^t \phi(B(u))dB_x(u)$ as

$$\int_s^t \phi(B(u))dB_x(u) = \left[\!\!\left[\int_s^t \phi_1(B(u))dB_x(u), \int_s^t \phi_2(B(u))dB_x(u)\right]\!\!\right] . \tag{3.5}$$

Concerning the first component of (3.5), we can see a similarity to the stochastic integral introduced in [5].

§ 4. Itô's formula and Lévy's Laplacian

We are now in a position to define the domain of the Lévy's Laplacian. The product $B_{x_1}(\cdot)^{p_1} \cdots B_{x_n}(\cdot)^{p_n}$, which has only formal significance, will be understood to be

$$\left[\!\!\left[:B_{x_1}(\cdot)^{p_1} \cdots B_{x_n}(\cdot)^{p_n}:, \sum_{j=1}^{n} C_1(p_j)(\cdot): \prod_{\substack{1 \leq \nu \leq n \\ \nu \neq j}} B_{x_\nu}(\cdot)^{p_\nu} B_{x_j}(\cdot)^{p_j - 2}: \right]\!\!\right] ,$$

where $C_1(p_j) = p_j(p_j - 1)/2$, $j = 1, 2, \cdots, n$ and $:B_{x_1}(\cdot)^{p_1} \cdots B_{x_n}(\cdot)^{p_n}:$ denotes the renormalization of $B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n}$. Then an integral

$$\int\underset{R^n}{\cdots}\int f(x_1, \cdots, x_n)B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n}dx_1\cdots dx_n$$

is given by

$$\Big[\!\Big[\int\underset{R^n}{\cdots}\int f(x_1, \cdots, x_n) : B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n} : dx_1\cdots dx_n,$$
$$\sum_{j=1}^n C_1(p_j)(\cdot)\int\underset{R^n}{\cdots}\int f(x_1, \cdots, x_n) : \prod_{\substack{1\leqq\nu\leqq n\\ \nu\neq j}} B_{x_\nu}(\cdot)^{p_\nu}B_{x_j}(\cdot)^{p_j-2} : dx_1\cdots dx_n\Big]\!\Big].$$

Set

$$\mathscr{D}_L = LS\Big\{\int\underset{R^n}{\cdots}\int f(x_1, \cdots, x_n)B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n}dx_1\cdots dx_n;\ f\in L^1(\boldsymbol{R}^n),$$
$$(p_1, \cdots, p_n)\in(N\cup\{0\})^n,\ n=0, 1, 2, \cdots\Big\}.$$

LEMMA 1. *We have* $\mathscr{D}_L\subset\mathscr{C}([0, \infty)\to[\![L^2]\!]^{(-\alpha)})$ *for any* $\alpha>5/6$.

Proof. Take

$$\phi(B(\cdot)) = \int\underset{R^n}{\cdots}\int f(x_1, \cdots, x_n)B_{x_1}(\cdot)^{p_1}\cdots B_{x_n}(\cdot)^{p_n}dx_1\cdots dx_n, \tag{4.1}$$
$$f\in L^1(\boldsymbol{R}^n),\quad p_1+\cdots+p_n=N.$$

It is sufficient to prove $\phi(B(\cdot))\in\mathscr{C}([0, \infty)\to[\![L^2]\!]^{(-\alpha)})$ for any $\alpha>5/6$. We will first prove $\phi(B(t))\in[L^2]^{(-\alpha)}$ for any $\alpha>5/6$ and $t\geqq 0$. Set

$$F=\int\underset{R^n}{\cdots}\int f(x_1, \cdots, x_n)I_{[0,t]}^{\otimes N}\otimes\bigotimes_{\nu=1}^n\delta_{x_\nu}^{\otimes p_\nu}dx_1\cdots dx_n$$

and set

$$G=\sum_{j=1}^n C_1(p_j)t\int\underset{R^n}{\cdots}\int f(x_1, \cdots, x_n)I_{[0,t]}^{\otimes(N-2)}\otimes\bigotimes_{\substack{1\leqq\nu\leqq n\\ \nu\neq j}}\delta_{x_\nu}^{\otimes p_\nu}\otimes\delta_{x_j}^{\otimes(p_j-2)}dx_1\cdots dx_n.$$

