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POLYHEDRICITY OF CONVEX SETS IN SOBOLEV SPACE $H_0^2(\Omega)$

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1. Introduction

We provide results on differential stability of metric projection in Sobolev space $H_0^2(\Omega)$ onto convex set

(1.1) $$K = \{f \in H_0^2(\Omega) \mid f(x) \geq \psi(x),\ x \in \Omega\}$$

where $\Omega \subset R^d$ is open, bounded domain.

We derive the form of tangent cone $T_K(f)$ for any element $f \in K$–see Theorem 1. The same argument can be used for convex set

$$K = \{f \in H_0^m(\Omega) \mid f \geq \psi\},\ m = 2, 3, \ldots$$

where $\psi \in H^m(\Omega)$, $\psi < 0$ on $\partial\Omega$.

In section 3 we provide necessary and sufficient conditions under which set K is polyhedric [5], [8] at a given point $f \in K$. The question of polyhedricity is addressed here since it implies directional differentiability of metric projection onto K with the explicit form of the differential [5], [8]. We refer the reader to [5], [8] for related results in the Sobolev space $H_0^1(\Omega)$. Some applications of the differential stability of metric projection onto convex sets in Sobolev spaces are presented in [6], [9]-[18].

We recall some properties of the Sobolev spaces and the notion of capacity [19]. The Sobolev spaces $H_0^1(\Omega)$ and $H_0^2(\Omega)$ are the closures of $C_0^\infty(\Omega)$ with norms

$$\| \varphi \|^2_{H_0^1(\Omega)} = \int_\Omega | \nabla\varphi |^2 dx$$

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respectively. If $\varphi \in H_0^2(\Omega)$, from the definition $D^\alpha \varphi \in H_0^1(\Omega)$ for each α with $|\alpha| = 1$. Functions in $H_0^1(\Omega)$ are defined quasi everywhere and are quasi continuous. These notions are made precise below.

The C_1-capacity of a compact set F is defined as

$$C_1(F) = \inf \left\{ \int |\nabla\varphi|^2 dx : \varphi \geq 1 \text{ on } F,\ 0 \leq \varphi \in C_0^\infty(R^d) \right\}$$

similarly C_2-capacity

$$C_2(F) = \inf \left\{ \int |\Delta\varphi|^2 dx : \varphi \geq 1 \text{ on } F,\ 0 \leq \varphi \in C_0^\infty(R^d) \right\}.$$

The capacity of a Borel set is then defined as the supremum of capacities of its compact subsets. A statement holds C_i-q.e., $i = 1,2$, if it holds except for a set of C_i-capacity zero. With this definition we have the following results:

1. Let $\varphi \in H_0^1(\Omega)$, and $\{\varphi_n\} \subset C_0^\infty(\Omega)$ converge to φ in $H_0^1(\Omega)$. Then a subsequence of $\{\varphi_n\}$ converge C_1-q.e. and this is a representative of φ.
2. Let $\varphi \in H_0^1(\Omega)$. Then φ has a quasicontinuous representative: There is a representative $\bar{\varphi}$ such that given $\varepsilon > 0$, there is an open set $U(\varepsilon)$ of C_1-capacity less than ε such that the restriction of $\bar{\varphi}$ to the complement of $U(\varepsilon)$ is continuous.
3. Any two quasi continuous representatives of $\varphi \in H_0^1(\Omega)$ agree C_1-q.e.
4. Every set of positive Lebesque measure has positive C_1-capacity.

We use standard notation throughout the paper [1], [19].

2. Tangent cone

We shall consider the metric projection onto the following convex set

$$K = \{f \in H_0^2(\Omega) \mid f(x) \geq \psi(x),\ x \in \Omega\} \tag{2.1}$$

with respect to the scalar product

$$(y, z) = \int_\Omega \Delta y(x) \Delta z(x)\, dx. \tag{2.2}$$

We assume that $\psi \in H^2(\Omega)$, $\psi(x) < 0$ on $\partial\Omega$, therefore set (2.1) is nonempty. The metric projection $z = P_K y$, $y \in H_0^2(\Omega)$, is given by the unique solution of the following variational inequality

(2.3) $$z \in K : \int_\Omega \Delta z(x)\Delta(\varphi - z)(x)dx \geq \int_\Omega \Delta y(x)\Delta(\varphi - z)(x)dx \quad \forall \varphi \in K.$$

We denote

(2.4) $$C_K(z) = \{\varphi \in H_0^2(\Omega) \mid \exists t > 0 \text{ such that } z + t\varphi \in K\}.$$

We derive the form of tangent cone $T_K(z) = \mathrm{cl} C_K(z)$ for any element z in convex set (2.1).

