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RIESZ CAPACITY AND REGULAR BOUNDARY POINTS FOR THE PARABOLIC OPERATOR OF ORDER α

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§1. Introduction

Let $R^{n+1} = R^n \times R$ be the $(n+1)$-dimensional Euclidean space with $n \geq 1$. We denote by $X = (x, t)$ a point in R^{n+1} with $x \in R^n$ and $t \in R$. Consider the parabolic operator on R^{n+1}:

$$L^{(\alpha)} = \frac{\partial}{\partial t} + (-\Delta)^{\alpha},$$

where $0 < \alpha \leq 1$ and Δ denotes the Laplacian on R^n.

For a closed set E in R^n, we put

$$T^{(\alpha)}(E) = \{(s^{1/2\alpha}x, -s)\,;\, x \in E,\, s \geq 0\}$$

and

$$\Omega^{(\alpha)}(E) = R^{n+1} \setminus T^{(\alpha)}(E).$$

In [EK] and [IN], it is shown that for a non-empty open set ω in R^n, the origin O is a regular boundary point of $\Omega^{(\alpha)}(\bar{\omega})$ for $L^{(\alpha)}$ (with respect to the Dirichlet problem). The purpose of this paper is to give a characterization of this type. Let $K_{2\alpha}(x, y)$ be the kernel on $R^n \times R^n$ of the form

$$K_{2\alpha}(x, y) = \begin{cases} 1 & \left(n = 1,\, \alpha > \dfrac{1}{2}\right) \\ \max\left(0, \log\dfrac{1}{|x-y|}\right) & (2\alpha = n) \\ |x-y|^{2\alpha - n} & (1 \leq 2\alpha < n) \\ \min(|x|^{2\alpha-2}, |y|^{2\alpha-n}) & \left(n = 1,\, \alpha < \dfrac{1}{2}\right) \\ |x-y|\,(|x-y|\,\theta(y, x-y) + |x-y|^{\frac{1}{2\alpha}})^{1-n-2\alpha} & \left(n \geq 2,\, \alpha = \dfrac{1}{2}\right), \end{cases}$$

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where $\theta(y, x-y)$ is the angle between y and $x-y$ or $\theta(y, x-y)=0$ according as $|y||x-y| \neq 0$ or $|y||x-y|=0$. We denote by $C_{2\alpha}(E)$ the capacity of a set E associated with $K_{2\alpha}$. Our main theorem is the following

THEOREM 1. *Let E be a closed set in R^n. Then the origin O is a regular boundary point of $\Omega^{(\alpha)}(E)$ for $L^{(\alpha)}$ if and only if $C_{2\alpha}(E)>0$.*

From Theorem 1, it follows immediately

THEOREM 2. *Let Ω be an open set in R^{n+1} and $(x_0, t_0) \in R^{n+1}$ a boundary point of Ω. If there exist $s_0>0$ and a closed set E in R^n with $C_{2\alpha}(E)>0$ and with*

$$T^{(\alpha)}_{(x_0,t_0)}(E, s_0) \cap \Omega = \emptyset,$$

where

$$T^{(\alpha)}_{(x_0,t_0)}(E, s_0) = \{(x_0+s^{1/2\alpha}x,\ t_0-s)\ ;\ x \in E,\ 0 \leq s \leq s_0\},$$

then (x_0, t_0) is a regular boundary point of Ω for $L^{(\alpha)}$.

We remark that for $1/2 \leq \alpha \leq 1$, $C_{2\alpha}(E)$ is the 2α-Riesz capacity of E. For $0<\alpha<1/2$, there is no relation between $C_{2\alpha}(E)$ and the Riesz capacity of E (see Section 4 for further discussions). Our typical applications of Theorem 1 is the following

COROLLARY 3. *Let H be a hyperplane in R^n and E a non-empty open subset of H. In the case $0 \in H$, the origin O is a regular boundary point of $\Omega^{(\alpha)}(\bar{E})$ for $L^{(\alpha)}$ if and only if $1/2<\alpha \leq 1$. In the case $0 \notin H$, O is a regular boundary point of $\Omega^{(\alpha)}(\bar{E})$ for $L^{(\alpha)}$ if and only if $\alpha \neq 1/2$.*

In particular, we have

COROLLARY 4. *Let $n=1$ and let E be a closed set in R. Then the origin O is a regular boundary point of $\Omega^{(\alpha)}(E)$ for $L^{(\alpha)}$ if and only if*

$$\begin{array}{ll} E \neq \emptyset & \text{for } 1/2<\alpha \leq 1, \\ C_1(E)>0 & \text{for } \alpha=1/2, \\ E \setminus \{0\} \neq \emptyset & \text{for } 0<\alpha<1/2, \end{array}$$

where $C_1(E)$ is the logarithmic capacity of E.

