ON THE PROJECTIVE VARIETIES ASSOCIATED WITH
SOME SUBRINGS OF THE RING OF THETANULLWERTE

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0. Introduction

Let (X, L) be a principally polarized abelian variety (ppav) of dimension g such that L is a symmetric line bundle, i.e. $i^*L \simeq L$ where i is the inversion map $i(x) = -x$. We shall denote by $X[2]$ the two torsion points of X which are fixed by i. For any x in $X[2]$ we have an isomorphism

$$(1) \qquad t_x^*(L)^2 \simeq L^2.$$

Here t_x is the translation map.

Let τ be a point of the Siegel upper half space $\mathbf{H}_g$ and X be the abelian variety $\mathbf{C}^g/(\tau \mathbf{Z}^g + \mathbf{Z}^g)$. As symmetric line bundle L we take $\mathbf{C}^g \times \mathbf{C}/(\tau \mathbf{Z}^g + \mathbf{Z}^g)$ with the action $(\tau a + b)(z, w) = (z + \tau a + b, \mathbf{e}(-(1/2)^t a \tau a + {}^t a z) w)$.

Here $\mathbf{e}(t)$ stands for $\exp(2\pi i t)$.

For $x = (1/2)(\tau m' + m'')$ in $X[2]$, $t_x^* L$ is still symmetric and we have that the theta function of characteristic m and modulus τ :

$$(2) \quad \vartheta_m(\tau, z) = \sum_{p \in \mathbf{Z}^g} \mathbf{e}\left((1/2)^t\left(p + \frac{m'}{2}\right)\tau\left(p + \frac{m'}{2}\right) + {}^t\left(p + \frac{m'}{2}\right)\left(z + \frac{m''}{2}\right)\right)$$

is up to a multiplicative constant the unique section of the above line bundle. Here m' and m'' are in $\{0, 1\}^g$.

Sometime, if it will be necessary, we shall write $\vartheta_m(\tau, z) = \vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)$.

It is a well known fact that a basis of $H^0(X, L^2)$ is given by the 2^g theta function $\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau, 2z)$ and from (1) we have theta relation

$$(3) \quad \vartheta_m(\tau, z)^2 = \sum_{\sigma} \mathbf{e}((1/2)^t(m' + \sigma)m'') \,\vartheta\begin{bmatrix} \sigma \\ 0 \end{bmatrix}(2\tau, 0)\, \vartheta\begin{bmatrix} m' + \sigma \\ 0 \end{bmatrix}(2\tau, 2z).$$

Let q be an even positive integer we shall denote by $\Gamma_g(q)$ the subgroup of $\Gamma_g = Sp(g, \mathbf{Z})$ defined by

$$(4) \qquad\qquad \sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv 1_{2g} \bmod q.$$

We shall further denote by $\Gamma_g(q, 2q)$ the subgroup of $\Gamma_g(q)$ defined by $\mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) \equiv 0 \bmod 2q$.

Evaluating at $z = 0$ the theta function $\vartheta_m(\tau, z)$ we get an holomorphic function $\vartheta_m = \vartheta_m(\tau)$ defined on $\mathbf{H}_g$, that is not identically zero if and only if the characteristic m is even, i.e. $^t m' m'' \equiv 0 \bmod 2$. We have exactly $2^{g-1}(2^g + 1) = N + 1$ even characteristics. We know, cf [6] and [8], that the maps Θ, Θ^2, Θ^4 sending τ in $(\cdots \vartheta_m(\tau) \cdots)$, $(\cdots \vartheta_m^2(\tau) \cdots)$, $(\cdots \vartheta_m^4(\tau) \cdots)$ from $\mathbf{H}_g$ to $\mathbf{P}^N$ factorize over $\Gamma_g(4,8)$, $\Gamma_g(2,4)$ and $\Gamma_g(2)$ respectively.

We know that all these maps are injective, cf [6] and [8].

Here we shall show that the injectivity of these maps extends to the Satake compactifications and we shall characterize the points of these compactifications in terms of vanishing of Thetanullwerte. Moreover we shall show that the characterization of the reducible points, obtained in this way, is ideal-theoretic. Under Θ, $\Gamma_g(4,8) \backslash \mathbf{H}_g$ is biholomorphic onto its image, cf [6]; Θ^2 and Θ^4 have the same property when g is less or equal to 2, cf [5]. We shall show that Θ^2 and Θ^4 are not immersions for $g \geq 4$ and that Θ^2 is an immersion when g is 3. This result still holds when we extend Θ^2 to the Satake compactification of $\Gamma_3(2,4) \backslash \mathbf{H}_3$.