THEOREM 1. *For any element $z \in K$, tangent cone $T_K(z)$ takes the form*

(2.5) $$T_K(z) = \{\varphi \in H_0^2(\Omega) \mid \varphi(x) \geq 0,\ C_2\text{-q.e. on } \Xi\}$$

where $\Xi = \{x \in \Omega \mid z(x) = \psi(x)\} \subset \Omega$.

Proof of Theorem 1. Note that $C_K(z)$ and hence also $T_K(z)$ is a convex cone containing all non-negative elements of $H_0^2(\Omega)$. Let an element $V \in H_0^2(\Omega)$ be given and suppose that $V \geq 0$ C_2-q.e. on Ξ. There exists the unique element $\phi_0 \in T_K(z)$ such that

(2.6) $$\| V - \phi_0 \|^2_{H_0^2(\Omega)} = \inf\{\| V - \phi_0 \|^2_{H_0^2(\Omega)} \mid \phi \in C_K(z)\}.$$

It is easy to see that for any $H_0^2(\Omega) \ni \phi \geq 0$, $t \geq 0$, $\phi_0 + t\phi \in T_K(z)$. Using (2.6) and standard arguments it follows

(2.7) $$(V - \phi_0, \phi)_{H_0^2(\Omega)} \leq 0,\ 0 \leq \phi \in H_0^2(\Omega)$$

hence there exists a non-negative Radon measure μ on Ω such that

(2.8) $$(V - \phi_0, \phi)_{H_0^2(\Omega)} \leq -\int \phi d\mu,\quad \phi \in C_0^\infty(\Omega).$$

This implies in particular that for $\phi \geq 0$

$$\int \phi d\mu = -(V - \phi_0, \phi)_{H_0^2(\Omega)} \leq \| V - \phi_0 \|_{H_0^2(\Omega)} \| \phi \|_{H_0^2(\Omega)}.$$

So by definition of C_2-capacity we see μ cannot charge sets of zero C_2-capacity. Since the measure may be large near the boundary it is not clear that (2.8) holds for all $\phi \in H_0^2(\Omega)$. We can circumvent this difficulty by repeated use of a result of L. I. Hedberg: Theorem 3.1 in [7]. First we show that (2.8) holds for any bounded $\phi \in H_0^2(\Omega)$ which is non-negative and has compact support. Indeed for suitable mollifiers ρ_n, $\phi * \rho_n \in C_0^\infty(\Omega)$, have compact support, and tend boundedly pointwise C_2-q.e. and in $H_0^2(\Omega)$ to ϕ. Since μ is Randon measure we may appeal to Lebesque dominated convergence to finish the claim. In the general case if $0 \leq \phi \in H_0^2(\Omega)$ by the above theorem of Hedberg, we can select $0 \leq w_k \leq 1$, $k = 1,2,\ldots$ such that $w_k\phi$ has compact support and is in L^∞ approximating ϕ in $H_0^2(\Omega)$. In particular $w_k\phi$ converges to ϕ C_2-q.e. By (2.8) we have

$$\int w_k\phi d\mu = -(V - \phi_0, w_k\phi)_{H_0^2(\Omega)}$$

is bounded, so by Fatou Lemma $\phi \in L^1(\mu)$. On the other hand $w_k\phi \leq \phi$ so domin ated convergence applies

$$-\int \phi d\mu = (V - \phi_0, \phi)_{H_0^2(\Omega)}, \quad 0 \leq \phi \in H_0^2(\Omega). \tag{2.9}$$