§2. Capacities and regular boundary points

Denote by $W^{(\alpha)}$ the fundamental solution of $L^{(\alpha)}$, that is,

$$W^{(\alpha)}(x, t) = \begin{cases} (2\pi)^{-n} \int_{R^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}x\cdot\xi)d\xi & t > 0 \\ 0 & t \leq 0, \end{cases}$$

where $x \cdot \xi$ is the inner product of x and ξ and $|\xi| = (\xi \cdot \xi)^{1/2}$. Put $\phi_\alpha(|x|) = W^{(\alpha)}(x, 1)$. Then ϕ_α is decreasing on $[0, \infty)$ and

(1) $$W^{(\alpha)}(x, t) = t^{-n/2\alpha}\phi_\alpha(t^{-1/2\alpha}|x|)$$

for $t > 0$. Furthermore, in the case $0 < \alpha < 1$, $\phi_\alpha(r)$ is of order $r^{-n-2\alpha}$ as $r \to \infty$.

Recall that for a closed set F in R^{n+1}, the α-parabolic capacity $\mathrm{cap}^{(\alpha)}(F)$ of F is defined by

$$\mathrm{cap}^{(\alpha)}(F) = \sup\left\{\int d\mu \,;\, \mu \in M_1(F)\right\},$$

where $M_1(F)$ is the set of all Radon measures $\mu \geq 0$ on R^{n+1} supported by F satisfying $W^{(\alpha)}*\mu \leq 1$ on R^{n+1} (see [N] and [W]). If F is compact, there exists a unique $\mu \in M_1(F)$ with $\int d\mu = \mathrm{cap}^{(\alpha)}(F)$, which is called the equilibrium measure of F with respect to $W^{(\alpha)}$. For $\lambda > 0$, we denote by $\tau_\lambda^{(\alpha)} : R^{n+1} \to R^{n+1}$ the 2α-parabolic dilation, that is, $\tau_\lambda^{(\alpha)}(x, t) = (\lambda x, \lambda^{2\alpha}t)$. By (1), we have

Proposition 5. *Let F be a compact set in R^{n+1} and $\lambda > 0$. Denote by μ and μ_λ the equilibrium measures of F and $\tau_\lambda^{(\alpha)}(F)$ with respect to $W^{(\alpha)}$, respectively. Then we have*

$$\mu_\lambda = \lambda^n \cdot \tau_\lambda^{(\alpha)}\mu$$

and

$$\mathrm{cap}^{(\alpha)}(\tau_\lambda^{(\alpha)}(F)) = \lambda^n \mathrm{cap}^{(\alpha)}(F),$$

where $\tau_\lambda^{(\alpha)}\mu$ is the image measure of μ by $\tau_\lambda^{(\alpha)}$.

We define the capacity associated with $K_{2\alpha}$ in the usual manner.

Definition 1. For $E \subset R^n$, we put

$$C_{2\alpha}(E) = \sup\left\{\int d\nu \,;\, \nu \in \mathfrak{M}_1(E)\right\},$$

where $\mathfrak{M}_1(E)$ is the set of all Radon measures $\nu \geq 0$ on R^n such that $\mathrm{supp}(\nu) \subset E$ and $\int K_{2\alpha}(x, y)\,d\nu(y) \leq 1$ for every $x \in R^n$.

DEFINITION 2 (see [IN]). Let Ω be an open set in R^{n+1} and X_0 a boundary point of Ω. Then X_0 is said to be regular for $L^{(\alpha)}$ (with respect to the Dirichlet problem) if

$$\lim_{X\in\Omega, X\to X_0} \varepsilon''_{X,\complement\Omega} = \varepsilon_{X_0} \text{ (vaguely)},$$

where $\varepsilon''_{X,\complement\Omega}$ is the balayaged measure of the point measure ε_X at X to $\complement\Omega$ (the complement of Ω) with respect to $\tilde{W}^{(\alpha)}(x, t) = W^{(\alpha)}(x, -t)$. Denote by $\tilde{M}_X(\complement\Omega)$ the vague closure of the set of all positive Radon measures μ on R^{n+1} which satisfies $\tilde{W}^{(\alpha)} * \mu \geq \tilde{W}^{(\alpha)} * \varepsilon_X$ on $\complement\Omega$. Then $\varepsilon''_{X,\complement\Omega}$ is the unique positive measure in $\tilde{M}_X(\complement\Omega)$ which satisfies $\tilde{W}^{(\alpha)} * \varepsilon''_{X,\complement\Omega} \leq \tilde{W}^{(\alpha)} * \mu$ on R^{n+1} for every $\mu \in \tilde{M}_X(\complement\Omega)$.