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1. Satake compactification

Let R be an associative ring with the unity, then we shall denote by $M_{p,q}(R)$ the R-module of the p by q matrices with coefficients in R. Let A and B be in $M_{p,q}(R)$ and $M_{s,t}(T)$ respectively. Then we define the tensor product of matrices

$$A \otimes B = \begin{bmatrix} Ab_{11} & \cdots & Ab_{1t} \\ \cdots & \cdots & \cdots \\ Ab_{s1} & \cdots & Ab_{st} \end{bmatrix}.$$

Among all the properties of the tensor product we recall that if rank $A = p$ and rank $B = s$, then rank $A \otimes B = ps$. We shall write $M_p(R)$ for $M_{p,p}(R)$. $S^g(R)$ shall denote the submodule of the symmetric matrices of $M_p(R)$. $GL(p, R)$ shall denote the subset of A in $M_p(R)$ such that $\det A$ is invertible. We shall de-

note by 1_p the unit matrix; a matrix σ in $GL(2g, R)$ is in $Sp(g, R)$ if

$$(5) \qquad {}^t\sigma \begin{bmatrix} 0 & 1_g \\ -1_g & 0 \end{bmatrix} \sigma = \begin{bmatrix} 0 & 1_g \\ -1_g & 0 \end{bmatrix}.$$

We shall express σ in $Sp(g, R)$ as

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, d are in $M_g(R)$.

We know that $Sp(g, \mathbf{R})$ acts transitively on $\mathbf{H}_g$ as

$$(6) \qquad \sigma \cdot \tau = (a\tau + b)(c\tau + d)^{-1}.$$

Let Γ be a subgroup of finite index of Γ_g, an analytic function ψ defined on $\mathbf{H}_g$, which satisfies the functional equation

$$(7) \qquad \psi(\sigma \cdot \tau) = \det(c\tau + d)^k \, \psi(\tau)$$

for every σ in Γ (plus a condition at the cusps for $g = 1$) is called a modular form of weight k (positive, half-integer) relative to Γ. The set of such functions form a finite dimensional subspace over $\mathbf{C}$ that we shall denote by $[\Gamma, k]$.

The graded ring

$$(8) \qquad A(\Gamma) = \bigoplus_{0 \le k < \infty} [\Gamma, k]$$

is a finitely generated over $\mathbf{C}$, integrally closed domain, it is called the ring of modular forms relative to Γ and the projective variety $\operatorname{Proj} A(\Gamma)$ associated to it is a compactification of $\Gamma \backslash \mathbf{H}_g$, in the sense that this is complex-analytically isomorphic to an open set in the Zariski topology of $\operatorname{Proj} A(\Gamma)$.

Usually $\operatorname{Proj} A(\Gamma)$ is called the Satake compactification of $\mathbf{H}_g / \Gamma$. Let us describe it more in detail, we refer to [2] for an accurate description. Let $\mathbf{S}_g$ be the subset of $M_{2g,g}(\mathbf{C})$ formed by all matrices

$$(9) \qquad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \; W_i \text{ in } M_g(\mathbf{C})$$

having the properties: rank $W = g$ and ${}^tW_1W_2$ is in $S^g(\mathbf{C})$; the group $GL(g, \mathbf{C})$ acts on $\mathbf{S}_g$ by multiplication on the right. Let $\mathbf{G}_g = \mathbf{S}_g / GL(g, \mathbf{C})$, then it is a closed submanifold of the grassmannian manifold of all g dimensional subspaces of $\mathbf{C}^{2g}$.

The map that send τ in the class $\begin{bmatrix} \tau \\ 1_g \end{bmatrix}$ is an embedding of $\mathbf{H}_g$ in $\mathbf{G}_g$.

$Sp(g, \mathbf{C})$ acts on $\mathbf{G}_g$ by multiplication on the left; this is the extension of the action (6), in fact we have

$$(10) \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \tau \\ 1_g \end{bmatrix} = \begin{bmatrix} a\tau + b \\ c\tau + d \end{bmatrix} = \begin{bmatrix} \sigma \cdot \tau \\ 1_g \end{bmatrix}.$$

Let $k \leq g$, then we have an embedding of $\mathbf{H}_k$ in $\mathbf{G}_g$ via the map

$$(11) \qquad j_{g,k}(\tau_k) = \begin{bmatrix} 1_g \\ \tilde{\tau}_k \end{bmatrix}, \ \tilde{\tau}_k = \begin{bmatrix} \tau_k^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We remark that in the case $k = g$, this agrees with the previous embedding. We call the image $j_{g,k}(\mathbf{H}_k)$ the k-dimensional standard boundary component of $\mathbf{H}_g$.

