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# AFFINE HYPERSURFACES WITH PARALLEL CUBIC FORM

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## §1. Introduction

As is well known, there exists a canonical transversal vector field on a nondegenerate affine hypersurface $M$. This vector field is called the affine normal. The second fundamental form associated to this affine normal is called the affine metric. If $M$ is locally strongly convex, then this affine metric is a Riemannian metric. And also, using the affine normal and the Gauss formula one can introduce an affine connection $\nabla$ on $M$ which is called the induced affine connection. Thus there are in general two different connections on $M$: one is the induced connection $\nabla$ and the other is the Levi Civita connection $\hat{\nabla}$ of the affine metric $h$. The difference tensor $K$ is defined by $K(X, Y) = K_X Y = \nabla_X Y - \hat{\nabla}_X Y$. The cubic form $C$ is defined by $C = \nabla h$ and is related to the difference tensor by

$$h(K_X Y, Z) = -\frac{1}{2} C(X, Y, Z).$$

The classical Berwald theorem states that $C$ vanishes identically on $M$, implying that the two connections coincide, if and only if $M$ is an open part of a nondegenerate quadric.

In this paper we will consider the condition $\hat{\nabla} C = 0$ for a 4-dimensional locally strongly convex affine hypersurface in $\mathbf{R}^5$. Clearly $\hat{\nabla} C = 0$ if and only if $\hat{\nabla} K = 0$. For surfaces this condition has been studied by M. Magid and K. Nomizu in [MN], where they proved the following;

THEOREM A [MN]. *Let $M^2$ be an affine surface in $\mathbf{R}^3$ with $\hat{\nabla} C = 0$. Then either $M$ is an open part of a nondegenerate quadric (i.e. $C = 0$) or $M$ is affine equivalent to an open part of the following surfaces*:

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(1) $xyz = 1$,
(2) $x(y^2 + z^2) = 1$,
(3) $z = xy + \frac{1}{3}y^3$ (*the Cayley surface*).

A generalization of this theorem to 3-dimensional locally strongly convex hypersurfaces in $\mathbf{R}^4$ is given by the first two authors in [DV1]. There the following classification theorem is proved.

THEOREM B [DV1]. *Let $M$ be a 3-dimensional affine locally strongly convex hypersurface in $\mathbf{R}^4$ with $\hat{\nabla} C = 0$. Then either $M$ is a part of a locally strongly convex quadric (i.e. $C = 0$) or $M$ is affine equivalent to an open part of one of the following two hypersurfaces*:

(1) $xyzw = 1$,
(2) $(y^2 - z^2 - w^2)^3 x^2 = 1$.

Comparing Theorem A and Theorem B with the classification of locally strongly convex homogeneous hyperspheres in $\mathbf{R}^3$ and $\mathbf{R}^4$ in [NS] and [DV3] (homogeneous in the sense used therein), we find that a locally strongly convex affine hypersphere in $\mathbf{R}^3$ or $\mathbf{R}^4$ is homogeneous if and only if it satisfies $\hat{\nabla} C = 0$. In [DV2] it is proved that the hypersurface in $\mathbf{R}^5$ with equation

$$\left(z - \frac{1}{2}x^2/u - \frac{1}{2}y^2/v\right)^4 u^3 v^3 = 1,$$

is a homogeneous hyperbolic affine hypersphere in $\mathbf{R}^5$. It however does not satisfy $\hat{\nabla} C = 0$. In the present paper we give a classification of all locally strongly convex affine hypersurfaces in $\mathbf{R}^5$ with $\hat{\nabla} C = 0$. In particular, our main result is the following theorem.

THEOREM 1. *Let $M$ be a 4-dimensional locally strongly convex affine hypersurface in $\mathbf{R}^5$ with $\hat{\nabla} C = 0$. Then either $M$ is an open part of a locally strongly convex quadric (i.e. $C = 0$) or $M$ is affine equivalent to an open part of one of the following three hypersurfaces*:

(1) $xyzwt = 1$,
(2) $(y^2 - z^2 - w^2 - t^2)^2 x = 1$,
(3) $(z^2 - w^2 - t^2)^3 (xy)^2 = 1$.

