N. Gupta and K. Tahara
Nagoya Math. J.
Vol. 100 (1985), 127–133

DIMENSION AND LOWER CENTRAL SUBGROUPS OF METABELIAN P-GROUPS

NARAIN GUPTA* AND KEN-ICHI TAHARA

To the memory of the late Takehiko Miyata

§1. Introduction

It is a well-known result due to Sjogren [9] that if G is a finitely generated p-group then, for all $n \leqq p-1$, the $(n+2)$-th dimension subgroup $D_{n+2}(G)$ of G coincides with $\gamma_{n+2}(G)$, the $(n+2)$-th term of the lower central series of G. This was earlier proved by Moran [5] for $n \leqq p-2$. For $p=2$, Sjogren's result is the best possible as Rips [8] has exhibited a finite 2-group G for which $D_4(G) \neq \gamma_4(G)$ (see also Tahara [10, 11]). In this note we prove that if G is a finitely generated metabelian p-group then, for all $n \leqq p$, $D^2_{n+2}(G) \subseteqq \gamma_{n+2}(G)$. It follows, in particular, that, for p odd, $D_{n+2}(G) = \gamma_{n+2}(G)$ for all $n \leqq p$ and all metabelian p-groups G.

§2. Notation and preliminaries

While the central idea of the proof of our main result stems from Gupta [1], with a slight repetition, it is equally convenient to give a self-contained proof using a less cumbersome notation.

Let $\mathfrak{f} = ZF(F-1)$ denote the augmentation ideal of the integral group ring ZF of a free group F freely generated by $x_1, x_2, \cdots, x_m$, $m \geqq 2$. For a fixed prime p, let $(p^{\alpha_1}, p^{\alpha_2}, \cdots, p^{\alpha_m})$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_m > 0$ be an m-tuple of p-powers, and let $S = \langle x_1^{p^{\alpha_1}}, x_2^{p^{\alpha_2}}, \cdots, x_m^{p^{\alpha_m}}, F' \rangle$ be the normal subgroup of F so that F/S is abelian. Set $\mathfrak{s} = ZF(S-1)$, the ideal of ZF generated by all elements $s-1$, $s \in S$. For $1 \leqq n \leqq p$, we shall need to investigate the structure of the subgroup $D_{n+2}(\mathfrak{f}\mathfrak{s}) = F \cap (1 + \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2})$ of F which consists of all elements $w \in F$ such that $w - 1 \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2}$. It is clear that $[F', S]\gamma_{n+2}(F) \subseteqq D_{n+2}(\mathfrak{f}\mathfrak{s})$.

Let $w \in D_{n+2}(\mathfrak{f}\mathfrak{s})$ be an arbitrary element. Then $w - 1 \in \mathfrak{f}^2$ and it

Received July 25, 1984.
*) Research supported by N.S.E.R.C., Canada.

follows that $w \in F'$. Thus, modulo F'', using the Jacobi identity, we may write w as

$$w \equiv w_1 w_2 \cdots w_{m-1}, \tag{1}$$

where

$$w_i = \prod_{j=i+1}^{m} [x_i, x_j]^{d_{ij}} \tag{2}$$

and $d_{ij} = d_{ij}(x_i, x_{i+1}, \cdots, x_m) \in ZF$. For $i = 1, 2, \cdots, m$, define homomorphisms $\theta_i\colon ZF \to ZF$ by $x_k \to 1$ if $k \leqq i$, $x_k \to x_k$ if $k > i$. Since the ideals $\mathfrak{f}$, $\mathfrak{s}$ are invariant under θ_i's, it follows, using $\theta_1, \theta_2, \cdots, \theta_{m-2}$ in succession, that if $w - 1 \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2}$ then $w_i - 1 \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2}$ for each i. For each $k = 1, 2, \cdots, m$, define

$$t(x_k) = 1 + x_k + \cdots + x_k^{p^{\alpha_k}-1}. \tag{3}$$

Then

$$\begin{aligned} t(x_k) &= \sum_{l=1}^{p^{\alpha_k}} \binom{p^{\alpha_k}}{l} (x_k - 1)^{l-1} \\ &\equiv p^{\alpha_k} + \binom{p^{\alpha_k}}{p} (x_k - 1)^{p-1} \bmod (\mathfrak{s} + \mathfrak{f}^p). \end{aligned} \tag{4}$$