A subset A of $\mathbf{G}_g$ is a k-dimensional rational boundary component if there exists σ in Γ_g such that $\sigma(A)$ is the k-dimensional standard component. We shall write A_k for the union of all k dimensional rational boundary components. We set

$$(12) \qquad \bar{A}_g = \overset{g}{\underset{k=0}{\cup}} A_k.$$

Then it is a well known fact that $\Gamma \backslash \bar{A}_g \simeq \operatorname{Proj} A(\Gamma)$. We call $\Gamma \backslash A_k$ the k-th stratum of $\operatorname{Proj} A(\Gamma)$ and shall denote by α the projection map from $\bar{A}_g$ to $\operatorname{Proj} A(\Gamma)$. In particular the Satake compactifications $\operatorname{Proj} A(\Gamma_g(4,8))$, $\operatorname{Proj} A(\Gamma_g(2,4))$ and $\operatorname{Proj} A(\Gamma_g(2))$ have that the k-th stratum is a disjoint union of a finite number of quasi projective varieties each complex analytically isomorphic to $\Gamma_k(4,8) \backslash \mathbf{H}_k$, $\Gamma_k(2,4) \backslash \mathbf{H}_k$, $\Gamma_k(2) \backslash \mathbf{H}_k$ respectively, $k = 0,1,\dots,(g-1)$.

Clearly each pair of isomorphic components are conjugate under the action of Γ_g. Therefore any point in the boundary can be obtained as

$$(13) \qquad \lim_{t \to +\infty} \begin{bmatrix} \tau_k & 0 \\ 0 & it1_{g-k} \end{bmatrix} = \begin{bmatrix} 1_g \\ \cdot \tilde{\tau}_k \end{bmatrix} := \tilde{\tau}_k$$

with τ_k in $\mathbf{H}_k$ modulo its appropriate relative subgroup $(\Gamma_k(4,8), \Gamma_k(2,4), \Gamma_k(2))$. We shall denote by $\alpha_{4,8}$, $\alpha_{2,4}$ and α_2 the projection maps from $\bar{A}_g$ to $\operatorname{Proj} A(\Gamma_g(4,8))$, $\operatorname{Proj} A(\Gamma_g(2,4))$, $\operatorname{Proj} A(\Gamma_g(2))$ respectively.

Let $m' = ({}^t m'_0, {}^t m'_1)$ and $m'' = ({}^t m''_0, {}^t m''_1)$ with m'_0, m''_0 in $\{0,1\}^k$ and m'_1, m''_1 in $\{0,1\}^{g-k}$, then we shall write $m = m_0 \oplus m_1$ for any τ_k in $\mathbf{H}_k$ we put

$$(14) \qquad \vartheta_m(\tilde{\tau}_k) = \lim_{t \to +\infty} \vartheta_m \begin{pmatrix} \tau_k & 0 \\ 0 & it1_{g-k} \end{pmatrix} = \begin{cases} \vartheta_{m_0}(\tau_k) & \text{if } m'_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

For any m in $\{0,1\}^{2g}$ and σ in Γ_g, we put

$$(15) \qquad \sigma \cdot m = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} + \begin{pmatrix} \text{diag}(c\,{}^t d) \\ \text{diag}(a\,{}^t b) \end{pmatrix}.$$

Let us denote by $\sigma \circ m$ the unique element in $\{0,1\}^{2g}$ that is congruent to $\sigma \cdot m$ mod 2; then (15) give rise to an action of Γ_g on $\{0,1\}^{2g}$.

We recall the transformation formula of theta functions; for every σ in Γ_g and characteristic m we have

$$(16) \qquad \vartheta_{\sigma \circ m}(\sigma \cdot \tau, \,{}^t(cz+d)^{-1}z)$$

$$= k(\sigma)\det(c\tau+d)^{1/2}\mathbf{e}(\varphi_m(\sigma))$$
$$\cdot \mathbf{e}((1/4)\,{}^t(\sigma \circ m)'(\sigma \cdot m - \sigma \circ m)'')\mathbf{e}((1/2)\,{}^t z(c\tau+d)^{-1}cz)$$

$$= k(\sigma)\,\det(c\tau+d)^{1/2}\mathbf{e}((1/2)\,{}^t z(c\tau+d)^{-1}cz)\,\chi_m(\sigma)\vartheta_m(\tau,z)$$

where $k(\sigma)$ is an eighth root of unity independent from m, τ and z, and

$$(17)$$
$$\varphi_m(\sigma) = (-1/8)\,({}^t m'\,{}^t b d m' - 2\,{}^t m'\,{}^t b c m'' + {}^t m''\,{}^t a c m'' - 2\,{}^t \text{diag}(a\,{}^t b)\,(dm' - cm''))$$

An immediate consequence of (14), (16) and (17) is that the maps

$$(18) \quad \Theta : \Gamma_g(4,8) \backslash \mathbf{H}_g \to \mathbf{P}^N, \; \Theta^2 : \Gamma_g(2,4) \backslash \mathbf{H}_g \to \mathbf{P}^N, \; \Theta^4 : \Gamma_g(2) \backslash \mathbf{H}_g \to \mathbf{P}^N$$

are well defined, extend to the respective Satake compactifications and are $\Gamma_g/\Gamma_g(4,8)$, $\Gamma_g/\Gamma_g(2,4)$, $\Gamma_g/\Gamma_g(2)$ equivariant.