All examples occurring in the previous theorems are special cases of the following class of hypersurfaces of $\mathbf{R}^{n+1}$ satisfying $\hat{\nabla} C = 0$ with equation

$$\prod_{i=1}^{k} (x_{i;p_i+1}^2 - \sum_{j=1}^{p_i} x_{i,j}^2)^{p_i+1}(y_1 \cdots y_{q+1})^2 = 1,$$

where $n = \sum_{i=1}^{k}(p_i + 1) + q$ and

$$(x_{1;1}, \ldots, x_{1;p_1+1}, x_{2;1}, \ldots, x_{2;p_2+1}, \ldots, x_{k;1}, \ldots, x_{k;p_k+1}, y_1, \ldots, y_{q+1})$$

are affine coordinates of $\mathbf{R}^{n+1}$. The theorems mentioned above show that this class gives all examples of locally strongly convex hypersurfaces with $\hat{\nabla} C = 0$ for $n =$ 2, 3, 4. This is however not true for $n = 5$, as follows from the discussions in [DV2].

All examples occurring are also homogeneous. This property remains true in all dimensions. We will prove this in the final section.

THEOREM 2. *Let $M$ be a nondegenerate affine hypersurface in $\mathbf{R}^{n+1}$ with $\hat{\nabla} C = 0$. Then $M$ is a locally homogeneous affine sphere.*

We will use the formalism and the notations of [N]. For a short survey of the preliminaries that we need in this paper, we refer to [DV1, §2].

## §2. The construction of an orthonormal basis

In this section, we consider an $n$-dimensional, locally strongly convex affine hypersurface $M$ in $\mathbf{R}^{n+1}$ which has parallel cubic form, i.e. which satisfies $\hat{\nabla} C = 0$. From [BNS], if follows that $M$ is an affine sphere, so the affine shape operator is $S = \lambda I$.

Since $\hat{\nabla} C = 0$ implies that $h(C, C)$ is constant, there are two cases. First if $h(C, C) = 0$, then $C = 0$, $h$ being definite, and therefore $M$ is an open part of a quadric. Otherwise, $C$ never vanishes, and we assume this for the remainder of this section.

Let $p \in M$. We now choose an orthonormal basis with respect to the affine metric $h$ at the point $p$ in the following way, similar as in [DV1]. Let $UM_p = \{u \in T_pM \mid h(u, u) = 1\}$. Since $M$ is locally strongly convex, $UM_p$ is compact. We define a function $f$ on $UM_p$ by $f(u) = h(K_u u, u)$. Let $e_1$ be an element of $UM_p$ at which the function $f$ attains an absolute maximum. If $f(e_1) = 0$, then $f$ is identically zero, and therefore, $K$ being symmetric, $K = 0$. This contradicts our assumption, so $f(e_1) > 0$.

Let $u \in UM_p$ such that $h(u, e_1) = 0$, and let $g$ be a function, defined by $g(t) = f(\cos(t)e_1 + \sin(t)u)$. Since $g$ attains an absolute maximum at $t = 0$, we have $g'(0) = 0$, which means that $h(K_{e_1}e_1, u) = 0$. So $e_1$ is an eigenvector of $K_{e_1}$, say with eigenvalue $\lambda_1$. Let $e_2, e_3, \dots, e_n$ be orthonormal vectors, orthogonal to $e_1$, which are the remaining eigenvectors of $K_{e_1}$ with respective eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_n$. Further, since $e_1$ is an absolute maximum of $f$, we know that $g''(0) \leq 0$ and if $g''(0) = 0$, then also $g'''(0) = 0$. This implies that

$$\lambda_1 - 2\lambda_i \geqq 0 \tag{2.1}$$

and

$$\text{if } \lambda_1 = 2\lambda_i, \text{ then } h(K_{e_i}e_i, e_i) = 0 \tag{2.2}$$

for $i \in \{2,3,\dots,n\}$. From the apolarity condition we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0. \tag{2.3}$$

Now $\hat{\nabla} K = 0$ implies that $\hat{R} \cdot K = 0$, and as in the proof of [DV1, Lemma 3.3], this implies that

$$(\lambda_1 - 2\lambda_i)(-\lambda - \lambda_i^2 + \lambda_1\lambda_i) = 0. \tag{2.4}$$