By the Wiener criterion and Proposition 5, we know the following characterization of the regular boundary points (see [EG] and [N]).

PROPOSITION 6. *For a closed set E in R^n, the following three conditions are equivalent:*
(1) *O is a regular boundary point of $\Omega^{(\alpha)}(E)$ for $L^{(\alpha)}$.*
(2) $\mathrm{cap}^{(\alpha)}(T^{(\alpha)}(E, s)) > 0$ *for every* $s > 0$.
(3) $\mathrm{cap}^{(\alpha)}(T^{(\alpha)}(E, s_1, s_2)) > 0$ *for some* $0 < s_1 < s_2$,

where

$$T^{(\alpha)}(E, s_1, s_2) = \{(s^{1/2\alpha}x, -s) \,;\, x \in E,\ s_1 \leq s \leq s_2\}.$$

§3. Proofs of Theorem 1 and Corollary 3

In order to discuss the α-parabolic capacity and the capacity associated with $K_{2\alpha}$, we estimate the following integral:

$$\int_{-2}^{-1} W^{(\alpha)}((x, -1) - ((-s)^{1/2\alpha}y, s))\,ds.$$

Since the 2α-Riesz kernel is the fundamental solution of $(-\Delta)^\alpha$, we have

LEMMA 7. *For* $|x| \leq 1/2$, *there exists a constant* $M > 1$ *such that*

$$M^{-1} R_{2\alpha}(x) \leq \int_{-2}^{-1} W^{(\alpha)}((x, -1)-(0, s))ds \leq M R_{2\alpha}(x),$$

where $R_{2\alpha}$ *is the* 2α*-Riesz kernel, that is,*

$$R_{2\alpha}(x) = \begin{cases} |x|^{2\alpha-n} & \text{for } 2\alpha < n \\ \max\left(0, \log\frac{1}{|x|}\right) & \text{for } 2\alpha = n \\ 1 & \text{for } 2\alpha > n. \end{cases}$$

In the proof of Theorem 1, the following lemma plays an essential role.

LEMMA 8. *For each* $0 < r_1 < r_2$, *there exists a constant* $M > 0$ *such that*

$$M^{-1} K_{2\alpha}(x, y) \leq \int_{-2}^{-1} W^{(\alpha)}((x, -1)-((-s)^{1/2\alpha}y, s))ds \leq M K_{2\alpha}(x, y)$$

for every $x, y \in R^n$ *with* $|x| \leq r_2$, $r_1 \leq |y| \leq r_2$.

Proof. For functions a and b on $\{(x, y) ; |x| \leq r_2, r_1 \leq |y| \leq r_2\}$, we write $a \approx b$ if

$$C^{-1}b \leq a \leq Cb$$

for some constant $C \geq 1$. Assume $\alpha \geq 1/2$. Then we have

$$\begin{aligned} (2) \qquad & \int_{-2}^{-1} W^{(\alpha)}((x, -1)-((-s)^{1/2\alpha}y, s))ds \\ &= \int_0^1 s^{-n/2\alpha}\phi_\alpha(s^{-1/2\alpha}|x-(s+1)^{1/2\alpha}y|)\, ds \\ &= \int_0^1 s^{-n/2\alpha}\phi_\alpha(|s^{-1/2\alpha}(x-y) - s^{-1/2\alpha}((s+1)^{1/2\alpha}-1)y|)\, ds \\ &\approx \int_0^1 s^{-n/2\alpha}\phi_\alpha(s^{-1/2\alpha}|x-y|)\, ds \\ &\approx K_{2\alpha}(x, y). \end{aligned}$$

