Y. Tanigawa
Nagoya Math. J.
Vol. 100 (1985), 83-96

CONSTRUCTION OF SIEGEL MODULAR FORMS OF DEGREE THREE AND COMMUTATION RELATIONS OF HECKE OPERATORS

YOSHIO TANIGAWA

In connection with the Shimura correspondence, Shintani [6] and Niwa [4] constructed a modular form by the integral with the theta kernel arising from the Weil representation. They treated the group $Sp(1) \times O(2, 1)$. Using the special isomorphism of $O(2, 1)$ onto $SL(2)$, Shintani constructed a modular form of half-integral weight from that of integral weight. We can write symbolically his case as "$O(2, 1) \to Sp(1)$". Then Niwa's case is "$Sp(1) \to O(2, 1)$", that is from the half-integral to the integral. Their methods are generalized by many authors. In particular, Niwa's are fully extended by Rallis-Schiffmann to "$Sp(1) \to O(p, q)$".

In [7], Yoshida considered the Weil representation of $Sp(2) \times O(4)$ and constructed a lifting from an automorphic form on a certain subgroup of $O(4)$ to a Siegel modular form of degree two. In this note, under the spirit of Yoshida, we consider $Sp(3) \times O(4)$ and construct a Siegel modular form of degree three. We use Kashiwara-Vergne's results [2] for the analysis of the infinite place. Roughly speaking, the representation (λ, V_λ) of $O(4)$ which corresponds to an irreducible component of the Weil representation determines the representation $\tau(\lambda)$ of $GL(3, C)$. Then we can make the $V_{\lambda'}$-valued theta series. By integrating the inner product of this theta series and a V_λ-valued automorphic form, we get a Siegel modular form (Proposition 1). The main results in this note are commutation relations of Hecke operators (Theorems 1, 2). By these formulas we can express the Andrianov's L-function by the product of the L-functions of original forms. It is desired that the relations of Theorems 1 and 2 are computed more naturally.

Received April 24, 1984.

§1. Weil representation and the results of Kashiwara and Vergne

Let v be a place of $\boldsymbol{Q}$. We fix a non-trivial additive character ψ_v of $\boldsymbol{Q}_v$. For a positive integer n, let $Sp(n, \boldsymbol{Q}_v)$ be a symplectic group of degree n i.e. $Sp(n, \boldsymbol{Q}_v) = \{g \in GL(2n, \boldsymbol{Q}_v) \mid {}^t gJg = J\}$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let (E, S) be a k-dimensional quadratic space E with a quadratic form $S[x] = {}^t xSx$. We put $X_R = M_{k,n}(R)$ for any ring R. We also put $S[x] = {}^t xSx$ for $x \in X_{\boldsymbol{Q}_v}$. The function $q(x) = \psi_v(\frac{1}{2}\operatorname{tr}(S[x]))$ defines a character of second degree on $X_{\boldsymbol{Q}_v}$. The associated self duality on $X_{\boldsymbol{Q}_v}$ is given by $\langle x, y\rangle = \psi_v(\operatorname{tr}({}^t ySx))$. We denote by dx the self-dual measure on $X_{\boldsymbol{Q}_v}$ with respect to $\langle\ ,\ \rangle$. The Fourier transform of Φ is defined by

$$\Phi^*(x) = \int_{X_{\boldsymbol{Q}_v}} \Phi(y)\langle x, y\rangle dy.$$

Then the Weil representation R_v of $Sp(n, \boldsymbol{Q}_v)$ is realized on $L^2(X_{\boldsymbol{Q}_v})$ and has the following forms for special elements (cf. Weil [9]):

(i) $R_v\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\Phi(x) = \psi_v(\operatorname{tr} bS[x])\Phi(x)$ for $b = {}^t b \in M_n(\boldsymbol{Q}_v)$

(ii) $R_v\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}\Phi(x) = |\det(a)|^{k/2}\Phi(xa)$ for $a \in GL(n, \boldsymbol{Q}_v)$

(iii) $R_v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi(x) = \Phi^*(x)$.

It is well known that for even k, R_v is equivalent to a true representation π_v of $Sp(n, \boldsymbol{Q}_v)$ (cf. Lion and Vergne [4] p. 212, Yoshida [8]).

Hereafter we choose an additive character so that $\psi_\infty = e^{2\pi i x}$, $x \in \boldsymbol{R}$ and $\psi_p = e^{-2\pi i \operatorname{Fr}(x)}$, $x \in \boldsymbol{Q}_p$ for each finite place p, where $\operatorname{Fr}(x)$ is the fractional part of $x \in \boldsymbol{Q}_p$.

In [2], Kashiwara and Vergne decompose the Weil representation R_∞ into irreducible components. We will recall briefly their results.

Let (E, S) be a positive definite quadratic space of dimension k. There are two groups acting on $L^2(X_{\boldsymbol{R}})$, the orthogonal group $O(S)$ of (E, S) and $Sp(n, \boldsymbol{R})$. The action of $O(S)$ is defined by

$$(\sigma\Phi)(x) = \Phi({}^t\sigma x) \qquad \text{for } \sigma \in O(S),$$

and that of $Sp(n, \boldsymbol{R})$ by the Weil representation. It is easily seen that they commute with each other. Therefore we can decompose $L^2(X_{\boldsymbol{R}})$ under $O(S)$. Let (λ, V_λ) be an irreducible unitary representation of $O(S)$. Denote by $L^2(X_{\boldsymbol{R}}; \lambda)$ the space of all V_λ-valued square integrable functions

$\phi(x)$ on X_R which satisfies $\phi(\sigma x) = \lambda(\sigma)\phi(x)$ for $\sigma \in O(S)$. Then $L^2(X_R) = \bigoplus_{\lambda \in \widehat{O(S)}} L^2(X_R; \lambda') \otimes V_\lambda$ where λ' is the contragradient representation of λ.