If $\lambda_1 = 2\lambda_i$ for all $i \in \{2,3,\dots,n\}$, then (2.3) implies that $\lambda_1 = 0$ which is a contradiction. Therefore there is a number $k$, $1 \leq k < n$ such that, after rearranging the ordering,

$$\lambda_2 = \lambda_3 = \cdots = \lambda_k = \frac{1}{2}\lambda_1 \quad \text{and} \quad \lambda_{k+1} < \frac{1}{2}\lambda_1, \dots, \lambda_n < \frac{1}{2}\lambda_1.$$

Moreover, if $i > k$, then (2.4) implies that

$$-\lambda - \lambda_i^2 + \lambda_1\lambda_i = 0. \tag{2.5}$$

Subtracting (2.4) for $i, j > k$, we obtain

$$(\lambda_i - \lambda_j)(\lambda_1 - (\lambda_i + \lambda_j)) = 0.$$

But for $i, j > k$ one can check that $\lambda_1 - (\lambda_i + \lambda_j) \neq 0$. Thus $\lambda_i = \lambda_j$ for $k < i, j \leq n$. Setting $\lambda_{k+1} = \cdots = \lambda_n = \mu$ and using (2.3) and (2.5), we have

$$\mu = -\frac{k+1}{2(n-k)}\lambda_1$$

$$-\lambda = \lambda_1^2 \frac{((k+1)^2 + 2(k+1)(n-k))}{4(n-k)^2}.$$

Therefore we have proved the following result.

PROPOSITION 2.1. *If $M$ is a locally strongly convex hypersurface of $\mathbf{R}^n$ with $\hat{\nabla} C = 0$, then $M$ is a hyperbolic affine sphere.*

## §3. Hypersurfaces in $\mathbf{R}^5$

From now on $M$ will be a hypersurface in $\mathbf{R}^5$. Then, using the notation of §2, we have the following cases.

*Case A*: $k = 1$. Then $\lambda_1 = \frac{3}{2}\sqrt{-\lambda}$, $\lambda_2 = \lambda_3 = \lambda_4 = -\frac{\sqrt{-\lambda}}{2}$.

So in this case $K_{e_i}$ has a 3-dimensional eigenspace corresponding to the eigenvalue $\mu$. Define the function $f_1$ to be the restriction of $f$ to this eigenspace and choose $e_2$ as the maximum of $f_1$, thus $h(K_{e_2}e_2, u) = 0$ where $u \in UM_p$ is orthogonal to both $e_1$ and $e_2$. Let the function $f_2$ be the restriction of $f$ to $B$ where $B = \{u \in UM_p \mid h(u, e_1) = h(u, e_2) = 0\}$. We can choose $e_3$ as an absolute maximum of $f_2$. Then $h(K_{e_3}e_3, u) = 0$ for $u \in UM_p$ with $h(u, e_1) = h(u, e_2) = h(u, e_3) = 0$. Finally we can adjust the sign of $e_4$ such that $h(K_{e_2}e_3, e_4) \geq 0$. Resuming, the difference tensor $K$ takes the following form:

$$K_{e_1}e_1 = \frac{3\sqrt{-\lambda}}{2} e_1,\ K_{e_2}e_2 = -\frac{\sqrt{-\lambda}}{2} e_1 + ae_2,$$

$$K_{e_3}e_3 = -\frac{\sqrt{-\lambda}}{2} e_1 + be_2 + de_3,\ K_{e_4}e_4 = -\frac{\sqrt{-\lambda}}{2} e_1 - (a + b)e_2 - de_3,$$

$$K_{e_1}e_2 = \frac{-\sqrt{-\lambda}}{2} e_2,\ K_{e_1}e_3 = \frac{-\sqrt{-\lambda}}{2} e_3,\ K_{e_1}e_4 = \frac{-\sqrt{-\lambda}}{2} e_4$$

$$K_{e_2}e_3 = be_3 + ce_4,\ K_{e_2}e_4 = ce_3 - (a + b)e_4,\ K_{e3}e_4 = ce_2 - de_4,$$

where $a$, $b$, $c$, $d$ are real numbers and by assumption $a \geq 0$, $c \geq 0$, $d \geq 0$. Note that if $a = 0$, then the function $f_2$ is identically zero, so also $b = c = d = 0$.