Assume $0 < \alpha < 1/2$. Then we have

$$\int_{-2}^{-1} W^{(\alpha)}((x, -1)-((-s)^{1/2\alpha}y, s))ds$$

$$= \int_0^1 s^{-n/2\alpha} \phi_\alpha (s^{-1/2\alpha} \,| x - (s+1)^{1/2\alpha} y |)\, ds$$

$$\approx \int_0^1 s^{-n/2\alpha} \phi_\alpha (s^{-1/2\alpha} \,| (2^{1/2\alpha} - 1) sy - (x - y) |)\, ds$$

$$\approx \int_0^1 s^{-n/2\alpha} \phi_\alpha (s^{-1/2\alpha} \,| s\tilde{y} - \tilde{x} |)\, ds \quad (\text{say} = I),$$

where $\tilde{y} = y/|y|$ and $\tilde{x} = (x-y)/(2^{1/2\alpha} - 1)|y|$. In the above calculation, we use the change of variables from s to s':

$$(s+1)^{1/2\alpha} = (2^{1/2\alpha} - 1)s' + 1.$$

We may assume that

$$|\tilde{x}| \le \min\left(\frac{1}{2}, \left(\frac{\pi}{6}\right)^{\frac{2\alpha}{1-2\alpha}}\right).$$

By separating into the following three cases of $\theta = \theta(\tilde{x}, \tilde{y})$, we estimate the integral I ; $\pi/6 \le \theta \le \pi$, $|\tilde{x}|^{1/2\alpha - 1} \le \theta \le \pi/6$ and $0 \le \theta \le |\tilde{x}|^{1/2\alpha - 1}$.

In the case $\theta \ge \pi/6$, we have, for $0 \le s \le 1$,

$$|s\tilde{y} - \tilde{x}| \approx |\tilde{x}| + s,$$

so that

$$I \approx \int_0^1 s^{-n/2\alpha} \phi_\alpha (s^{-1/2\alpha} (|\tilde{x}| + s))\, ds \tag{3}$$

$$\approx \int_0^{|\tilde{x}|} s^{-n/2\alpha} \phi_\alpha (s^{-1/2\alpha} |\tilde{x}|)\, ds + \int_{|\tilde{x}|}^1 s^{-n/2\alpha} \phi_\alpha (s^{1-1/2\alpha})\, ds$$

$$\approx \int_0^{|\tilde{x}|} s\, |\tilde{x}|^{-n-2\alpha}\, ds + \int_{|\tilde{x}|}^1 s^{1-n-2\alpha}\, ds$$

$$\approx \begin{cases} 1 & \text{for } n = 1 \\ |\tilde{x}|^{2-n-2\alpha} & \text{for } n \ge 2. \end{cases}$$

In the case $0 \le \theta \le \pi/6$, we have

$$|s\tilde{y} - \tilde{x}| \approx |\tilde{x}| \quad \text{for} \quad 0 \le s \le |\tilde{x}|/3$$

and

$$|s\tilde{y} - \tilde{x}| \approx s \quad \text{for} \quad s \ge 2|\tilde{x}|,$$

so that

(4)
$$\approx \int_0^{|\tilde{x}|/3} + \int_{2|\tilde{x}|}^1 s^{-n/2\alpha}\phi_\alpha(s^{1-1/2\alpha}\,|\,s\tilde{y}-\tilde{x}\,|)ds$$
$$\approx \begin{cases} 1 & \text{for } n=1 \\ |\,\tilde{x}\,|^{2-n-2\alpha} & \text{for } n\geq 2. \end{cases}$$

In the case $|\,\tilde{x}\,|^{1/2\alpha-1}\leq\theta\leq\pi/6$, denoting by $\tilde{x}\cdot\tilde{y}$ the inner product of $\tilde{x}$ and $\tilde{y}$, we have for $|\,\tilde{x}\,|/3\leq s\leq 2\,|\,\tilde{x}\,|$,

$$s^{-1/2\alpha}\,|\,s\tilde{y}-\tilde{x}\,|\geq s^{-1/2\alpha}\,|\,\tilde{x}\,|\sin\theta\approx\theta\,|\,\tilde{x}\,|^{1-1/2\alpha}\geq 1,$$

so that

$$\int_{|\tilde{x}|/3}^{2|\tilde{x}|} s^{-n/2\alpha}\phi_\alpha(s^{-1/2\alpha}\,|\,s\tilde{y}-\tilde{x}\,|)ds$$
$$\approx \int_{|\tilde{x}|/3}^{2|\tilde{x}|} |\,\tilde{x}\,|\,|\,s\tilde{y}-\tilde{x}\,|^{-n-2\alpha}\,ds$$
$$\approx \int_{|s-\tilde{x}\cdot\tilde{y}|\leq|\tilde{x}|\theta} |\,\tilde{x}\,|\,(|\,\tilde{x}\,|\theta)^{-n-2\alpha}\,ds$$
$$+\int_{|s-\tilde{x}\cdot\tilde{y}|\leq|\tilde{x}|\theta,|\tilde{x}|/3\leq s\leq 2|\tilde{x}|} |\,\tilde{x}\,|\,|\,s-\tilde{x}\cdot\tilde{y}\,|^{-n-2\alpha}\,ds$$
$$\approx |\,\tilde{x}\,|\,(|\,\tilde{x}\,|\,\theta)^{1-n-2\alpha}.$$