Now let $\phi \in C_0^\infty(\Omega)$, $0 \leq \phi \leq 1$, then $\phi(z - \psi) \in H_0^2(\Omega)$. We show that

$$\phi_0 + t\phi(z - \psi) \in T_K(z), \ -1 \leq t \leq 1.$$

It is sufficient to show that for any $\varphi \in C_K(z)$, it follows $\varphi + t\phi(z - \psi) \in C_K(z)$. Now $\varepsilon\varphi + z - \psi \geq$ in Ω for some $\varepsilon > 0$, hence for $s > 0$, $\frac{s}{1-s} < \varepsilon$ we have

$$s[\varphi + t\phi(z - \psi)] + z - \psi \geq 0, \text{ in } \Omega$$

since $(1 + st\phi)(z - \psi) \geq (1 - s)(z - \psi)$. Using this in (2.6) with ϕ replaced by $\phi_0 + t\phi(z - \psi)$ we obtain

$$(V - \phi_0, \phi(z - \psi))_{H_0^2(\Omega)} = 0$$

which, because $\phi(z - \psi)$ has compact support and belongs to $H_0^2(\Omega)$ means

$$\int \phi(z - \psi) d\mu = 0$$

hence

$$\mu(x : z > \psi) = 0$$

i.e. μ is concentrated on Ξ. Our next step is to show that $\phi_0 = 0$ μ-a.e. To this end using the fact that $T_K(z)$ is a cone and taking $t\phi_0$ for ϕ in (2.6) we get

$$(V - \phi_0, \phi_0)_{H_0^2(\Omega)} = 0. \tag{2.10}$$

Now we use Hedberg's result once more. Choose w_k, $0 \leq w_k \leq 1$ such that $w_k\phi_0$ has compact support and converges to ϕ_0 in $H_0^2(\Omega)$. Since $\phi_0 \geq 0$ on Ξ and μ is concentrated on Ξ, $w_k\phi_0 \leq \phi_0$ μ-a.e. So using the same argument as above we get

$$0 = (V - \phi_0, \phi_0)_{H_0^2(\Omega)} = -\int \phi_0 d\mu$$

i.e. that $\phi_0 = 0$ μ-a.e.

Finally since $\phi_0 = 0$ μ-a.e and $V \geq 0$ C_2-q.e. on Ξ we can repeat the above argument to get

$$(V - \phi_0, V - \phi_0)_{H_0^2(\Omega)} = -\int (V - \phi_0) d\mu = -\int V d\mu.$$

But the right hand side is ≤ 0 because $V \geq 0$, thus $V = \phi_0$.

Remark 1. For $d = 1, 2, 3$ proof of Theorem 1 simplifies since by Sobolev embedding theorem $H_0^2(\Omega) \subset C(\bar{\Omega})$. It is clear that

$$T_K(u) \subset \{\varphi \in H_0^2(\Omega) \mid \varphi(x) \geq 0, \text{ on } \Xi\}$$

therefore it is sufficient to show that any element $V(\cdot) \geq 0$ on Ξ actually belongs to $T_K(u)$. Ξ is compact, hence there exists $0 \leq \eta \in C_0^\infty(\Omega)$, $\eta \equiv 1$ on Ξ. Since by Sobolev embedding theorem u, ψ, $V \in C(\bar{\Omega})$ therefore for any $\varepsilon > 0$ there exists $t > 0$ such that

$$t(V + \varepsilon\eta) + u - \psi \geq 0, \quad \text{in } \Omega.$$

Thus

$$V + \varepsilon\eta \in C_K(u), \ \forall \varepsilon > 0$$

and

$$V + \varepsilon\eta \to V \text{ in } H_0^2(\Omega) \quad \text{strongly with } \varepsilon \downarrow 0$$

hence $V \in \overline{C_K(u)} = T_K(u)$.