*Case B*: $k = 2$. Then $\lambda_1 = 4\sqrt{\frac{-\lambda}{21}}$, $\lambda_2 = 2\sqrt{\frac{-\lambda}{21}}$ and $\lambda_3 = \lambda_4 = -3\sqrt{\frac{-\lambda}{21}}$.

Here, we can choose $e_3$ in the direction of $K_{e_2}e_2$, such that $h(K_{e_2}e_2, e_4) = 0$ and $h(K_{e_2}e_2, e_3) \geq 0$. Also, because of (2.2) we know that $h(K_{e_2}e_2, e_2) = 0$. Here the difference tensor takes the following form:

$$K_{e_1}e_1 = 4\sqrt{\frac{-\lambda}{21}} e_1,\ K_{e_2}e_2 = 2\sqrt{\frac{-\lambda}{21}} e_1 + ae_3,$$

$$K_{e_3}e_3 = -3\sqrt{\frac{-\lambda}{21}}\, e_1 + be_2 + de_3 + fe_4,$$

$$K_{e_4}e_4 = -3\sqrt{\frac{-\lambda}{21}}\, e_1 - be_2 - (a+d)e_3 - fe_4,$$

$$K_{e_1}e_2 = 2\sqrt{\frac{-\lambda}{21}}\, e_2,\ K_{e_1}e_3 = -3\sqrt{\frac{-\lambda}{21}}\, e_3,\ K_{e_1}e_4 = -3\sqrt{\frac{-\lambda}{21}}\, e_4,$$

$$K_{e_2}e_3 = ae_2 + be_3 + ce_4,\ K_{e_2}e_4 = ce_3 - be_4,\ K_{e_3}e_4 = ce_2 + fe_3 - (a+d)e_4$$

where $a, b, c, d, e, f$ are real numbers and by assumption $a \geq 0$.

*Case* $C$: $k=3$. In this case, we have $\lambda_1 = 2\sqrt{\frac{-\lambda}{24}}$, $\lambda_2 = \lambda_3 = \sqrt{\frac{-\lambda}{24}}$ and $\lambda_4 = -4\sqrt{\frac{-\lambda}{24}}$.

Now we put

$$u = -\frac{1}{2}e_1 + \frac{\varepsilon\sqrt{3}}{2}e_4, \quad \varepsilon = \pm 1.$$

Then we notice that

$$f(u) = h(K_u u, u) = -\frac{1}{8}\lambda_1 - \frac{9}{8}\lambda_4 + \frac{3\sqrt{3}}{8}\varepsilon f(e_4).$$

If we choose $\varepsilon$ such that $\varepsilon f(e_4)$ is positive, then

$$f(u) \geq -\frac{1}{8}\lambda_1 + \frac{18}{8}\lambda_1 = \frac{17}{8}\lambda_1 > \lambda_1,$$

which contradicts the maximality of $\lambda_1$. Thus this case cannot occur.

Expressing the equation $\hat{R}\cdot K = 0$, using the expression for $K$ obtained in cases A and B, we obtain the following system of equations.

*Case* A.
(A1) $4a^2b - 12ab^2 + 8b^3 - 20ac^2 + 8bc^2 - 5\lambda a + 10\lambda b = 0,$
(A2) $c(12a^2 + 4ab + 4b^2 + 4c^2 + 5\lambda) = 0,$
(A3) $d(16ab + 20b^2 + 28c^2 + 24d^2 + 15\lambda) = 0,$
(A4) $b(4b^2 + 4c^2 - 4ab + 5\lambda) = 0,$
(A5) $(a+2b)(4ab + 4b^2 + 4c^2 + 12d^2 + 5\lambda) = 0,$
(A6) $d(12a^2 + 28ab + 20b^2 + 4c^2 + 5\lambda) = 0,$
(A7) $abc = 0$;

*Case* B.

(B1) $b = 0$,

(B2) $c = 0$,

(B3) $af = 0$,

(B4) $3a^2 + \frac{10\lambda}{7} = 0$,

(B5) $2a^2 - ad + \frac{5}{7}\lambda = 0$.

Now we will solve these systems explicitly.

**Solving the equations in Case A.** We already noted that if $a = 0$ then $b = c = d = 0$. This is a solution of the system.
We call it solution (S1).

Suppose $a \neq 0$, then from (A7) it follows that $b = 0$ or $c = 0$. So we can consider the following cases.