We can now prove,

LEMMA 2.1. *Let w_i be as in (2) with $w_i - 1 \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2}$ and $n \leqq p$. Then, modulo $\mathfrak{s} + \mathfrak{f}^n$, $d_{ij} \equiv t(x_i)a_{ij} \equiv t(x_j)b_{ij}$, where $t(x_i)$, $t(x_j)$ are given by (3), $a_{ij} \in Z$ and $b_{ij} \in ZF$. Moreover, if $\alpha_i = \alpha_j$ then $b_{ij} \in Z$.*

Proof. Expansion of $w_i - 1$ shows

$$\sum_{j=i+1}^{m} \{(x_i - 1)(x_j - 1) - (x_j - 1)(x_i - 1)\} d_{ij} \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2}. \tag{5}$$

Since $\mathfrak{f}$ is a free right ZF-module on $x_1 - 1, x_2 - 1, \cdots, x_m - 1$, it follows from (5) that, for all $j = i + 1, \cdots, m$,

$$(x_j - 1)(x_i - 1) d_{ij} \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2},$$

which yields

$$(x_i - 1) d_{ij} \in \mathfrak{s} + \mathfrak{f}^{n+1} \tag{6}$$

and, in turn,

$$d_{ij} \in t(x_i)ZF + \mathfrak{s} + \mathfrak{f}^n, \tag{7}$$

where $t(x_i)$ is given by (3). Since $n \leqq p$, (4) induces that, for $k \geqq i$, $t(x_i)(x_k - 1) \equiv p^{\alpha_i - \alpha_k} p^{\alpha_k}(x_k - 1) \equiv 0 \bmod (\mathfrak{s} + \mathfrak{f}^n)$. Thus (7) implies $d_{ij} \equiv$

$t(x_i)a_{ij} \bmod (\mathfrak{s} + \mathfrak{f}^n)$ with $a_{ij} \in Z$. Substituting in (5) gives

$$(x_i - 1) \sum_{j=i+1}^{m} (x_j - 1) d_{ij} \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2} .$$

and, as before,

$$\sum_{j=i+1}^{m} (x_j - 1) d_{ij} \in \mathfrak{s} + \mathfrak{f}^{n+1} .$$

Using the homomorphisms $\theta_{i+1}, \cdots, \theta_{m-1}$ in turn, gives

$$(x_j - 1) d_{ij} \in \mathfrak{s} + \mathfrak{f}^{n+1} \tag{8}$$

for all $j = i + 1, \cdots, m$, since $d_{ij} \equiv t(x_i)a_{ij} \bmod (\mathfrak{s} + \mathfrak{f}^n)$ with $a_{ij} \in Z$. Thus

$$d_{ij} \in t(x_j)ZF + \mathfrak{s} + \mathfrak{f}^n, \tag{9}$$

and if $\alpha_i = \alpha_j$ then, as before, $d_{ij} \equiv t(x_j)b_{ij} \bmod (\mathfrak{s} + \mathfrak{f}^n)$ with $b_{ij} \in Z$. This completes the proof of the lemma.

Now, let $\dfrac{\partial}{\partial x_k} d$ be a free partial derivative of $d \in ZF$ with respect to x_k. Then we prove,

LEMMA 2.2. *$\dfrac{\partial}{\partial x_k} d_{ij} \in p^{\alpha_k}ZF + \mathfrak{s} + \mathfrak{f}^{n-1}$, $i < k$, and*

$$\frac{\partial}{\partial x_i} d_{ij} \in \begin{cases} p^{\alpha_i}ZF + \mathfrak{s} + \mathfrak{f}^{n-1} & \text{if } \alpha_i = \alpha_j \\ p^{\alpha_i}ZF + p^{\alpha_i - 1}(x_i - 1)^{p-2}ZF + \mathfrak{s} + \mathfrak{f}^{n-1} & \text{if } \alpha_i > \alpha_j . \end{cases}$$