A polynomial $Q(x)$ on X_R is said to be pluriharmonic if $\Delta_{ij} Q = 0$ for all i, j. Here $\Delta_{ij} = \sum_{\ell=1}^{k} (\partial/\partial x_{\ell i})(\partial/\partial x_{\ell j})$. Let $\mathfrak{h}$ be the space of all such polynomials. $GL(n, C) \times O(S)$ acts on $\mathfrak{h}$ by $Q(x) \to Q(\sigma^{-1} x a)$ for $(a, \sigma) \in GL(n, C) \times O(S)$. For an irreducible representation (λ, V_λ) of $O(S)$, we denote by $\mathfrak{h}(\lambda)$ the space of all V_λ-valued pluriharmonic polynomials $Q(x)$ such that $Q(\sigma x) = \lambda(\sigma) Q(x)$ for $\sigma \in O(S)$. As above, we have $\mathfrak{h} = \bigoplus_{\lambda \in \widehat{O(S)}} \mathfrak{h}(\lambda') \otimes V_\lambda$. We define $\tau(\lambda)$ as the representation of $GL(n, C)$ on $\mathfrak{h}(\lambda)$ by the right translation.

On the other hand, the special representation of $Sp(n, R)$ is defined as follows. Let (τ, V) be an irreducible representation of $GL(n, C)$ and $\delta(a) = \det(a)$ be a one dimensional representation. Let $Sp(n, R)_2$ be the two fold covering group of $Sp(n, R)$. Then for $h \in Z$, we define the representation $T(\tau, h)$ of $Sp(n, R)_2$ in $\mathcal{O}(H_n, V)$, the space of all V-valued holomorphic functions $f(Z)$ on the Siegel upper half plane H_n, by

$$(T(\tau, h)(g)f)(Z) = \delta(CZ + D)^{-h/2} \tau({}^t(CZ + D)) f((AZ + B)(CZ + D)^{-1})$$

for $\tilde{g}^{-1} = (g, \delta(CZ + D)^{1/2}) \in Sp(2, R)_2$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

THEOREM A (Kashiwara and Vergne). *Let the notation be as above. Suppose that* $\mathfrak{h}(\lambda) \neq \{0\}$, *then*

(i) $\tau(\lambda)$ *is irreducible*

(ii) $L^2(X_R; \lambda)$ *is equivalent to* $(T(\tau(\lambda), k), \mathcal{O}(H_n, \mathfrak{h}(\lambda)))$.

The correspondence $\lambda \to \tau(\lambda)$ is also determined explicitly in their paper.

For any $Q \in \mathfrak{h}(\lambda)$ and $Z \in H_n$, we put

$$f_{Q,Z}(x) = Q(x) e^{\pi \sqrt{-1} \operatorname{tr}(Z S[x])}.$$

$f_{Q,Z}$ is a V_λ-valued function on X_R. We also put $\tau = \tau(\lambda)$ and $V_\tau = \mathfrak{h}(\lambda)$.

THEOREM B (Lion and Vergne). *Let* $f_{Q,Z}$ *be as above, then for any* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, R)$,

$$R_\infty(g) f_{Q,Z} = \det(CZ + D)^{-k/2} f_{\tau({}^t(CZ+D)^{-1})Q,\, g(Z)}.$$

This theorem is easily proved by checking the above formula for the generators of the form $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Especially for $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, it is obtained by acting the diffierential operator $Q((1/2\pi i)(\partial/\partial x))$ on both sides of the theta formula.

§ 2. Shintani-Yoshida's construction of Siegel modular form of degree three

Let D be a quaternion algebra over $\boldsymbol{Q}$ which does not split only at ∞ and 2. We denote by $a \to a^*$ the canonical involution of D. Let R be a maximal order in D and Z the center of D. Let (ξ_ν, V_ν) be the symmetric tensor representation of $GL(2, \boldsymbol{C})$ of degree ν. We put $\sigma_\nu(g) = (\xi_\nu \cdot \iota)(g)N(g)^{-\nu/2}$ for $g \in D_\infty^\times$, where ι is an embedding of $D_\infty^\times$ into $GL(2, \boldsymbol{C})$. Let A be the adele ring of rational field $\boldsymbol{Q}$ and $D_A^\times$ be the adelization of $D^\times$. Then an automorphic form on $D_A^\times$ of the type (R, σ_ν) is a V_ν-valued function φ on $D_A^\times$ with the following properties:

(i) $\varphi(\gamma g) = \varphi(g)$ for any $\gamma \in D_{\boldsymbol{Q}}^\times$ and $g \in D_A^\times$,

(ii) $\varphi(gk) = \sigma_\nu(k)\varphi(g)$ for any $k \in D_\infty^\times$ and $g \in D_A^\times$,

(iii) $\varphi(gk) = \varphi(g)$ for any $k \in (R \otimes \boldsymbol{Z}_p)^\times$ and $g \in D_A^\times$ where p is any finite place of $\boldsymbol{Q}$,

(iv) $\varphi(zg) = \varphi(g)$ for any $z \in Z_A^\times$ and $g \in D_A^\times$.

We put $(E, S) = (D, \text{norm})$ as a quadratic space over $\boldsymbol{Q}$. So the dimension of E is four. $D^\times \times D^\times$ acts on E by $\rho(a, b)x = a^*xb$, $(a, b) \in D^\times \times D^\times$. Under this action, the group $G' = \{(a, b) \in D^\times \times D^\times \mid N(a) = N(b) = 1\}$ operates isometrically on E, and is considered as a subgroup of $O(S)$.

Let $G = Sp(3)$ be a symplectic group of degree 3. We put $K_p = Sp(3, \boldsymbol{Z}_p)$ for any finite place p and $K_\infty =$ the stabilizer of $\sqrt{-1}$ in $G_{\boldsymbol{R}}$. We get the local (true) Weil representation π_v of G_v corresponding to the quadratic space E and the additive character ψ_v defined in Section 1. The global Weil representation π is also defined in the usual way.