Then what we should prove can be reduced to show that two series

$$\sum_{j_1,k_1,\cdots,j_N,k_N=0}^\infty\Big\{\prod_{\nu=1}^N(2j_\nu+1)(2k_\nu+1)\Big\}^{-\alpha}\Big\langle F, \frac{1}{N!}\sum_\sigma\xi_{\sigma(j_1,k_1)}\otimes\cdots\otimes\xi_{\sigma(j_N,k_N)}\Big\rangle^2 \tag{4.2}$$

and

$$\sum_{j_1,k_1,\cdots,j_{N-2},k_{N-2}=0}^\infty\Big\{\prod_{\nu=1}^{N-2}(2j_\nu+1)(2k_\nu+1)\Big\}^{-\alpha} \tag{4.3}$$
$$\times\Big\langle G, \frac{1}{(N-2)!}\cdot\sum_\tau\xi_{\tau(j_1,k_1)}\otimes\cdots\otimes\xi_{\tau(j_{N-2},k_{N-2})}\Big\rangle^2$$

converge for any $\alpha > 5/6$, where σ and τ extend over the set of all possible permutations. It is easily checked that

$$\Big\langle F, \frac{1}{N!}\sum_{\sigma} \xi_{\sigma(j_1,k_1)} \otimes \cdots \otimes \xi_{\sigma(j_N,k_N)}\Big\rangle^2 \leqq t^{2N} \|f\|^2_{L^1(\mathbf{R}^n)} \|\xi_{j_1}\|^2_\infty \cdots \|\xi_{j_N}\|^2_\infty \|\xi_{k_1}\|^2_\infty \cdots \|\xi_{k_N}\|^2_\infty \tag{4.4}$$

and

$$\Big\langle G, \frac{1}{(N-2)!} \cdot \sum_{\tau} \xi_{\tau(j_1,k_1)} \otimes \cdots \otimes \xi_{\tau(j_{N-2},k_{N-2})}\Big\rangle^2 \leqq t^{2(n+N-2)} \Big\{\sum_{i,j=1}^{n} C_1(p_i)C_1(p_j)\Big\} \times \|f\|^2_{L^1(\mathbf{R}^n)} \|\xi_{j_1}\|^2_\infty \cdots \|\xi_{j_{N-2}}\|^2_\infty \|\xi_{k_1}\|^2_\infty \cdots \|\xi_{k_{N-2}}\|^2_\infty , \tag{4.5}$$

where $\|\cdot\|_{L^1(\mathbf{R}^n)}$ is the $L^1(\mathbf{R}^n)$-norm and $\|\cdot\|_\infty$ is the maximum norm. By E. Hille and R. S. Phillips [3], p 571, (21.3.3), it holds that

$$\|\xi_j\|^2_\infty = O(j^{-1/6}), \qquad j > 0 . \tag{4.6}$$

From (4.4), (4.5) and (4.6), follows the convergence of two series (4.2) and (4.3) for any $\alpha > 5/6$. Next, we prove the continuity of $\phi(B(\cdot))$. Set $\phi(B(\cdot)) = [\![\phi_1(B((\cdot)), \phi_2(B(\cdot))]\!]$. Then $\|\phi(B(t))\|^2_{[[L^2]](-\alpha)} = \|\phi_1(B(t))\|^2_{(L^2)(-\alpha)} + \|\phi_2(B(t))\|^2_{(L^2)(-\alpha)}$. It is clear that, for any $\alpha > 5/6$ and $0 \leqq s \leqq t$, $\|\phi_1(B(t)) - \phi_1(B(s))\|^2_{(L^2)(-\alpha)} \leqq N! \sum_{j_1,k_1,\dots,j_N,k_N=0}^{\infty} \{\prod_{\nu=1}^{N} (2j_\nu + 1)(2k_\nu + 1)\}^{-\alpha}(t-s)$ {polynomial in $(t-s)$} $\times \|f\|^2_{L^1(\mathbf{R}^n)} \|\xi_{j_1}\|^2_\infty \cdots \|\xi_{j_N}\|^2_\infty \|\xi_{k_1}\|^2_\infty \cdots \|\xi_{k_N}\|^2_\infty$. Similar evaluation can be obtained for $\phi_2(B(t)) - \phi_2(B(s))$. Thus follows the continuity of $\phi(B(\cdot))$.

Q.E.D.

LEMMA 2. *For any $\phi(B(\cdot))$ in $\mathscr{D}_L$, the $W(t, x)$-derivative $\partial_{s,x}\phi(B(t))$ exists and is independent of the choice of s in the interval $(0, t)$.*