Proof. We have

$$\frac{\partial}{\partial x_k}(\mathfrak{s}) \subseteqq \mathfrak{s} + p^{\alpha_k}ZF; \quad \frac{\partial}{\partial x_k}(\mathfrak{f}^n) \subseteqq \mathfrak{f}^{n-1} .$$

Thus since $d_{ij} \equiv t(x_i)a_{ij} \bmod (\mathfrak{s} + \mathfrak{f}^n)$ with $a_{ij} \in Z$, it follows that

$$\frac{\partial}{\partial x_k} d_{ij} \equiv 0 \bmod (p^{\alpha_k}ZF + \mathfrak{s} + \mathfrak{f}^{n-1}) .$$

By (4) and $d_{ij} \equiv t(x_i)a_{ij} \bmod (\mathfrak{s} + \mathfrak{f}^n)$, we have

$$\frac{\partial}{\partial x_i} d_{ij} \equiv a_{ij}\binom{p^{\alpha_i}}{p}(p - 1)(x_i - 1)^{p-2} \bmod (p^{\alpha_i}ZF + \mathfrak{s} + \mathfrak{f}^{n-1}) .$$

Since $p^{\alpha_i - 1}$ divides $\dbinom{p^{\alpha_i}}{p}$, $\dfrac{\partial}{\partial x_i} d_{ij} \equiv 0 \bmod (p^{\alpha_i - 1}(x_i - 1)^{p-2}ZF + p^{\alpha_i}ZF + \mathfrak{s} + \mathfrak{f}^{n-1})$. If $\alpha_i = \alpha_j$ then $b_{ij} \in Z$, and we may differentiate $d_{ij} \equiv t(x_j)b_{ij}$ with respect to x_i to obtain the desired result.

Next, we need to expand $[x_i, x_j]^{d_{ij}} - 1$ modulo $(\mathfrak{f}^2\mathfrak{s} + \mathfrak{f}^{n+2})$. We first observe,

$$
\begin{aligned}
&[x_i, x_j]^{x_i^{\beta_i} x_{i+1}^{\beta_{i+1}} \cdots x_m^{\beta_m}} - 1 \\
&\equiv x_m^{-\beta_m} \cdots x_{i+1}^{-\beta_{i+1}} x_i^{-\beta_i}([x_i, x_j] - 1) x_i^{\beta_i} x_{i+1}^{\beta_{i+1}} \cdots x_m^{\beta_m} \\
&\equiv ([x_i, x_j] - 1) x_i^{\beta_i} x_{i+1}^{\beta_{i+1}} \cdots x_m^{\beta_m} - \sum_{k=i}^{m} \beta_k (x_k - 1)([x_i, x_j] - 1) x_i^{\beta_i} x_{i+1}^{\beta_{i+1}} \cdots x_m^{\beta_m} \\
&\equiv ([x_i, x_j] - 1) x_i^{\beta_i} \cdots x_m^{\beta_m} - \sum_{k=i}^{m} (x_k - 1)([x_i, x_j]^{x_k(\partial/\partial x_k)(x_i^{\beta_i} \cdots x_m^{\beta_m})} - 1) .
\end{aligned}
$$

Thus,

$$
[x_i, x_j]^{d_{ij}} - 1 \equiv ([x_i, x_j] - 1) d_{ij} - \sum_{k=i}^{m} (x_k - 1)([x_i, x_j]^{x_k(\partial/\partial x_k) d_{ij}} - 1) .
$$