3. Differentiability of metric projection

We derive a result on the differentiability of metric projection P_K in the Hilbert space $H = H_0^2(\Omega)$ onto convex closed set $K \subset H$ of the form (2.1). Here we assume for the sake of simplicity that $d = 1,2,3$, hence by the Sobolev embedding theorem it follows that $H^2(\Omega) \subset C(\bar{\Omega})$, the latter embedding is compact [1] for bounded domain Ω with smooth boundary $\partial\Omega$. We use the following notation. For any given element $u \in K$ we denote

$$C_K(u) = \{\phi \in H \mid \exists\, t > 0 \text{ such that } u + t\phi \in K\}. \tag{3.1}$$

The tangent cone $T_K(u)$ to K at u is the closure of set (3.1)

$$T_K(u) = \mathrm{cl}(C_K(u)). \tag{3.2}$$

Let us consider set K defined in section 1. We shall address the question of polyhedricity of K, see Definition 1 below. Let $T_K(f)$ be the tangent cone to K at $f \in K$. It is clear that $T_K(f)$ is the closure in the space $H_0^2(\Omega)$ of the convex cone

$$C_K(f) = \{v \in H_0^2(\Omega) \mid \exists\, t > 0 \text{ such that } f(x) + tv(x) \geq \psi(x) \text{ in } \Omega\}. \tag{3.3}$$

For a given element $g \in H_0^2(\Omega)$, such that $f = P_K(g)$ let us define the following convex cone in the space $H_0^2(\Omega)$

$$S = T_K(f) \cap [g - P_K(g)]^{\perp} = T_K(f) \cap [f - g]^{\perp}. \tag{3.4}$$

DEFINITION 1. The set $K \subset H_0^2(\Omega)$ is polyhedric at $f \in K$, if for any $g \in H_0^2(\Omega)$ such that $f = P_K g$ it follows

$$T_K(f) \cap [f - g]^{\perp} = \mathrm{cl}(C_K(f) \cap [f - g]^{\perp}) \tag{3.5}$$

here cl stands for the closure.

Remark 2. Let us recall [5], [8] that if condition (3.5) is satisfied for given elements $(f, g) \in H_0^2(\Omega) \times H_0^2(\Omega)$, $f = P_K(g)$ then for all $h \in H_0^2(\Omega)$ and for $t > 0$ small enough

$$P_K(g + th) = P_K g + tP_S h + o(t). \tag{3.6}$$

In such a case the metric projection P_K is conically differentiable, in the notation of [8], at $g \in H_0^2(\Omega)$. It turns out that condition (3.5) is satisfied if and only if the support of non-negative Radon measure defined below by (3.9) is admissible in the following way.

DEFINITION 2. Compact F is admissible if for any element $\varphi \in H_0^2(\Omega)$, $\varphi = 0$ on F implies $\varphi \in H_0^2(\Omega \setminus F)$.

We denote by $B(x, r)$, $x \in R^d$, $r > 0$ the ball of radius r and center x, $|A|$ denotes the Lebesque measure of any set $A \subset R^d$.

PROPOSITION 1. *Let $F \subset \Omega$ be compact and assume that the following holds: for C_1-quasi every $x \in F$,*

$$|F \cap B(x, r)| > 0.$$

Then F is admissible.

Proof of Proposition 1. By Theorem 1.1 in [7] it is sufficient to show the following: let $\varphi \in H_0^2(\Omega)$ and $\varphi = 0$ C_2-q.e. on F. Then $\nabla\varphi = 0$ C_1-q.e. on F. Now $\varphi \in H_0^1(\Omega)$ so by a standard result, $\nabla\varphi = 0$ a.e. on F. Since $\varphi \in H_0^2(\Omega)$, each component of $\nabla\varphi$ belongs to $H_0^1(\Omega)$ and hence has a finely continuous version [19]. If for $x \in F$, $|\nabla\varphi|(x) > 0$ then in a fine neighborhood of x the same inequality will obtain. Since finely open sets have positive measure, and since $\nabla\varphi = 0$ a.e. on F, this violates our condition on F. Thus $\nabla\varphi = 0$ C_1-q.e. on F.

Denote by $\nu \geq 0$ the Radon measure defined as follows

$$\int \varphi d\nu = \int_\Omega \Delta(g - f)\Delta\varphi dx, \ \forall \varphi \in C_0^\infty(\Omega). \tag{3.9}$$

THEOREM 2. *We have*

$$\begin{aligned} &\mathrm{cl}(C_K(f) \cap [f - g]^\perp) \\ &= \{\varphi \in H_0^2(\Omega \setminus F) \mid \varphi \geq 0 \ \textit{on} \ \Xi \setminus \mathrm{spt}\, \nu\} \end{aligned} \tag{3.10}$$

where spt $\nu \subset \Xi$ *is compact,* spt ν *denotes the support of Radon measure* ν.