We are going to define a lifting from an automorphic form on G'_A to that on G_A. As before we let $X = M_{4,3}$. For any finite place p, let f_p be the characteristic function of $X_{\boldsymbol{Z}_p}$. For the infinite place ∞, let $\sigma_{n_1} \otimes \sigma_{n_2}$ be an irreducible representation of $G'_{\boldsymbol{R}}$ such that $n_1 \equiv n_2 \pmod 2$. We put $m_1 = (n_1 + n_2)/2$, $m_2 = |n_1 - n_2|/2$ and λ the irreducible representation of $O(S)_{\boldsymbol{R}}$ with the signature (m_1, m_2). Then $\sigma_{n_1} \otimes \sigma_{n_2}$ is naturally included in λ. Let $\tau(\lambda')$ be the representation of $GL(3, \boldsymbol{C})$ which corresponds to λ'. For any $Q \in \mathfrak{h}(\lambda')$, we put $f_Q = \prod_{v\neq\infty} f_v \times f_{Q,\sqrt{-1}} \in \mathcal{S}(X_A) \otimes V_{\lambda'}$ where $f_{Q,\sqrt{-1}} = Q(x)e^{-\pi \operatorname{tr}(S[x])}$. Now we define the theta series by

$$\theta_{f_Q}(g, h) = \sum_{x \in X_Q} (\pi(g)f_Q)(\rho(h)x)$$

for $g \in G_A$, $h \in G'_A$. Then from Theorem B, we get

$$\theta_{f_Q}(gk, h) = \theta_{f_{Q'}}(g, h), \qquad \text{for any } k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_\infty, \tag{2.1}$$

where $Q' = (\delta^2 \otimes \tau(\lambda'))({}^t(-B\sqrt{-1} + A)^{-1})Q$.

Let φ_1 and φ_2 be automorphic forms on $D_A^\times$ of type (R, σ_{n_1}) and (R, σ_{n_2}) respectively. Then $\varphi = \varphi_1 \otimes \varphi_2$ can be regarded as a V_λ-valued automorphic form on G'_A. Define a function of G_A by

$$\Phi_{f_Q}(g) = \int_{G'_Q \backslash G'_A} (\theta_{f_Q}(g, h), \varphi(h))dh.$$

Here (,) is the natural inner product on $V_{\lambda'}$ and V_λ. Take a basis $B = \{Q_1, \cdots, Q_m\}$ of $\mathfrak{h}(\lambda')$ and fix it. The matrix representation of $\tau(\lambda')$ with respect to B is also denoted by the same letter. Finally we define the C^m-valued function on G_A by

$$\Phi_B(g) = (\Phi_{f_{Q_1}}(g), \cdots, \Phi_{f_{Q_m}}(g)). \tag{2.2}$$

The next Proposition follows at once by the definitions.

PROPOSITION 1. *Let the notation be as above. Then $\Phi_B(g)$ is a Siegel modular form with respect to the representation $\delta^2 \otimes \tau(\lambda')$; it satisfies the following properties,*

(i) $\Phi_B(\gamma g) = \Phi_B(g)$ *for any* $\gamma \in G_Q$, $g \in G_A$,

(ii) $\Phi_B(gk) = \Phi_B(g)(\delta^2 \otimes \tau(\lambda'))({}^t(-B\sqrt{-1} + A)^{-1})$ *for any* $k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_\infty$, $g \in G_A$,

(iii) $\Phi_B(gk) = \Phi_B(g)$ *for any* $k \in K_p$, $g \in G_A$, *where p is any finite place.*

To transform into classical notation, we put $j(g, Z) = (\delta^2 \otimes \tau(\lambda'))\cdot({}^t(CZ + D))$ for any $Z \in H_3$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(3, \boldsymbol{R})$. Then $j(g, Z)$ satisfies the cocycle relation $j(g_1g_2, Z) = j(g_2, Z)j(g_1, g_2(Z))$. For any point $Z \in H_3$ we choose an element $g \in Sp(3, \boldsymbol{R})$ such that $g(\sqrt{-1}) = Z$ and put $g' = 1_f \cdot g \in G_A$, where 1_f is an element of the finite part of G_A such that all the p-component is equal to 1. Then $F(Z) = \Phi_B(g')j(g, \sqrt{-1})$ satisfies the transformation formula $F(\gamma(Z)) = F(Z)j(\gamma, Z)$ for $\gamma \in Sp(3, \boldsymbol{Q}) \cap \prod_{p\neq\infty} K_p$.

§ 3. Hecke operators

Let $\tilde{G}$ be the group of symplectic similitude of degree 3 i.e.

$$\tilde{G}_F = \{g \in GL(6, F) \mid {}^t gJg = m(g)J,\ m(g) \in F^\times\}$$

for any field F. In order to consider Hecke operators we must extend the function on $G_Q \backslash G_A$ to the function on $\tilde{G}_A$. For that purpose we will adopt Yoshida's standard extension. Put $\tilde{K}_p = \tilde{G}_{Q_p} \cap GL(6, Z_p)$ and $\tilde{G}_{\infty,+} = \{g \in GL(6, R) \mid {}^t gJg = m(g)J,\ m(g) > 0\}$. By the approximation theorem, we have $\tilde{G}_A = \tilde{G}_Q \cdot \prod_{p \neq \infty} \tilde{K}_p \cdot \tilde{G}_{\infty,+}$. Let ν be an element of $\tilde{G}_A$ such that $\nu_p = \begin{pmatrix} 1_3 & 0 \\ 0 & \mu_p 1_3 \end{pmatrix}$, $\mu_p \in Z_p^\times$, for each finite place p and $\nu_\infty = \mu_\infty 1_6$, $\mu_\infty \in R_+^\times$, for the infinite place ∞. Then by the approximation theorem, any element g of $\tilde{G}_A$ can be written as $g = \gamma k \nu$ with $\gamma \in \tilde{G}_Q$ and $k \in \prod_{p \neq \infty} K_p \times G_\infty$. Suppose that Φ is a function on G_A which is left invariant under G_Q. We define a function $\tilde{\Phi}$ on $\tilde{G}_A$ by $\tilde{\Phi}(g) = \Phi(k)$ for $g = \gamma k \nu$. It is shown in [7] that this is well-defined and left invariant under $\tilde{G}_Q$.