(a) $\underline{b = c = 0}$. This is not possible since (A1) then implies that $\lambda a = 0$.

(b) $\underline{b = 0,\ c \neq 0}$. Then (A1) implies that $c^2 = -\frac{\lambda}{4}$ and (A2) implies that $a^2 = \frac{-\lambda}{3}$. Moreover, (A5) gives $d^2 = \frac{-\lambda}{3}$. All the other equations are satisfied. Setting $u = -\frac{1}{\sqrt{5}}(e_2 + e_3 - \sqrt{3}e_4)$, we notice that

$$h(K_u u, u) = \sqrt{\frac{-5\lambda}{3}} > \sqrt{\frac{-\lambda}{3}} = a,$$

but this contradicts the fact that $f_2$ attains an absolute maximum at $e_2$.

(c) $\underline{b \neq 0,\ c = 0,\ d = 0}$. The equation (A4) implies that $4ab = 4b^2 + 5\lambda$. Substituting this in (A5) we obtain that $a + 2b = 0$, implying that $b < 0$. Using (A4) we get $b^2 = -\frac{5\lambda}{12}$. We can compute easily that all the the other equations are satisfied. We will call this solution of the system (S2).

(d) $\underline{b \neq 0,\ c = 0,\ d \neq 0}$. Again (A4) implies that $4ab = 4b^2 + 5\lambda$. Substituting this into (A6) we get that $(a + 2b)(3a + 2b) = 0$.

If $a = -2b$, then $b = -\sqrt{\frac{-5\lambda}{12}}$. Substituting this into (A3) gives us $d = \sqrt{\frac{-5\lambda}{6}}$. All the other equations are satisfied. We call this solution (S3).

If $3a+2b=0$, then $b^2=-\frac{3\lambda}{4}$. From (A5) we then obtain that $d=\frac{\sqrt{-3\lambda}}{3}$

Setting $u=\frac{1}{\sqrt{2}}(-e_2+e_3)$ we get

$$h(K_u u, u)=\frac{3}{4}\sqrt{-\frac{3\lambda}{2}}>\sqrt{\frac{-\lambda}{3}},$$

which again contradicts the fact that $f_2$ attains an absolute maximum at $e_2$.

**Solving the equations in Case B.** From the equation (B4) we conclude that $a=\sqrt{-\frac{10}{21}\lambda}\neq 0$. Thus (B3) implies that $f=0$. From (B5) we get $d=\frac{1}{2}\sqrt{-\frac{10\lambda}{21}}$ and all the other equations are satisfied.

If $u=-\cos\alpha e_3-\sin\alpha e_4$, $\alpha\in\mathbf{R}$, such that $\tan\alpha=\sqrt{\frac{7}{3}}$ then

$$h(K_u u, u)=3\sqrt{3}\sqrt{\frac{-\lambda}{21}}>\lambda_1$$

which contradicts the fact that $\lambda_1$ is an absolute maximum.

**The three possible shapes for $K$.** Corresponding to the three possible solutions of the system (A), the following shapes for $K$ can occur at $p$.

(S1)

$$K_{e_1}e_1=\frac{3\sqrt{-\lambda}}{2}e_1,$$

$$K_{e_2}e_2=K_{e_3}e_3=K_{e_4}e_4=-\frac{\sqrt{-\lambda}}{2}e_1,$$

$$K_{e_1}e_2=-\frac{\sqrt{-\lambda}}{2}e_2,\ K_{e_1}e_3=\frac{\sqrt{-\lambda}}{2}e_3,\ K_{e_1}e_4=-\frac{\sqrt{-\lambda}}{2}e_4,$$

$$K_{e_2}e_3=K_{e_2}e_4=K_{e_3}e_4=0\,;$$

(S2)

$$K_{e_1}e_1=\frac{3\sqrt{-\lambda}}{2}e_1,$$

$$K_{e_2}e_2=\frac{-\sqrt{-\lambda}}{2}e_1+2\sqrt{\frac{-5\lambda}{12}}e_2,$$

$$K_{e_3}e_3 = -\frac{\sqrt{-\lambda}}{2}e_1 - \sqrt{\frac{-5\lambda}{12}}e_2,$$
$$K_{e_4}e_4 = -\frac{\sqrt{-\lambda}}{2}e_1 - \sqrt{\frac{-5\lambda}{12}}e_2,$$

$$K_{e_1}e_2 = -\frac{\sqrt{-\lambda}}{2}e_2,\ K_{e_1}e_3 = -\frac{\sqrt{-\lambda}}{2}e_3,\ K_{e_1}e_4 = -\frac{\sqrt{-\lambda}}{2}e_4,$$

$$K_{e_2}e_3 = -\sqrt{-\frac{5\lambda}{12}}e_3,\ K_{e_2}e_4 = -\sqrt{-\frac{5\lambda}{12}}e_4,\ K_{e_3}e_4 = 0\,;$$