Now, modulo $(\mathfrak{f}^2\mathfrak{s} + \mathfrak{f}^{n+2})$

$$
\begin{aligned}
([x_i, x_j] - 1) d_{ij} &\equiv x_i^{-1} x_j^{-1}\{(x_i - 1)(x_j - 1) - (x_j - 1)(x_i - 1)\} d_{ij} \\
&\equiv \{(x_i - 1)(x_j - 1) - (x_j - 1)(x_i - 1)\} d_{ij} \\
&\quad - (x_i - 1)\{(x_i - 1)(x_j - 1) - (x_j - 1)(x_i - 1)\} d_{ij} \\
&\quad - (x_j - 1)\{(x_i - 1)(x_j - 1) - (x_j - 1)(x_i - 1)\} d_{ij} \\
&\equiv (x_i - 1)(x_j - 1) d_{ij} - (x_j - 1)(x_i - 1) d_{ij} , \\
&\qquad \text{by (6) and (8)} \\
&\equiv (x_i - 1)(x_j - 1) t(x_j) b_{ij} - (x_j - 1)(x_i - 1) t(x_i) a_{ij} , \\
&\qquad \text{by Lemma 2.1} \\
&\equiv (x_i - 1)(x_j^{p^{\alpha_j} b_{ij}} - 1) - (x_j - 1)(x_i^{p^{\alpha_i} a_{ij}} - 1) .
\end{aligned}
$$

Thus we have,

LEMMA 2.3. *Modulo* $(\mathfrak{f}^2\mathfrak{s} + \mathfrak{f}^{n+2})$,

$$
\begin{aligned}
[x_i, x_j]^{d_{ij}} - 1 \equiv &(x_i - 1)(x_j^{p^{\alpha_j} b_{ij}} - 1) - (x_j - 1)(x_i^{p^{\alpha_i} a_{ij}} - 1) \\
&- \sum_{k=i}^{m} (x_k - 1)([x_i, x_j]^{x_k(\partial/\partial x_k) d_{ij}} - 1) .
\end{aligned}
$$

Finally, using (6) and (8), we have, for any x_k, mod $[F', S]\gamma_{n+3}(F)$,

$$
\begin{aligned}
[[x_i, x_j]^{d_{ij}}, x_k] &\equiv [x_i, x_j, x_k]^{d_{ij}} \\
&\equiv [x_i, x_k, x_j]^{d_{ij}} [x_k, x_j, x_i]^{d_{ij}} \\
&\equiv [x_i, x_k]^{(-1+x_j) d_{ij}} [x_k, x_j]^{(-1+x_i) d_{ij}} \\
&\equiv 1 .
\end{aligned}
$$

Thus we have,

LEMMA 2.4 (Gupta [2]). $[D_{n+2}(\mathfrak{f}\mathfrak{s}), F] \subseteq [F', S]\gamma_{n+3}(F)$ *for all* $n \geqq 0$.

This completes our preliminary discussions.

§ 3. The main theorem

Let G be a finitely generated metabelian p-group. Then G admits a presentation of the form

$$G = F/R = \langle x_1, x_2, \cdots, x_m; x_1^{p^{\alpha_1}}\zeta_1, x_2^{p^{\alpha_2}}\zeta_2, \cdots, x_m^{p^{\alpha_m}}\zeta_m, \zeta_{m+1}, \cdots, F''\rangle,$$

where $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_m > 0$ (see for instance [4], page 149). Let S be the normal subgroup of F generated by $x_1^{p^{\alpha_1}}, x_2^{p^{\alpha_2}}, \cdots, x_m^{p^{\alpha_m}}$ and F', then it follows that $S' \subseteq R \subseteq S$. In terms of the free group rings, the dimension subgroup $D_{n+2}(G) = D_{n+2}(\mathfrak{r})/R$, where $\mathfrak{r} = ZF(R-1)$ and $D_{n+2}(\mathfrak{r}) = F \cap (1 + \mathfrak{r} + \mathfrak{f}^{n+2})$. Then $R\gamma_{n+2}(F) \subseteq D_{n+2}(\mathfrak{r})$. If $z \in D_{n+2}(\mathfrak{r})$, then $z - 1 \in \mathfrak{r} + \mathfrak{f}^{n+2}$ implies that $zr - 1 \in \mathfrak{f}\mathfrak{r} + \mathfrak{f}^{n+2}$ for some $r \in R$. It follows that $D_{n+2}(G) = \gamma_{n+2}(G)$ if and only if $D_{n+2}(\mathfrak{f}\mathfrak{r}) = F \cap (1 + \mathfrak{f}\mathfrak{r} + \mathfrak{f}^{n+2}) \subseteq R\gamma_{n+2}(F)$. We now prove our main result.