We put $S_p = \{g \in M_6(Z_p) \mid {}^t gJg = m(g)J,\ m(g) \neq 0\}$. For the Hecke pair $(\tilde{K}_p, S_p)$ we denote by $\mathscr{L}(\tilde{K}_p, S_p)$ the corresponding Hecke ring. It is well known that the complete representatives of the double cosets $\tilde{K}_p \backslash S_p / \tilde{K}_p$ is given by

$$\alpha = \begin{bmatrix} p^{d_1} & & & & & \\ & p^{d_2} & & & 0 & \\ & & p^{d_3} & & & \\ & & & p^{e_1} & & \\ & 0 & & & p^{e_2} & \\ & & & & & p^{e_3} \end{bmatrix}$$

where $d_1 \leqq d_2 \leqq d_3 \leqq e_3 \leqq e_2 \leqq e_1$ and $m(\alpha) = p^{d_i + e_i}$ for any i. We denote the element $\tilde{K}_p \alpha \tilde{K}_p$ of $\mathscr{L}(\tilde{K}_p, S_p)$ by $T(p^{d_1}, p^{d_2}, p^{d_3}, p^{e_1}, p^{e_2}, p^{e_3})$ and put $m(\tilde{K}_p \alpha \tilde{K}_p) = m(\alpha)$. For a non-negative integer n, we define the Hecke operator of degree p^n by $T(p^n) = \sum \tilde{K}_n \alpha \tilde{K}_p$ where the summation is taken over all distinct double cosets $\tilde{K}_p \alpha \tilde{K}_p$ with $m(\tilde{K}_p \alpha \tilde{K}_p) = p^n$.

$\mathscr{L}(\tilde{K}_p, S_p)$ is a polynomial ring generated by $T_0 = T(1, 1, 1, p, p, p)$, $T_1 = T(1, 1, p, p^2, p^2, p)$, $T_2 = T(1, p, p, p^2, p, p)$ and $T_3 = T(p, p, p, p, p, p)$. Define a local Hecke series by $D_p(s) = \sum_{n=0}^{\infty} T(p^n) p^{-ns}$.

THEOREM C (Andrianov). *Let the notation be as above and put* $t = p^{-s}$. *Then*

$$D_p(s) = \left[\sum_{n=0}^{6} (-1)^{n+1} e(n) t^n\right] \times \left[\sum_{n=0}^{8} (-1)^n f(n) t^n\right]^{-1}, \tag{3.1}$$

where

$$e(0) = -1,\ e(1) = 0,\ e(2) = p^2(T_2 + (p^4 + p^2 + 1)T_3),\ e(3) = p^4(p+1)T_0T_3,$$
$$e(4) = p^7(T_2T_3 + (p^4 + p^2 + 1)T_3^2),\ e(5) = 0,\ e(6) = -p^{15}T_3^3;\ f(0) = 1,$$
$$f(1) = T_0,\ f(2) = pT_1 + p(p^2+1)T_2 + (p^5 + p^4 + p^3 + p)T_3,$$
$$f(3) = p^3(T_0T_2 + T_0T_3),\ f(4) = p^6T_0^2T_3 + p^6T_2^2 - 2p^7T_1T_3 - 2p^6(p-1)T_2T_3$$
$$- (p^{12} + 2p^{11} + 2p^{10} + 2p^7 - p^6)T_3^2,\ f(5) = p^6T_3f(3),\quad f(6) = p^{12}T_3^2f(2),$$
$$f(7) = p^{18}T_3^3f(1),\ f(8) = p^{24}T_3^4.$$

For $\tilde{K}_p\alpha\tilde{K}_p \in \mathscr{L}(\tilde{K}_p, S_p)$, let $\tilde{K}_p\alpha\tilde{K}_p = \cup\alpha_i\tilde{K}_p$ be a right cosets decomposition. α_i may be considered as an element of $\tilde{G}_A$ by the canonical embedding $\tilde{G}_{Q_p} \hookrightarrow \tilde{G}_A$. Let Φ be a Siegel modular form and $\tilde{\Phi}$ its standard extension. We define the action of $\tilde{K}_p\alpha\tilde{K}_p$ on $\tilde{\Phi}$ by $(\tilde{\Phi}\,|\,\tilde{K}_p\alpha\tilde{K}_p)(g) = \sum_i \tilde{\Phi}(g\alpha_i)$, which does not depend on the choice of representatives α_i. Suppose that Φ is an eigenfunction of all Hecke operators: $\tilde{\Phi}\,|\,T(m) = \lambda(m)\tilde{\Phi}$ for all $m \in \boldsymbol{Z}$, $m > 0$. Then by the Theorem C of Andrianov, we have

$$\sum_{n=0}^{\infty} \lambda(p^n)n^{-ns} = G_{p,\Phi}(p^{-s})H_{p,\Phi}(p^{-s})^{-1},$$

where $G_{p,\Phi}$ (resp. $H_{p,\Phi}$) is the polynomial given by the numerator (resp. denominator) in (3.1) after replacing the $e(n)$ (resp. $f(n)$) with the corresponding eigenvalues. We define the Andrianov's L-function by the Euler product

$$L(s, \Phi) = \prod_p H_{p,\Phi}(p^{-s})^{-1}.$$

On the other hand, any odd prime p is unramified in D. Therefore $D_p = D \otimes \boldsymbol{Q}_p$ is isomorphic to $M_2(\boldsymbol{Q}_p)$. Hence we get the p-part of Hecke operators in the usual way. If φ is an automorphic form $D_A^\times$ such that $\varphi\,|\,T(p) = \lambda'(p)\varphi$ for all $p \neq 2$, we define the L-function of φ by

$$L(s, \varphi) = \prod_{p\neq 2} \frac{1}{1 - \lambda'(p)p^{-s} + p^{1-2s}}.$$

Note that in this paper we don't set any normalization in the definitions of Hecke operators and L-functions.