(S3)

$$K_{e_1}e_1 = \frac{3\sqrt{-\lambda}}{2}e_1,$$

$$K_{e_2}e_2 = -\frac{\sqrt{-\lambda}}{2}e_1 + 2\sqrt{-\frac{5\lambda}{21}}e_2,$$

$$K_{e_3}e_3 = -\frac{\sqrt{-\lambda}}{2}e_1 - \sqrt{\frac{-5\lambda}{12}}e_2 + \sqrt{\frac{-5\lambda}{6}}e_3,$$

$$K_{e_4}e_4 = -\frac{\sqrt{-\lambda}}{2}e_1 - \sqrt{\frac{-5\lambda}{12}}e_2 - \sqrt{\frac{-5\lambda}{6}}e_3,$$

$$K_{e_1}e_2 = -\frac{\sqrt{-\lambda}}{2}e_2,\ K_{e_1}e_3 = -\frac{\sqrt{-\lambda}}{2}e_3,\ K_{e_1}e_4 = -\frac{\sqrt{-\lambda}}{2}e_4,$$

$$K_{e_2}e_3 = -\sqrt{\frac{-5\lambda}{12}}e_3,\ K_{e_2}e_4 = -\sqrt{\frac{-5\lambda}{12}}e_4,\ K_{e_3}e_4 = \sqrt{\frac{-5\lambda}{12}}e_4.$$

The following lemma can be proved as [DV1, Lemma 3.4].

LEMMA 3.1. *If the case* (S3) *holds at* $p$ *then all the sectional curvatures are zero, moreover* $h(K, K) = -\frac{67}{12}\lambda$. *If the case* (S1) *holds at* $p$ *then* $h(K, K) = -\frac{9}{2}\lambda$ *and if the case* (S2) *holds at* $p$ *then* $h(K, K) = -\frac{26}{3}\lambda$.

We therefore can conclude that, if (S1), respectively (S2) or (S3), is true at a point $p$, then it is true for every point on $M$. If (S3) is true on $M$, then we can apply the main theorem of [VLS] and obtain that $M$ is affine equivalent to an open part of the hypersurface (1) of Theorem 1.

Having this basis $\{e_i\}$ at a point $p$, we can translate it parallelly along geodesics through $p$ and obtain a local frame $\{E_i\}$ on a normal neighborhood of $p$. Since $\hat{\nabla}K = 0$, $K$ will have the same expression in any point as in $p$. This is stated in the following lemmas, which can be proved similarly as Lemma 3.5 and Lemma 3.6 of [DV1].

LEMMA 3.2. *Let $M$ be a 4-dimensional locally strongly convex affine hypersurface in $\mathbf{R}^5$ with $\hat{\nabla} C = 0$. If* (S1) *holds at every point of $M$, then there exists a local basis $\{E_1, E_2, E_3, E_4\}$, orthonormal with respect to $h$, such that:*

(1) *at any $p \in M$, $f$ attains its maximum value at $E_1(p)$,*

(2) *at any $p \in M$, $\{E_1(p), E_2(p), E_3(p), E_4(p)\}$ satisfies* (S1)

(3) *$\hat{\nabla}_X E_1 = 0$, for any vector field $X$ on $M$.*

*Moreover, $(M, h)$, considered as a Riemannian manifold, is locally isometric to $\mathbf{R} \times H$, where $H$ is the 3-dimensional hyperbolic space of constant negative sectional curvature $\frac{5\lambda}{4}$. After the identification, $E_1$ is tangent to $\mathbf{R}$.*