We shall denote by $\bar\Theta$, $\bar\Theta^2$, $\bar\Theta^4$ these extensions.

Clearly the images of these maps are the projective varieties $\text{Proj}\,\mathbf{C}[\vartheta_m]$, $\text{Proj}\,\mathbf{C}[\vartheta_m^2]$ $\text{Proj}\,\mathbf{C}[\vartheta_m^4]$. We recall that the map $\bar\Theta^2$ can be obtained, up to a projectivity, composing the map.

$$(19) \qquad \bar\Theta_2 : \text{Proj}\,A(\Gamma_g(2,4)) \to \mathbf{P}^{2^g-1}$$

$$\tau \qquad \to \left(\cdots \vartheta \begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau) \cdots \right)$$

with the Veronese map; therefore $\text{Proj}\,\mathbf{C}[\vartheta_m^2] \cong \text{Proj}\,\mathbf{C}\left[\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}\right]$.

2. Fiber of the projection maps

In this section we shall use several times results obtained in [5] and [8]. Let us recall some of them. We set $C = \Gamma_g(2)/\Gamma_g(2,4)$ and $G' = \Gamma_g(2,4)/\Gamma_g(4,8)$.

If i, j are distinct positive integer at most equal to g, we replace the (i, j)

coefficient of 1_g by 2. We take this matrix as a, we put $b = c = 0$, $d = {}^t a^{-1}$, the corresponding matrix A_{ij} is an element of $\Gamma_g(2,4)$. If $1 \leq i \leq g$ we replace the (i, i) coefficient of 1_g by -1. We take this matrix as a, we put $b = c = 0$, $a = d$, the corresponding matrix A_{ii} is in $\Gamma_g(2,4)$. If $1 \leq i \leq g$ we replace the (i, i) coefficient of the matrix 0 by 2. We take this matrix as b, we put $a = d = 1_g$, $c = 0$; the corresponding matrix B_{ii} is in $\Gamma_g(2)$. If $1 \leq i < j \leq g$ we replace the (i, j) and the (j, i) coefficient of the 0 matrix by 2. We take this matrix as b, we put $a = d = 1_g$, $c = 0$; the corresponding matrix B_{ij} is in $\Gamma_g(2,4)$. Finally we set $C_{ij} = {}^t B_{ij}$. We recall from [5] the following fact. The $g(2g + 1)$ matrices A_{ij} $(1 \leq i \leq j \leq g)$, B_{ij}, C_{ij} $(1 \leq i < j \leq g)$, B_{ii}^2, $C_{ii}^2 (1 \leq i \leq g)$ are a basis of G'; the $2g$ matrices B_{ii}, C_{ii} $(1 \leq i \leq g)$ are a basis of C. Moreover in [8] we proved that the group of characters $\hat{G}'$ and $\hat{C}$ are generated by all χ_m and χ_m^2 respectively, m arbitrary characteristic.

One can deduce the definition of χ_m from (16).

These facts are used to prove the injectivity of the map Θ^2 and Θ^4, in fact they conduce to the injectivity of Θ of [8] Theorem 3. Since $\bar{\Theta}$ is injective of [7], using the same method used in the above cited theorem we get

THEOREM 1.   *The maps $\bar{\Theta}^2$ and $\bar{\Theta}^4$ are injective.*

We have the following diagram

$$
\begin{array}{ccc}
\operatorname{Proj} A(\Gamma_g(4,8)) & \rightarrow & \mathbf{P}^N \\
\downarrow p & & \downarrow Sq \\
\operatorname{Proj} A(\Gamma_g(2,4)) & \rightarrow & \mathbf{P}^N \\
\downarrow q & & \downarrow Sq \\
\operatorname{Proj} A(\Gamma_g(2)) & \rightarrow & \mathbf{P}^N
\end{array}
$$

where the horizontal arrows are injective, Sq is the map squaring the coordinates and p and q are the usual projection map. We are interested in computing the cardinality of the fiber of the maps p and q.

For any point P in $\operatorname{Proj} A(\Gamma_g(4,8))$ and Q in $\operatorname{Proj} A(\Gamma_g(2,4))$ we define St_P and F_Q as the subgroups of G' and C fixing the point P and Q respectively.