THEOREM 3.1. $D_{n+2}^2(\mathfrak{f}\mathfrak{r}) \subseteq R\gamma_{n+2}(F)$ *for all* $n \leqq p$.

Proof. Let $w \in D_{n+2}(\mathfrak{f}\mathfrak{r})$. Then $w - 1 \in \mathfrak{f}\mathfrak{r} + \mathfrak{f}^{n+2} \subseteq \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2}$, and by Lemma 2.1,

$$w \equiv \prod_{1 \leqq i < j \leqq m} [x_i, x_j]^{d_{ij}} \bmod F'',$$

where $d_{ij} \equiv t(x_i)a_{ij} \equiv t(x_j)b_{ij} \bmod (\mathfrak{s} + \mathfrak{f}^n)$. Now, $w - 1 \in \mathfrak{f}\mathfrak{r} + \mathfrak{f}^{n+2}$ implies $w - 1 \in \mathfrak{f}\mathfrak{r} + \mathfrak{f}^2\mathfrak{s} + \mathfrak{f}^{n+2}$. Then it follows by Lemma 2.3, that

$$(10) \qquad w - 1 \equiv \sum_{k=1}^{m} (x_k - 1)(y_k u_k^{-1} - 1) \equiv 0 \bmod (\mathfrak{f}\mathfrak{r} + \mathfrak{f}^2\mathfrak{s} + \mathfrak{f}^{n+2}),$$

where

$$y_k = \prod_{i<k} x_i^{-p^{\alpha_i} a_{ik}} \prod_{k<j} x_j^{p^{\alpha_j} b_{jk}}, \qquad u_k = \prod_{\substack{i<j \\ i \leqq k}} [x_i, x_j]^{x_k(\partial/\partial x_k) d_{ij}}.$$

From (10) it follows that for each $k = 1, 2, \cdots, m$,

$$y_k u_k^{-1} - 1 \in \mathfrak{r} + \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+1},$$

which yields, in turn, using $\mathfrak{f}\mathfrak{r} \subseteq \mathfrak{f}\mathfrak{s}$,

$$y_k u_k^{-1} r_k - 1 \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+1}$$

with some $r_k \in R$, and by Lemma 2.4, for all $k = 1, 2, \cdots, m$,

$$[x_k, y_k u_k^{-1} r_k] \in R\gamma_{n+2}(F)\,,$$

which reduces to

$$[x_k, y_k u_k^{-1}] \in R\gamma_{n+2}(F)$$

and hence

$$[x_k, u_k^{-1}][x_k, y_k] \in R\gamma_{n+2}(F)\,. \tag{11}$$

Next, $[x_k, u_k^{-1}] \equiv [x_k, u_k]^{-1} \bmod R\gamma_{n+2}(F)$, and $[x_k, u_k]$ is a product of commutators of the form

$$[x_k, [x_i, x_j]^{x_k(\partial/\partial x_k) d_{ij}}], \quad 1 \leqq i \leqq k, \quad 1 \leqq i < j \leqq m\,.$$

By Lemma 2.2, for either $i < k$ or $i = k$ and $\alpha_i = \alpha_j$,

$$\begin{aligned}[x_k, [x_i, x_j]^{x_k(\partial/\partial x_k) d_{ij}}] &\equiv [x_k, [x_i, x_j]^{p^{\alpha_k} x_k v}] \text{ for some } v \in ZF\,,\\ &\equiv [x_k^{p^{\alpha_k}}, [x_i, x_j]^{x_k v}]\\ &\equiv 1 \bmod [F', S]\gamma_{n+2}(F)\,.\end{aligned}$$