LEMMA 3.3. *Let $M$ be a 4-dimensional locally strongly convex affine hypersurface in $\mathbf{R}^5$ with $\hat{\nabla} C = 0$. If* (S2) *holds at every point of $M$, then there exists a local basis $\{E_1, E_2, E_3, E_4\}$ orthonormal with respect to $h$, such that:*

(1) *at any $p \in M$, $f$ attains its maximum value at $E_1(p)$,*

(2) *at any $p \in M$, $\{E_1(p), E_2(p), E_3(p), E_4(p)\}$ satisfies* (S2),

(3) *$\hat{\nabla}_X E_1 = \hat{\nabla}_X E_2 = 0$, for any vector field $X$ on $M$.*

*Moreover, $(M, h)$, considered as a Riemannian manifold is isometric to $\mathbf{R} \times \mathbf{R} \times H$, where $H$ is the hyperbolic plane of constant negative sectional curvature $\frac{5\lambda}{3}$. After the identification, $E_1$ is tangent to the first $\mathbf{R}$-component and $E_2$ is tangent to the second.*

## §4. Proof of Theorem 1

Using [DV2], it is easy to compute that the hypersurface (2) of Theorem 1 satisfies the data of Lemma 3.2, and that the hypersurface (3) satisfies Lemma 3.3 for some appropriate choice of $\lambda$.

Let $M$ satisfy Lemma 3.2, and suppose that $F : M \to \mathbf{R} \times H$ is an isometry (we should rather consider a suitable open subset of $M$, but we don't really worry about this). Let $f : \mathbf{R} \times H \to \mathbf{R}^{n+1}$ be the immersion giving the hypersurface (2), where we apply a homothetic transformation to make sure that both scaling factors $\lambda$ are the same, and let $g : M \to \mathbf{R}^{n+1}$ denote the immersion of $M$.

Let $\{E_1, E_2, E_3, E_4\}$ be the frame on $M$ satisfying Lemma 3.2. Then it can be seen easily that $\{F_*E_1, F_*E_2, F_*E_3, F_*E_4\}$ is a frame on $\mathbf{R} \times H$ such that the difference tensor of $\mathbf{R} \times H$ has the form (S1). Hence $F$ preserves both the affine metric $h$ and the cubic form $C$. Applying the fundamental uniqueness theorem of affine differential geometry, for instance [D, Theorem 3.5], we obtain that there is an affine transformation $A : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ such that $A(g) = f(F)$. This means, forgetting about the immersions, that $M$ is affine equivalent to an open part of (2).

If $M$ satisfies Lemma 3.3, we can show similarly that it is affine equivalent to an open part of (3).

## §5. Proof of Theorem 2

The fact that $M$ is an affine sphere follows from [BNS]. If $M$ satisfies $\hat{\nabla} C = 0$, then $\hat{\nabla}\hat{R} = 0$. Let $p$, $q \in M$ and let $\{e_i\}$ be any orthonormal basis of $T_pM$. We can translate it parallelly along geodesics through $p$ and obtain a local frame $\{E_i\}$ on a normal neighborhood of $p$. Since $\hat{\nabla} K = 0$ and $\hat{\nabla}\hat{R} = 0$, the numbers $c_{ijk} = h(K(E_i, E_j), E_k)$ and $r_{ijkl} = h(\hat{R}(E_i, E_j)E_k, E_l)$ will be constants. If we translate $\{e_i\}$ parallelly to $q$, we obtain an orthonormal basis $\{f_i\}$ of $T_qM$. Let $L: T_pM \to T_qM$ be the linear isometry mapping $e_i$ onto $f_i$. Then $L$ preserves curvature, such that from [O'N, Theorem 8.14] we know that there is an isometry $f: U \to M$ from an open $U$ around $p$ such that $f(p) = q$ and $f_{*p} = L$. Let $F_i = f_* E_i$, then the frame $\{F_i\}$ is obtained from $\{f_i\}$ by parallel translation as above. Moreover $c_{ijk} = h(K(F_i, F_j), F_k)$ and $r_{ijkl} = h(\hat{R}(F_i, F_j)F_k, F_l)$ are the same constants. Therefore $f$ preserves both $h$ and $K$, so by the fundamental uniqueness theorem, we again obtain that there is an equiaffine transformation $A$ of $\mathbf{R}^{n+1}$ such that $A(x) = f(x)$ for all $x \in U$. Hence $M$ is locally homogeneous.

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