We define $\hat{H}_P$ and $\hat{D}_Q$ as the subgroup of $\hat{G}'$ resp. C' generated by all $\chi_m \chi_n$, $\chi_m^2 \chi_n^2$ respectively such that the product $\vartheta_m \vartheta_n$ doesn't vanish at $\alpha_{4,8}^{-1}(P)$ and $\alpha_{2,4}^{-1}(Q)$ resp. In [8] we remarked that St_P and $\hat{H}_P$ are dual and the same is true for F_Q and $\hat{D}_Q$ moreover we proved that, for the points of the g-th stratum, we have $\# q^{-1}(R) = \# \hat{D}_Q$ for any Q such that $q(Q) = R$. This was a consequence of the proof of Theorem 3.

Therefore, now as consequence of Theorem 1, we can extend this result to all points of the Satake compactifications. We can prove the following

THEOREM 2. *Let R denote a point of the k-th stratum of* $\operatorname{Proj} A(\Gamma_g(2))$; *then we have*

$$(21) \qquad\qquad \# q^{-1}(R) = 2^{g+k}.$$

Proof. We have to show that $\# \hat{D}_Q = 2^{g+k}$ for any Q in the k-th stratum of $\operatorname{Proj} A(\Gamma_g(2,4))$. Already in [8] we proved the statement relatively to the g-th stratum. In view of Lemma 10 of [8] that states $\hat{D}_{\sigma \cdot Q} = \hat{D}_Q^\sigma$ we can assume in $a_2^{-1}(R)$ points of the form (13) and $\vartheta_0(\tilde{\tau}_k) \neq 0$. Therefore to prove the proposition, it is enough to find $g + k$ linearly independent (mod 2) characteristics whose associate Thetanullwerte don't vanish at $\hat{\tau}_k$ cf [8]. We know that there are $2k$ linearly independent characteristics $n_1, \ldots, n_{2k}$ in $\{0,1\}^{2k}$ such that $\vartheta_n(\tau_k) \neq 0$ $1 \leq i \leq 2k$. Let $m = n_i \oplus m_1$, $1 \leq i \leq 2k$ and $m_1' = 0$, then we have $\vartheta_m(\tilde{\tau}_k) \neq 0$ for all characteristics m of the above form, thus we can get at least $(g + k)$ linearly independent characteristics such that the associate Thetanullwerte don't vanish at $\hat{\tau}_k$. Using (10) it is immediate that all B_{ti}, $k + 1 \leq i \leq g$, belong to F_Q, therefore $\# F_Q \geq 2^{g-k}$, thus we get $\# \hat{D}_Q = 2^{g+k}$.

For studying the fiber of the map p we need a refinement of the decomposition (12). Let $\underline{r} = (r_1, r_2, \ldots, r_h)$ be a partition of $k \leq g)$, we put

$$(22) \qquad\qquad \mathbf{H}_{\underline{r}} = j_{g,k}(\mathbf{H}_{i_1} \times \mathbf{H}_{r_2} \times \cdots \times \mathbf{H}_{r_h})$$

and

$$(23) \qquad\qquad A_{\underline{r}} = \Gamma_g \cdot \mathbf{H}_{\underline{r}}$$

let $\underline{s}$ be a refinement of $\underline{r}$, we shall write $\underline{s} < \underline{r}$, then we have $A_{\underline{s}} \subseteq A_{\underline{r}}$ if and only if $\underline{s} < \underline{r}$.

Clearly A_k is equal to $A_{(k)}$. For any partition $\underline{r}$ we shall denote by $A_{\underline{r}}^0$ the set of points in $A_{\underline{r}}$ that are not contained in any refinement of $\underline{r}$, then we have

$$(24) \qquad\qquad A_k = \overset{\displaystyle\cdot}{\underset{\underline{r}, \text{ partition of } k}{\cup}} A_{\underline{r}}^0.$$

We call $A_{(g)}^0$ the set of irreducible points of $A_g = \mathbf{H}_g$, $A_g \setminus A_g^0$, the set of reducible points and $\Gamma \setminus A_{\underline{r}}^0$ the $\underline{r}$ component of the k-th stratum.