Proof. It is sufficient to prove this Lemma for a functional given by (4.1). Set $\Xi(t, x) = \int_0^t \xi(r, x)dr$ for $\xi \in S(\mathbf{R}^2)$. Then by Lemma 1, the $\mathscr{S}$-transform of $\phi(B(t))$ is given by

$$\mathscr{S}[\phi(B(t))](\xi) = \Big[\!\Big[\int\!\!\cdots\!\!\int_{\mathbf{R}^n} f(x_1, \cdots, x_n) \prod_{\nu=1}^{n} \Xi(t, x_\nu)^{p_\nu} dx_1 \cdots dx_n ,$$

$$\sum_{j=1}^{n} C_1(p_j) t \int\!\!\cdots\!\!\int_{\mathbf{R}^n} f(x_1, \cdots, x_n) \prod_{\substack{1\leqq \nu \leqq n \\ \nu \neq j}} \Xi(t, x_\nu)^{p_\nu} \Xi(t, x_j)^{p_j - 2} dx_1 \cdots dx_n \Big]\!\Big], \quad \xi \in S(\mathbf{R}^2).$$

Hence,

$$\frac{\delta}{\delta\xi(s,x)}\mathscr{S}[\phi(B(t))](\xi)$$

$$= \Bigg[\Bigg[\sum_{j=1}^{n} p_j \Xi(t,x)^{p_j-1}\int_{R^{n-1}}\cdots\int f(x_1,\cdots,x_{j-1},x,x_{j+1},\cdots,x_n)$$

$$\times \prod_{\substack{1\leq\nu\leq n\\ \nu\neq j}} \Xi(t,x_\nu)^{p_\nu}dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n\,,$$

$$\sum_{k=1}^{n} C_1(p_k)t\sum_{\substack{1\leq j\leq n\\ j\neq k}} p_j\Xi(t,x)^{p_j-1}\int_{R^{n-1}}\cdots\int f(x_1,\cdots,x_{j-1},x,x_{j+1},\cdots,x_n)$$

$$\times \prod_{\substack{1\leq\nu\leq n\\ \nu\neq j,k}} \Xi(t,x_\nu)^{p_\nu}\Xi(t,x_k)^{p_k-2}dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n$$

$$+\sum_{j=1}^{n} C_1(p_j)(p_j-2)t\Xi(t,x)^{p_j-3}\int_{R^{n-1}}\cdots\int f(x_1,\cdots,x_{j-1},x,x_{j+1},\cdots,x_n)$$

$$\times \sum_{\substack{1\leq\nu\leq n\\ \nu\neq j}} \Xi(t,x_\nu)^{p_\nu}dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n\Bigg]\Bigg]\,,\qquad \xi\in S(R^2)\,.$$

By the definition of $\partial_{s,x}\phi(B(t))$ and by the above form, we can see that $\partial_{s,x}\phi(B(t))$ is independent of the choice of s in $(0,t)$. Q.E.D.

By Lemma 2, we may denote $\partial_{s,x}$ simply by ∂_x, when it acts on $\mathscr{D}_L$.

THEOREM. *If $\phi(B(\cdot))$ is in $\mathscr{D}_L$, then*

$$\phi(B(t))-\phi(B(s))=\int_R\int_s^t\partial_x\phi(B(u))dB_x(u)dx+\frac{1}{2}\cdot\frac{1}{dx}\cdot\int_s^t\Delta_L\phi(B/u))du \tag{4.7}$$

holds for $0\leqq s\leqq t$.

Proof. It is suffices for us to prove (4.7) for an element $\phi(B(t))$ of the form (4.1). The $\mathscr{S}$-transform of $\partial_x\phi(B(t))$ is given in the proof of Lemma 2. Hence we can easily compute the $\mathscr{S}$-transform of $\int_R\int_s^t\partial_x\phi(B(u))dB_x(u)dx$ by the definition of the stochastic integral. The $\mathscr{S}$-transform of $\frac{1}{dx}\int_s^t\Delta_L\phi(B(u))du$ is given by

$$\mathscr{S}\left[\frac{1}{dx}\int_s^t\Delta_L\phi(B(u))du\right](\xi)=\Bigg[\Bigg[0,\ 2\sum_{j=1}^{n}C_1(p_j)\int_{R^n}\cdots\int f(x_1,\cdots,x_n)$$

$$\times\int_s^t\prod_{\substack{1\leq\nu\leq n\\ \nu\neq j}}\Xi(u,x_\nu)^{p_\nu}\Xi(u,x_j)^{p_j-2}du\,dx_1\cdots dx_n\Bigg]\Bigg]\,,\qquad \xi\in S(R^2)\,.$$

By comparing $\mathscr{S}[\phi(B(t))](\xi)-\mathscr{S}[\phi(B(s))](\xi)$ with

$$\mathscr{S}\left[\int_R\int_s^t\partial_x\phi(B(u))dB_x(u)dx\right](\xi)+\frac{1}{2}\mathscr{S}\left[\frac{1}{dx}\int_s^t\Delta_L\phi(B(u))du\right](\xi)\,,$$

we obtain (4.7). Q.E.D.

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