The following Proposition will be used in Section 4.

PROPOSITION D (Yoshida). *Let V be a vector space over $\boldsymbol{R}$ and $f = \prod f_v$, where f_p is a characteristic function of X_{Z_p} for finite p and f_∞ is an element of $\mathscr{S}(X_{\boldsymbol{R}}) \otimes V$. Define the theta series by $\theta_f(g, h) = \sum_{x\in X_Q} \pi(g)f(\rho(h)x)$ for*

$(g, h) \in G_A \times O(S)_A$. *For a double coset* $\tilde{K}_p\alpha\tilde{K}_p$ *with* $m(\alpha) = p^{d_1+e_1}$, *let* $\tilde{K}_p\alpha\tilde{K}_p = \cup\alpha_i\tilde{K}_p$ *be a right cosets decomposition.*

(i) *When* $d_1 + e_1$ *is odd,* $d_1 + e_1 = 2t + 1$ $(t \in Z)$, *we put*

$$z = p^{-t}\begin{pmatrix} 1_3 & 0 \\ 0 & p^{-1}1_3 \end{pmatrix} \in \tilde{G}_Q.$$

Then for any element $g \in \tilde{G}_A$ *we have*

$$\sum_i \tilde{\theta}_f(g\alpha_i, h) = \tilde{\theta}_{f'}\left(\begin{pmatrix} 1_3 & 0 \\ 0 & p^{-1}1_3 \end{pmatrix}_\infty g, h\right)$$

where $f' = \prod_{v\neq p} f_v \times f'_p$ *and* $f'_p = \sum_i \pi_p(z_p\alpha_i)f_p$.

(ii) *When* $d_1 + e_1$ *is even,* $d_1 + e_1 = 2t (t \in Z)$, *we put* $z = p^{-t}1_6 \in \tilde{G}_Q$. *Then for any element* $g \in \tilde{G}_A$, *we have*

$$\sum_i \tilde{\theta}_f(g\alpha_i, h) = \tilde{\theta}_{f'}(g, h)$$

where $f' = \prod_{v\neq p} f_v \times f'_p$ *and* $f'_p = \sum_i \pi_p(z_p\alpha_i)f_p$.

§ 4. Local computation of Hecke operators

In this section we will compute the action of Hecke operators on Φ_B explicitly. It is enough to determine $f'_p = \sum_i \pi_p(z_p\alpha_i)f_p$ by Proposition D. First note that, if we put $\Gamma = Sp(3, Z)$, the left cosets decomposition $\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$ corresponds to the right cosets decomposition $\tilde{K}_p\alpha\tilde{K}_p = \bigcup_i m(\alpha)\alpha_i^{-1}\tilde{K}_p$. It is well known that the representatives $\{\alpha_i\}$ can be given by

$$\alpha_{ijk} = \begin{pmatrix} A_i & B_{ik} \\ 0 & D_i \end{pmatrix}\begin{pmatrix} U_{ij} & 0 \\ 0 & {}^tU_{ij}^{-1} \end{pmatrix}$$

where $A_i = \begin{pmatrix} p^{a_{i1}} & & 0 \\ & p^{a_{i2}} & \\ 0 & & p^{a_{i3}} \end{pmatrix}$, $0 \leqq a_{i1} \leqq a_{i2} \leqq a_{i3}$ with $D_i = m(\alpha)A_i^{-1}$ integral, B_{ik} is taken over the complete set of representatives of integral matrices mod D_i with

$$\begin{pmatrix} A_i & B_{ik} \\ 0 & D_i \end{pmatrix} \in \Gamma\alpha\Gamma, \quad \text{and} \quad SL(3, Z) = \bigcup_j (SL(3, Z) \cap A_i^{-1}SL(3, Z)A_i)U_{ij}.$$

Suppose that $m(\alpha) = p$ or p^2. For each i, we define the function on X_{Q_p} by

$$f_p^{(i)}(x) = \sum_k \psi_p(\mathrm{tr}(-B_{ik}D_i^{-1}S[x]))f_p(x).$$

Then by Proposition D, we have

$$
\begin{aligned}
f'_p(x) &= \sum_{ijk} \pi_p(z_p m(\alpha)\alpha_{ijk}^{-1}) f_p(x) \\
&= \sum_{ijk} \pi_p\left(\begin{pmatrix} U_{ij}^{-1} & 0 \\ 0 & {}^tU_{ij} \end{pmatrix}\right) \cdot \pi_p\left(\begin{pmatrix} pA_i^{-1} & 0 \\ 0 & m(\alpha)/pD_i^{-1} \end{pmatrix}\begin{pmatrix} 1 & -B_{ik}D_i^{-1} \\ 0 & 1 \end{pmatrix}\right) f_p(x) \\
&= \sum_{i,j} \pi_p\left(\begin{pmatrix} U_{ij}^{-1} & 0 \\ 0 & {}^tU_{ij} \end{pmatrix}\right) |\det(pA_i^{-1})|^2 \\
&\quad \times \sum_k \psi_p(\mathrm{tr}(-B_{ik}D_i^{-1}S[xpA_i^{-1}])) f_p(xpA_i^{-1}) \\
&= \sum_i |\det(pA_i^{-1})|^2 \sum_j f_p^{(i)}(xp(A_iU_{ij})^{-1}).
\end{aligned}
$$

Henceforth we write the above $f'_p(x)$ by $(f_p \mid \tilde{K}_p\alpha\tilde{K}_p)(x)$ to clarify the operation of $\tilde{K}_n\alpha\tilde{K}_p$.

First we deal with the Hecke operator of degree p.