Let $\underline{s} = (s_1, \ldots, s_l)$, $1 \geq h$ be a partition of $j < k$, then we shall write $\underline{s} << \underline{r}$ if there exist k indices $0 = i_0 < i_1 < i_2 < \ldots i_k = l$ such that $s_{i_\alpha} + s_{i_\alpha + 1}$

$+ \cdots + s_{i_{\alpha+1}} < r_{\alpha+1}, \ 0 \le \alpha \le k - 1.$
We put $\bar{\mathbf{H}}_{\underline{r}} = \mathbf{H}_{\underline{r}} \ \underset{\underline{s} < < \underline{r}}{\cup \ \cup} \ \mathbf{H}_{\underline{s}}.$

THEOREM 3.   *Let Q denote a point of the $\underline{r} = (r_1, r_2, \ldots r_h)$ component of the k-th stratum of $\mathrm{Proj} A(\Gamma_g(2,4))$; then we have*

$$\# p^{-1}(Q) = 2^{(1/2)(g+k)(g+k+1)-h}.$$

In [8] we proved the theorem relatively to points in the g-th stratum. The same method, using the results of Theorem 2, extends to all other strata reducing ourselves to computing the cardinality of the fiber over a point of the g-th stratum. In this case we obtain $\# p^{-1}(Q) = 2^{g(2g+1)-h}$. At this point the proof is a consequence of that of Proposition 4 of [8] and of the decomposition $(1/2)(g + k)$

$$(g + k + 1) - h = (k(2k + 1) - h) + \binom{g - k + 1}{2} + 2k(g - k).$$

Let $x_1, x_2, \ldots, x_h, x_{h+1}$ be characteristics in $\{0,1\}^{2r_1}, \ldots \{0,1\}^{2r_h}, \{0,1\}^{2(g-k)}$ respectively then we put

(25) $m = x_1 \oplus x_2 \ldots \oplus x_{h+1}.$

If one of the x_i, $1 \le i \le h$, is an odd characteristic or $x'_{h+1} \ne 0$, then it is a well known fact that the associate Thetanullwerte vanish on $\mathbf{H}_{\underline{r}}$. Let $\bar{V}_{\underline{r}}$ be the subset of $\bar{A}_g$ defined by the vanishing of all above described Thetanullwerte, then generalizing Theorem 5 of [8] we have

THEOREM 4. $\bar{V}_{\underline{r}} = \Gamma_g(2) \cdot \bar{\mathbf{H}}_{\underline{r}}$

Clearly a similar statement holds for the conjugate components.
From now to the end of this section we shall consider the case $k = g$.
Let as recall from [6] p. 187 the heat equation; differantiating the theta functions $\vartheta_m(\tau, z)$ respect to $z_1, \ldots, z_g, \tau_{11}, \tau_{12}, \ldots, \tau_{gg}$ we have

(26) $(2\pi i)^p \prod_{k=1}^{p} (\delta^{j_k}_{i_k} + 1) \partial^p \vartheta_m / \partial \tau_{i_1 j_1} \cdots \partial \tau_{i_p j_p}$
$$= \partial^{2p} \vartheta_m(\tau, z) / \partial z_{i_1} \partial z_{j_1} \cdots \partial z_{i_p} \partial z_{j_p}.$$

Moreover we know that at all point of $\mathbf{H}_g$

(27) $\mathrm{rank} \ ^t(\partial \vartheta_m(\tau, 0) / \partial z_1, \ldots, \partial \vartheta_m(\tau, 0) / \partial z_g)_{m \ \mathrm{odd}} = g.$

THEOREM 5.   i) $V_{\underline{r}} = \bar{V}_{\underline{r}} \cap \mathbf{H}_g$ *is a complex submanifold of $\Gamma_g(4,8) \setminus \mathbf{H}_g$.*

ii) $\Gamma_g(4,8) \setminus V_r$ is a complex submanifold of $\Gamma_g(4,8) \setminus \mathbf{H}_g$.

Proof. It is clear that (ii) descends from (i). Let us prove (i). We know that set theoretically $V_r = \Gamma_g(2)\mathbf{H}_r$, therefore we have codim $V_r = \Pi_{1 \leq i < j \leq h} r_i r_j$.

We shall show that this number is the rank of the matrix

$$(28) \qquad (\partial \vartheta_m(\tau, 0) / \partial \tau_{ij})_{1 \leq i \leq j \leq g},$$

with m of the form (25) and τ in V_r. Applying the transformation formula (16) we can assume τ in $\mathbf{H}_r$, i.e.

$$(29) \qquad \tau = \begin{bmatrix} \tau_1 & 0 & . & 0 \\ 0 & \tau_2 & 0 & . \\ . & . & . & 0 \\ 0 & . & 0 & \tau_h \end{bmatrix}, \ \tau_i \in \mathbf{H}_{r_i}.$$