Proof of Theorem 2. It is clear that

$$\mathrm{cl}(C_K(f) \cap [f - g]^\perp) \subset S = T_K(f) \cap [f - g]^\perp \tag{3.11}$$

and in view of Theorem 1

$$S = \{\varphi \in H_0^2(\Omega) \mid \varphi = 0 \text{ on spt}\, \nu, \ \varphi \geq 0 \text{ on } \Xi \setminus \mathrm{spt}\, \nu\}. \tag{3.12}$$

Let us observe that

$$H^2(\Omega) \ni f - \psi \geq 0, \text{ and } f - \psi = 0 \text{ on compact set } \Xi \tag{3.13}$$

therefore it can be shown [20]

$$\nabla(f-\psi)=0 \ C_1\text{-q.e. on } \Xi. \tag{3.14}$$

Let $\varphi \in C_K(f) \cap [f-g]^{\perp}$ then for some $t>0$

$$t\varphi+f-\psi \geq 0 \text{ on } \Omega, \text{ and } \varphi=0 \text{ q.e. on spt } \nu. \tag{3.15}$$

It follows that $\nabla[t\varphi+f-\psi]=0$ C_1-q.e. on spt ν i.e. that $\nabla\varphi=0$ C_1-q.e. on spt ν. Clearly the same conclusion obtains for any element in $\mathrm{cl}(C_K(f) \cap [f-g]^{\perp})$ therefore

$$\mathrm{cl}(C_K(f) \cap [f-g]^{\perp}) \subset H_0^2(\Omega \setminus \text{spt } \nu). \tag{3.16}$$

Now we can use the same argument as in the proof of Theorem 1 to show that if V is an arbitrary element in set

$$\{\varphi \in H_0^2(\Omega \setminus \text{spt } \nu) \mid \varphi \geq 0 \text{ on } \Xi\} \tag{3.17}$$

and φ_0 denotes the projection of V onto $\mathrm{cl}(C_K(f) \cap [f-g]^{\perp})$ then $V=\varphi_0$. Thus

$$\mathrm{cl}(C_K(f) \cap [f-g]^{\perp}) = \{\varphi \in H_0^2(\Omega \setminus \text{spt } \nu) \mid \varphi \geq 0 \text{ on spt } \nu\}. \tag{3.18}$$

THEOREM 3. *Set K is polyhedric at $f \in K$ if and only if $C_1(\Xi)=0$, where $\Xi=\{x \in \Omega \mid f(x)=\psi(x)\}$.*

Proof. We show that in (3.9) we can have any nonnegative Radon measure $\nu \in H^{-2}(\Omega)$ with spt $\nu \subset \Xi$. Let such $\nu \geq 0$ be given. Let $g \in H_0^2(\Omega)$ satisfy

$$\int_\Omega \Delta g \Delta\varphi dx = \int_\Omega \Delta f \Delta \varphi dx - \int \varphi d\nu, \ \forall \varphi \in H_0^2(\Omega). \tag{3.19}$$

We have $f=P_K g$. To see it let us observe that

$$\int \varphi d\nu \geq 0, \ \forall \varphi \in T_K(f) \tag{3.20}$$

since $\eta - f \in T_K(f)$, $\forall \eta \in K$ it follows

$$\int (\eta - f) d\nu \geq 0, \ \forall \eta \in K \tag{3.21}$$

hence

$$\int (\eta-f)d\nu = \int_\Omega \Delta(f-g)\Delta(\eta-f)dx \geq 0, \ \forall \eta \in K \tag{3.22}$$

which shows that $f=P_K g$. Therefore condition (3.5) can be satisfied if and only if

$C_1(\Xi) = 0.$

COROLLARY 1. Assume that $F = \text{spt}\,\nu$ is admissible then (3.5) and (3.6) hold, where cone S is defined by (3.12).

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