THEOREM 1. *We assume that p is an odd prime number. Put*

$$
\begin{aligned}
G_0(x) = &\sum_{v=0}^{p-1} f_p\left(\rho\left(\begin{pmatrix} p & v \\ 0 & 1 \end{pmatrix}, 1\right)x\right) + f_p\left(\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, 1\right)x\right) \\
&+ \sum_{v=0}^{p-1} f_p\left(\rho\left(1, \begin{pmatrix} p & v \\ 0 & 1 \end{pmatrix}\right)x\right) + f_p\left(\rho\left(1, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)x\right).
\end{aligned}
$$

Then for an element $T_0 = T(1,1,1,p,p,p)$ of $\mathscr{L}(\tilde{K}_p, S_p)$, we have

$$
(4.1) \qquad (f_p \mid T_0)(x) = \frac{p+1}{p} G_0(x).
$$

Proof. We will write $Y = M_{4,1}$ for simplicity. We prove the above equality case by case. First note that for any $a \in GL(3, \boldsymbol{Z}_p)$ both sides of the equality are invariant under $x \to xa$. We will frequently use this remark for $a =$ permutation matrices, $\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc. Now let us write down all A_i and U_{ij}.

(i) $A_1 = 1_3$ and $\{U_{1j}\} = \{1_3\}$

(ii) $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}$ and $\{U_{2j}\} = \left\{\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; 0 \leqq \alpha, \beta, \gamma \leqq p-1\right\}$

(iii) $A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$ and $\{U_{3j}\} = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \gamma \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \right.$

$$\left. 0 \leqq \alpha, \beta, \gamma \leqq p-1 \right\}$$

(iv) $A_4 = p1_3$ and $\{U_{4j}\} = \{1_3\}$.

We put $S[x] = (u_{ij})$ and define the subsets of X_{Z_p} by

$$V_1 = \{x \in X_{Z_p} \mid u_{ij} \in pZ_p \text{ for all } i \text{ and } j\},$$
$$V_2 = \{x \in X_{Z_p} \mid u_{11}, u_{12}, u_{22} \in pZ_p\},$$
$$V_3 = \{x \in X_{Z_p} \mid u_{11} \in pZ_p\}.$$

Let ϕ_i denote the characteristic function of V_i. Then we have

$$f_p^{(1)} = p^6\phi_1, \quad f_p^{(2)} = p^3\phi_2, \quad f_p^{(3)} = p\phi_3, \quad f_p^{(4)} = f_p.$$

Therefore for $x = (x_1, x_2, x_3) \in X_{Q_p}$, $x_i \in Y_{Q_p}$, we have

$$\begin{aligned}(f_p \mid T_0) = \phi_1(px) &+ p^{-1}\{\sum_{0 \leqq \alpha, \beta \leqq p-1} \phi_2(px_1, px_2, -\alpha x_1 - \beta x_2 + x_3) \\ &+ \sum_{0 \leqq \gamma \leqq p-1} \phi_2(px_1, px_3, \gamma x_1 - x_2) + \phi_2(px_2, px_3, x_1)\} \\ &+ p^{-1}\{\sum_{0 \leqq \alpha, \beta \leqq p-1} \phi_3(px_1, -\alpha x_1 + x_2, -\beta x_1 + x_3) \\ &+ \sum_{0 \leqq \gamma \leqq p-1} \phi_3(px_2, -\gamma x_2 + x_3, x_1) + \phi_3(px_3, x_1, x_2)\} + f_p(x).\end{aligned}$$

By the above remark, we have only to consider the following cases.

Case 1. We assume $x \in X_{Z_p}$. In this case, all the terms occur, so that $(f_p \mid T_0)(x) = (2(p+1)^2)/p$ and $G_0(x) = 2(p+1)$.

Case 2. We assume $px \notin X_{Z_p}$. In this case, all the terms vanish, so that both sides of (4.1) equal to zero.

Case 3. We assume that $x_1 \notin Y_{Z_p}$ and $px_1, x_2, x_3 \in Y_{Z_p}$. Then

$$(f_p \mid T_0)(x) = \begin{cases} 2(p+1)/p & \text{if } u_{11} \in p^{-1}Z_n \\ 0 & \text{otherwise.} \end{cases}$$

Now let us compute $G_0(x)$. By the above remark we may assume that

$$x_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \alpha \in p^{-1}Z_p, \quad \alpha \notin Z_p.$$

Then we have

$$\begin{pmatrix} p & v \\ 0 & 1 \end{pmatrix}^* x_1 = \begin{pmatrix} \alpha & -v\beta \\ 0 & p\beta \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^* x_1 = \begin{pmatrix} p\alpha & 0 \\ 0 & \beta \end{pmatrix},$$

$$x_1 \begin{pmatrix} p & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p\alpha & v\alpha \\ 0 & \beta \end{pmatrix}, \qquad x_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & p\beta \end{pmatrix}.$$

Note that $\beta \in \boldsymbol{Z}_p$ if and only if $u_{11} \in p^{-1}\boldsymbol{Z}_p$. Therefore we have

$$G_0(x) = \begin{cases} 2 & \text{if } u_{11} \in p^{-1}\boldsymbol{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

Case 4. We assume that $x_1, x_2 \notin Y_{Z_p}$, $px_1, px_2, x_3 \in Y_{Z_p}$, and there is no $s \in \boldsymbol{Z}$ such that $sx_1 + x_2 \in Y_{Z_p}$. Then we have

$$(f_p \mid T_0)(x) = \begin{cases} (p+1)/p & \text{if } u_{11}, u_{12}, u_{22} \in p^{-1}\boldsymbol{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, let $x_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $x_2 = \begin{pmatrix} \alpha' & \gamma' \\ \delta' & \beta' \end{pmatrix}$ with $\alpha \notin \boldsymbol{Z}_p$. As in the Case 3, $G_0(x) = 0$ if $u_{11} \notin p^{-1}\boldsymbol{Z}_p$. Hence we can suppose that $\beta \in \boldsymbol{Z}_p$. We have only to consider the following two terms:

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^* x_2 = \begin{pmatrix} p\alpha' & p\gamma' \\ \delta' & \beta' \end{pmatrix} \quad \text{and} \quad x_2 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p\alpha' & \gamma' \\ p\delta' & \beta' \end{pmatrix}.$$