If $i = k$ and $\alpha_i > \alpha_j$, then by Lemma 2.2, for some v, $w \in ZF$,

$$\begin{aligned}[x_i, [x_i, x_j]^{x_i(\partial/\partial x_i) d_{ij}}] &\equiv [x_i, [x_i, x_j]^{x_i p^{\alpha_i} v + p^{\alpha_i - 1}(x_i - 1)^{p-2} w}]\\ &\equiv [[x_i, x_j]^{(x_i-1)^{p-2} \cdot p^{\alpha_i - 1} w}, x_i]^{-1}\\ &\equiv [x_j^{p^{\alpha_j}}, \underbrace{x_i, \cdots, x_i}_{p}]^{p^{\alpha_i - 1 - \alpha_j} w} \bmod [F', S]\gamma_{n+2}(F)\\ &\equiv [\zeta_j, \underbrace{x_i, \cdots, x_i}_{p}]^{p^{\alpha_i - 1 - \alpha_j} w} \bmod R\gamma_{n+2}(F)\\ &\equiv 1 \bmod R\gamma_{n+2}(F)\,.\end{aligned}$$

Thus (11) is reduced to $[x_k, y_k] \in R\gamma_{n+2}(F)$. However,

$$\begin{aligned}[x_k, y_k] &\equiv \prod_{i<k} [x_i^{p^{\alpha_i} a_{ik}}, x_k] \prod_{k<j} [x_k, x_j^{p^{\alpha_j} b_{kj}}]\\ &\equiv \prod_{i<k} [x_i, x_k]^{d_{ik}} \prod_{k<j} [x_k, x_j]^{d_{kj}} \bmod [F', S]\gamma_{n+2}(F)\,.\end{aligned}$$

Thus

$$w^2 \equiv \prod_{k=1}^{m} [x_k, y_k] \equiv 1 \bmod R\gamma_{n+2}(F)\,.$$

This completes the proof of our main theorem.

As a corollary we obtain,

THEOREM 3.2. *Let G be a finitely generated metabelian p-group. Then*

(a) $D_{n+2}(G) = \gamma_{n+2}(G)$ *for all* $n \leqq p - 1$,
(b) *if* $p = 2$, $D_4^2(G) \subseteqq \gamma_4(G)$,
(c) *if* p *is odd*, $D_{p+2}(G) = \gamma_{p+2}(G)$.

For $p = 3$, part (a) of Theorem 3.2 was first proved by Passi [6]; part (b) is due to Losey [3]. We refer the reader to Passi [7] for a general background on the dimension subgroup problem.

REFERENCES

[1] N. Gupta, On the dimension subgroups of metabelian groups, J. Pure Appl. Algebra, **24** (1982), 1–6.
[2] ——, Sjogren's Theorem for dimension subgroups—The metabelian case, Annals of Math. Study (1985), to appear.
[3] G. Losey, N-series and filtration of the augmentation ideal, Canad. J. Math., **26** (1974), 962–977.
[4] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory, Interscience, New York, 1966.
[5] S. Moran, Dimension subgroups mod n, Proc. Camb. Phil. Soc., **68** (1970), 579–582.
[6] I.B.S. Passi, Dimension subgroups, J. Algebra, **9** (1968), 152–182.
[7] ——, Group Rings and Their Augmentation Ideals, Springer Lecture Notes in Math., **715** (1979), Springer-Verlag, Berlin-Heidelberg-New York.
[8] E. Rips, On the fourth integer dimension subgroup, Israel J. Math., **12** (1972), 342–346.
[9] J. A. Sjogren, Dimension and lower central subgroups, J. Pure Appl. Algebra, **14** (1979), 175–194.
[10] K. Tahara, On the structure of $Q_3(G)$ and the fourth dimension subgroup, Japan. J. Math. (New Ser.), **3** (1977), 381–394.
[11] ——, The fourth dimension subgroups and polynomial maps, II, Nagoya Math. J., **69** (1978), 1–7.

Narain Gupta
Department of Mathematics
University of Manitoba
Winnipeg, R3T 2N2
Canada

Ken-Ichi Tahara
Department of Mathematics
Aichi University of Education
Kariya, 448
Japan