We know that all $\partial \vartheta_m(\tau, 0) / \partial \tau_{ij}$ are 0 if more than two x_i's are odd. Let us assume x_1 and x_2 odd characteristics in the decomposition (25), then we have

$$(30) \qquad \partial \vartheta_m(\tau, 0) / \partial \tau_{ij} = \begin{cases} \prod_{i=3}^{h} \vartheta_{x_i}(\tau_i) \, (\partial \vartheta_{x_1}(\tau_1, 0) / \partial z_i) \, (\partial \vartheta_{x_2}(\tau_2, 0) / \partial z_j) \\ \qquad \text{if } 1 \leq i \leq r_1 < j \leq r_1 + r_2 \\ 0 \quad \text{otherwise.} \end{cases}$$

Let us take $x_3, \ldots, x_h$, even characteristics in such way that

$$\prod_{i=3}^{h} \vartheta_{x_i}(\tau_i) \neq 0.$$

This is always possible; now if we allow to the x_i's of varying in the set of odd resp. even characteristics according as $i \leq 2$ or $i \geq 3$; we get a submatrix A_{12} of (28). It is an immediate consequence of (27) and of the tensor product of matrices that this submatrix has rank $r_1 r_2$. Iterating this process we obtain similar matrices A_{ij}, $1 \leq i \leq j \leq g$, of rank $r_i r_j$. We remark that the matrix (28) is of the form

$$\begin{pmatrix} 0 & . & . & . & . & . & . & . & . & 0 \\ . & A_{12} & 0 & . & . & . & . & . & . & . \\ . & 0 & A_{13} & 0 & . & . & . & . & . & . \\ . & . & . & & . & . & . & . & . & . \\ . & . & . & 0 & A_{1g} & 0 & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & A_{23} & 0 & . & . \\ 0 & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & 0 & A_{g-1,g} & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \end{pmatrix}$$

that has the required rank. In this case too a similar statement holds for conjugate components. Finally we remark that all irreducible component of V_r are two by two disjoint.

3. Comparison of projective varieties

We shall prove that in genus 3 case $\operatorname{Proj} \mathbf{C}[\vartheta^2_m]$ is a normal projective variety, therefore $\bar{\Theta}^2$ will be an immersion.

It is a well known fact, cf. [2] and [3] that, for $g \geq 3$, the set of regular points of $\operatorname{Proj} A(\Gamma_g(2,4))$ coincides with. $\Gamma_g(2,4) \setminus A^0_g$, moreover at these points we have that the rank of

$$(31) \quad \left(\vartheta \begin{bmatrix} m' \\ 0 \end{bmatrix} (2\tau, 0), \, \partial\vartheta \begin{bmatrix} m' \\ 0 \end{bmatrix} (2\tau, 0) / \partial\tau_{11}, \ldots, \partial\vartheta \begin{bmatrix} m' \\ 0 \end{bmatrix} (2\tau, 0) / \partial\tau_{gg} \right)_{m' \in \{0,1\}^g}$$

is $(1/2)g(g+1) + 1 = \dim \operatorname{Proj} A(\Gamma_g(2,4)) + 1$; then Θ^2 is an immersion at the regular points.

THEOREM 5. *Let us assume $g = 3$, then $\operatorname{Proj} \mathbf{C}[\vartheta^2_m]$ is a normal variety, therefore $\bar{\Theta}^2$ is an immersion.*

Proof. We have $\operatorname{Proj} \mathbf{C}[\vartheta^2_m] \simeq \operatorname{Proj} \mathbf{C}\left[\vartheta \begin{bmatrix} m' \\ 0 \end{bmatrix} \right]$. It is an hypersurface of degree 16 in $\mathbf{P}^7$ cf [3], therefore it is Cohen–Macaulay. Since the subvariety of the singular points has codimension 2, it is regular in codimension 1, thus it is normal cf. [4] p. 187. As B. Runge pointed out [9] since $\operatorname{Proj} \mathbf{C}\left[\vartheta \begin{bmatrix} m' \\ 0 \end{bmatrix} \right]$ is a complete intersection, this variety is projectively normal.

Now, we want to disprove that Θ^2 and Θ^4 are immersions when $g \geq 4$; then we shall consider these maps at some special point; in particular let τ_0 be equal to the diagonal matrix

$$(32) \qquad \tau_0 = \begin{bmatrix} \tau_{11} & 0 & . & 0 \\ 0 & \tau_{22} & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \tau_{gg} \end{bmatrix}.$$

We put $Q_0 = \alpha_{2,4}(\tau_0)$ and $R_0 = \alpha_0(\tau_0)$. Let D be the finite group generated by A_{ii}, $1 \leq i \leq g$. It is a consequence of [8] that all σ in $\Gamma_g(2)$ fixing τ_0 are the elements of D. Moreover there exists σ_1 in $Sp(g, \mathbf{R})$ such that σ_1 centralizes all ele-

ments of D and $\sigma_1 \cdot \tau_0 = i1_g$. Let

$$(33) \qquad\qquad \sigma_2 = \begin{pmatrix} 1_g & -i1_g \\ 0 & 1_g \end{pmatrix}$$

then $\sigma = \sigma_2 \cdot \sigma_1$ sends τ_0 to 0 in $S^g(\mathbf{C})$ and fix all elements of D. Clearly σ is an analytic automorphism of $\mathbf{G}_g$.