When $\beta' \notin \boldsymbol{Z}_p$ we have $G_0(x) = 0$. On the other hand, if $\beta' \in \boldsymbol{Z}_p$, the above condition implies that there does not occur the case that both γ' and δ' belong to $\boldsymbol{Z}_p$. So that we have

$$G_0(x) = \begin{cases} 1 & \text{if } \gamma' \in \boldsymbol{Z}_p \text{ and } \delta' \notin \boldsymbol{Z}_p, \text{ or } \gamma' \notin \boldsymbol{Z}_p \text{ and } \delta' \in \boldsymbol{Z}_p \\ 0 & \text{if } \gamma', \delta' \in p^{-1}\boldsymbol{Z}_p - \boldsymbol{Z}_p. \end{cases}$$

Anyway, $G_0(x) = 1$ if and only if $\beta, \beta' \in \boldsymbol{Z}_p$, $\gamma'\delta' \in p^{-1}\boldsymbol{Z}_p$ and $\gamma' \notin \boldsymbol{Z}_p$ or $\delta' \notin \boldsymbol{Z}_p$, which is equivalent to $u_{11}, u_{12}, u_{22} \in p^{-1}\boldsymbol{Z}_p$. Otherwise $G_0(x) = 0$. Therefore we get the equality (4.1) in this case.

Case 5. We assume that $x_i \notin Y_{Z_p}$, $px_i \in Y_{Z_p}$ for $i = 1, 2, 3$, and for any pair (i, j) there is no $r \in \boldsymbol{Z}$ such that $rx_i + x_j \in Y_{Z_p}$ and there are no $s, t \in \boldsymbol{Z}$ such that $sx_1 + tx_2 + x_3 \in Y_{Z_p}$. Then we have $(f_p \mid T_0)(x) = \phi_1(px)$. We shall see that it is equal to zero. Let $x_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $x_2 = \begin{pmatrix} \alpha' & \gamma' \\ \delta' & \beta' \end{pmatrix}$ and $x_3 = \begin{pmatrix} \alpha'' & \gamma'' \\ \delta'' & \beta'' \end{pmatrix}$ with $\alpha \notin \boldsymbol{Z}_p$. Suppose that $u_{ij} \in p^{-1}\boldsymbol{Z}_p$ for all i and j. Then we have $\beta \in \boldsymbol{Z}_p$, $\beta' \in \boldsymbol{Z}_p$, $\gamma'\delta' \in p^{-1}\boldsymbol{Z}_p$, $\beta'' \in \boldsymbol{Z}_p$, $\gamma''\delta'' \in p^{-1}\boldsymbol{Z}_p$ and $\gamma'\delta'' + \gamma''\delta' \in p^{-1}\boldsymbol{Z}_p$.

Subcase 1. We assume $\gamma' \notin \boldsymbol{Z}_p$. We have $\delta' \in \boldsymbol{Z}_p$ and $\delta'' \in \boldsymbol{Z}_p$. But then there exist r and s in $\boldsymbol{Z}$ such that $rx_1 + sx_2 + x_3 \in Y_{\boldsymbol{Z}_p}$, which contradicts our assumption.

Subcase 2. We assume $\delta' \notin \boldsymbol{Z}_p$. This is also impossible as above.

Subcase 3. We assume $\gamma', \delta' \in \boldsymbol{Z}_p$. From $x_2 \notin Y_{\boldsymbol{Z}_p}$ we have $\alpha' \notin \boldsymbol{Z}_p$, but then there exists r in $\boldsymbol{Z}$ such that $rx_1 + x_2 \in Y_{\boldsymbol{Z}_p}$, which also contradicts our assumption. Therefore we have $(f_p \mid T_0)(x) = 0$.

On the other hand, by the same method as Case 4, we have $G_0(x) = 1$ if and only if

$$\beta,\ \beta',\ \beta'' \in \boldsymbol{Z}_p,$$

and

$$\gamma' \in p^{-1}\boldsymbol{Z}_p - \boldsymbol{Z}_p, \quad \delta' \in \boldsymbol{Z}_p, \quad \gamma'' \in p^{-1}\boldsymbol{Z}_p - \boldsymbol{Z}_p, \quad \delta'' \in \boldsymbol{Z}_p, \quad \text{or}$$
$$\gamma' \in \boldsymbol{Z}_p, \quad \delta' \in p^{-1}\boldsymbol{Z}_p - \boldsymbol{Z}_p, \quad \gamma'' \in \boldsymbol{Z}_p, \quad \delta'' \in p^{-1}\boldsymbol{Z}_p - \boldsymbol{Z}_p.$$

If this is true, there exist s and t in $\boldsymbol{Z}$ such that $sx_1 + tx_2 + x_3 \in Y_{\boldsymbol{Z}_p}$, which also contradicts our assumption, so that we have $G_0(x) = 0$. This completes the proof. q.e.d.

Let φ_i be an automorphic form on $D_A^\times$. We constructed the Siegel modular form Φ_B of degree 3 for some fixed basis B of $\mathfrak{h}(\lambda)$ in Proposition 1. The following corollary is an easy consequence of Theorem 1.

COROLLARY 1. *Let p be an odd prime number. Suppose that φ_i is an eigenfunction of $T(p)$ with the eigenvalue $\lambda_i(p)$. Then Φ_B is also an eigenfunction of T_0 with eigenvalue $p^2(p+1)(\lambda_1(p) + \lambda_2(p))$.*

Next we deal with the Hecke operators of degree p^2. To state the commutation relations for Hecke operators T_1 and T_2, we introduce two functions:

$$G_1(x) = \sum_{v_1, v_2 = 0}^{p-1} f_p\left(\rho\left(\begin{pmatrix} p & v_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p & v_2 \\ 0 & 1 \end{pmatrix}\right)x\right) + \sum_{v_1 = 0}^{p-1} f_p\left(\rho\left(\begin{pmatrix} p & v_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)x\right)$$
$$+ \sum_{v_2 = 0}^{p-1} f_p\left(\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} p & v_2 \\ 0 & 1 \end{pmatrix}\right)x\right) + f_p\left(\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)x\right)$$

$$G_2(x) = \sum_{v=0}^{p^2-1} f_p\left(\rho\left(\begin{pmatrix} p^2 & v \\ 0 & 1 \end{pmatrix}, 1\right)x\right) + \sum_{v=1}^{p-1} f_p\left(\rho\left(\begin{pmatrix} p & v \\ 0 & p \end{pmatrix}, 1\right)x\right) + f_p\left(\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, 1\right)x\right)$$
$$+ \sum_{v=0}^{p^2-1} f_p\left(\rho\left(1, \begin{pmatrix} p^2 & v \\ 0 & 1 \end{pmatrix}\right)x\right) + \sum_{v=1}^{p-1} f_p\left(\rho\left(1, \begin{pmatrix} p & v \\ 0 & p \end{pmatrix}\right)x\right) + f_p\left(\rho\left(1, \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}\right)x\right).$$

THEOREM 2. *Let the notation be as above. We assume that p is an odd prime number. Let $T_2 = T(1, p, p, p^2, p, p)$ and $T(p^2)$ be elements of Hecke ring $\mathscr{L}(\tilde{K}_p, S_p)$ defined in Section 3. Then*

$$(4.2)\qquad (f_p \mid T_2)(px) + f_p(px) = p^2\{G_1(x) + (p^2 + p + 1)f_p(px)\}$$

$$(4.3)\qquad (f_p \mid T(p^2))(px) = p^4(p^2 + p + 1)G_2(x) + p^5(p + 2)G_1(x) + p^5(2p + 1)f_p(px).$$

The proofs of (4.2) and (4.3) are similar to that of (4.1) but more complicated, so we omit them here.

COROLLARY 2. *Let φ_i be an automorphic form on $D_A^\times$ for $i = 1, 2$ and Φ_B be the Siegel modular form constructed by them. Suppose that φ_i be an eigenfunction of $T(1, p)$ with eigenvalue $\lambda_i(p)$, $i = 1, 2$. Then*

(i) $\Phi_B \mid T_2 = (p^2\lambda_1(p)\lambda_2(p) + p^4 + p^3 + p^2 - 1)\Phi_B$

(ii) $\Phi_B \mid T(p^2) = \{p^4(p^2 + p + 1)(\lambda_1(p)^2 + \lambda_2(p)^2) + p^5(p + 2)\lambda_1(p)\lambda_2(p) - p^4(2p^3 + 2p^2 + 3p + 2)\}\Phi_B.$

In fact, φ_i is also an eigenfunction of $T(1, p^2)$ with the eigenvalue $\mu_i(p^2) = \lambda_i(p)^2 - (p + 1)$. Then (i) and the following (ii)′ are easy consequences of Theorem 2:

(ii)′ $\Phi_B \mid T(p^2) = \{p^4(p^2 + p + 1)(\mu_1(p^2) + \mu_2(p^2)) + p^5(p + 2)\lambda_1(p)\lambda_2(p) + p^5(2p + 1)\}\Phi_B.$

We get (ii) at once from (ii)′.

It is clear that the Hecke operator $T_3 = T(p, p, p, p, p, p)$ acts trivially on f_p so we have $\Phi_B \mid T_3 = \Phi_B$.

By Theorem C of Andrianov, we know the following relation:

$$pT_1 = T_0^2 - T(p^2) - p(p^2 + p + 1)T_2 - p(p^5 + p^4 + 2p^3 + p^2 + p + 1)T_3.$$

This gives us the eigenvalue of T_1:

$$\Phi_B \mid T_1 = \{p^4(\lambda_1(p)^2 + \lambda_2(p)^2) + p^2(p^3 + p^2 + p - 1)\lambda_1(p)\lambda_2(p) + p^2(p^4 - p^3 - p^2 - 2p - 1)\}\Phi_B.$$

Let $f(n)$ be as defined in Theorem C and $\lambda(n)$ the corresponding eigenvalue: $\Phi_B | f(n) = \lambda(n)\Phi_B$. Then, using these formulas, we have

$$H_{p,\Phi_B}(t) = \sum_{n=0}^{\infty} \lambda(n)t^n = \prod_{i=1}^{2} (1 - \lambda_i(p)p^3t + p^7t^2)(1 - \lambda_i(p)p^2t + p^5t^2).$$

Therefore we get the following theorem.

THEOREM 3. *Let the notation and assumptions be as in Corollary* 2. *Define the L-function of* φ_i *by*

$$L(s, \varphi_i) = \prod_{p \neq 2} (1 - \lambda_i(p)p^{-s} + p^{1-2s})^{-1}.$$

Then, up to the Euler 2-*factor, the L-function of* Φ_B *can be expressed by*

$$L(s, \Phi_B) = \prod_{i=1}^{2} L(s-3, \varphi_i)L(s-2, \varphi_i).$$

REFERENCES

[1] A. N. Andrianov, Shimura's conjecture for Siegel's modular group of genus 3, Soviet Math. Dokl., **8** (1967), No. 6, 1474–1478.
[2] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent. Math., **44** (1978), 1–47.
[3] G. Lion and M. Vergne, The Weil representation, Maslov index and theta series, Birkhäuser, Progress in Math., **6** (1980).
[4] S. Niwa, Modular forms of half integral weight and the integral of certain theta functions, Nagoya Math. J., **56** (1974), 147–161.
[5] G. Shimura, On modular forms of half integral weight, Ann. of Math., **97** (1973), 440–481.
[6] T. Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J., **58** (1975), 83–126.
[7] H. Yoshida, Siegel's modular forms and the arithmetic of quadratic forms, Invent. Math., **60** (1980), 193–248.
[8] ——, Weil's representations of the symplectic groups over finite fields, J. Math. Soc. Japan, **31** (1979), 399–426.
[9] A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math., **111** (1964), 143–211.

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464
Japan