Let x be an element of D, then it is of the form $a = d = y$, $b = c = 0$, with y diagonal matrix with -1 and 1 as entries.

Let A be in $S^g(\mathbf{C})$, then the action of x at a neighborhood of 0 is given by

$$x \cdot A = yAy.$$

Let Y be the quotient space $S^g(\mathbf{C})/D$ and let us consider all monomials of the form

$$(34) \qquad\qquad a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1} \quad 1 \le i_1, i_2, i_3, \ldots, i_n \le g,$$

they are invariant for the actions of D on $S^g(\mathbf{C})$. Let us denote by m the maximal ideal of the homomorphic local ring $O_{0,Y}$.

LEMMA 1. i) *The dimension of the tangent space T_0Y at 0 of Y is $t_g = g\binom{g}{2} + (1/2)\sum_{h=3}^{g}(h-1)!\binom{g}{h}$.*

ii) *A basis of m/m^2 consists of all monomials of the form (34) of degree $\le g$ with the indices appearing with multiplicity 2.*

Proof. (i) follows from (ii), since there is a correspondence between k cycles of g elements and the monomials decribed in (ii). Moreover two cycles determine the same monomial if and only if they are one the inverse of the other. Thus we get $\dim_{\mathbf{C}} T_0Y = t_g$.

(ii) is a consequence of the fact that these monomials are a minimal set of generators for the ring of the invariants $\mathbf{C}[a_{ij}]^D$ $1 \le i \le j \le g$.

Clearly σ induces an isomorphism between

$$(35) \qquad T_{R_0}\,\mathrm{Proj}\,A(\Gamma_g(2)), \; T_{Q_0}\,\mathrm{Proj}\,A(\Gamma_g(2,4)) \text{ and } T_0Y.$$

We have

THEOREM 6. i) *Θ_2 is not a biholomorphic map when $g \ge 4$*
ii) *Θ^4 is not a biholomorphic map when $g \ge 4$.*

Proof. i) We know that a holomorphic map $\varphi : W \to Z$ is an immersion at P in W if and only if the jacobian map

(36) $$T_P \varphi : T_P W \to T_{\varphi(P)} Z$$

is injective cf. [1]. Let us prove (i). Clearly in this case we have $\dim T_{\varphi(Q_0)} Z \leq 2^g - 1$ and

(37) $$t_g = t_{g-1} + g + (1/2) \sum_{h=2}^{(g-1)} h! \binom{g-1}{h} > t_{g-1} + 2^{g-1}$$

for $g \geq 4$. Moreover $t_3 = 7$; this proves (i).

The method, that we shall use to prove (ii), works for the first case too, but, as we shall see, it is less elegant.

We have to write explicitly part of the matrix associated to the jacobian map. We have to consider the t_g by $(N + 1)$ matrix

(38) $$(\partial^n \vartheta_m^4(\tau) / \partial \tau_{i_1 i_2} \cdots \partial \tau_{i_n i_1})_{m \text{ even}}$$

where $\partial \tau_{i_1 i_2} \cdots \vartheta \tau_{i_n i_1}$ is the basis of $T_{R_0} \operatorname{Proj} A(\Gamma_g(2))$ related to the basis of $T_0 Y$. We shall say that a characteristic m is of type 0 if ${}^t m' m'' = 0$. Using the heat equations (26) and evaluating at $z = 0$ and $\tau = \tau_0$ an elementary and tedious computation gives us

(39) $$4\pi^4 \partial^4 \vartheta_m^4(\tau_0, 0) / \partial \tau_{12} \partial \tau_{23} \partial \tau_{34} \partial \tau_{41}$$
$$= \begin{cases} \vartheta_m^3(\tau_0) \partial^8 \vartheta_m(\tau_0, 0) / (\partial z_1)^2 (\partial z_2)^2 (\partial z_3)^2 (\partial z_4)^2 \\ \text{if } m \text{ is of type 0} \\ 0 \qquad \text{otherwise.} \end{cases}$$

We remark that

(40) $$\partial^4 \vartheta_m(\tau_0, 0) / \partial \tau_{12} \partial \tau_{23} \partial \tau_{34} \partial \tau_{41} = \partial^4 \vartheta_m(\tau_0, 0) / \partial \tau_{12} \partial \tau_{24} \partial \tau_{43} \partial \tau_{13}$$

for all characteristics m; therefore the matrix (38) has rank less than t_g. This proves the theorem.

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