Therefore combining this relation with (4), in the case $|\,\tilde{x}\,|^{1/2\alpha-1}\leq\theta\leq\pi/6$, we have

(5)
$$I\approx|\,\tilde{x}\,|\,(|\,\tilde{x}\,|\,\theta)^{1-n-2\alpha}.$$

In the case $\theta\leq|\,\tilde{x}\,|^{1/2\alpha-1}$, we have

$$\int_{|\tilde{x}|/3}^{2|\tilde{x}|} s^{-n/2\alpha}\phi_\alpha(s^{-1/2\alpha}\,|\,s\tilde{y}-\tilde{x}\,|)ds$$
$$\approx \int_{|\tilde{x}|/3}^{2|x|} |\,\tilde{x}\,|^{-n/2\alpha}\phi_a(|\,\tilde{x}\,|^{-1/2\alpha}\,|\,s\tilde{y}-\tilde{x}\,|)ds$$
$$\approx \int_{|s-\tilde{x}\cdot\tilde{y}|\leq|\tilde{x}|^{1/2\alpha}} |\,\tilde{x}\,|^{-n/2\alpha}\,ds$$
$$+\int_{(|s-\tilde{x}\cdot\tilde{y}|\geq|\tilde{x}|^{1/2\alpha},|\tilde{x}|/3\leq s\leq 2|\tilde{x}|)} |\,\tilde{x}\,|\,|\,s-\tilde{x}\cdot\tilde{y}\,|^{-n-2\alpha}\,ds$$

$$\approx |\tilde{x}|^{(1-n)/2\alpha}.$$

This and (4) imply

$$I \approx |\tilde{x}|^{(1-n)/2\alpha}. \tag{6}$$

Thus Lemma 8 is shown by (2), (3), (5) and (6).

To prove Theorem 1, we use the following

PROPOSITION 9. *Let μ be a positive Radon measure on $R^n \times [0, \infty)$. For $s > 0$, we put $\mu_s = \mu * \varepsilon_{(0,s)}$. If for any $s > 0$,*

$$0 \neq \mu|_{R^n \times (s,\infty)} \leq \mu_s,$$

then for every $0 < s_1 < s_2$, there exists a positive Radon measure $\nu \neq 0$ on R^n such that

$$\mu \geq \nu \otimes dt \quad on \quad R^n \times (s_1, s_2).$$

Here $\mu|_{R^n \times (s,\infty)}$ is the restriction of μ to $R^n \times (s, \infty)$.

Proof. Let $\varphi \geq 0$ be a continuous function with $\mathrm{supp}(\varphi) \subset (s_1 + 1, s_1 + 2)$ and $\int \varphi(t)dt = 1$. Define the positive measure ν on R^n by

$$\int f(x)d\nu(x) = \int f(x)\varphi(t)d\mu(x, t)$$

for every continuous function on R^n with compact support. Then ν is a required measure.

Applying the transformation:

$$(\xi, \tau) = (\exp(-t)x, -a\exp(-2\alpha t))$$

to the measure μ in the preceding proposition, we have

LEMMA 10. *Let $a > 0$ and $\mu \geq 0$ a measure on $R^n \times [-a, 0)$. If for any $0 < \lambda < 1$,*

$$0 \neq \mu|_{R^n \times (-a\lambda^{2\alpha},0)} \leq \tau_\lambda^{(\alpha)}\mu,$$

then for any $-a < s_1 < s_2 < 0$, there exists a positive Radon measure $\nu \neq 0$ on R^n

such that

$$\int f d\mu \geq \int \int_{s_1}^{s_2} f((-s)^{-1/2\alpha} y,\ s) ds\, d\nu(y)$$

for every continuous function $f \geq 0$ *on* R^{n+1} *with compact support.*

Now we give the proofs of Theorem 1 and Corollary 3.

Proof of Theorem 1. First suppose that $C_{2\alpha}(E) > 0$. Then there exists a positive Radon measure $\nu \neq 0$ on R^n such that $\mathrm{supp}(\nu) \subset E$ is compact and that

$$\int K_{2\alpha}(x,\ y) d\nu(y) \leq 1$$

for every $x \in R^n$. When $0 < \alpha < 1/2$, we may assume that $0 \notin \mathrm{supp}(\nu)$, because $K_{2\alpha}(0,\ 0) = \infty$. Define the measure μ on R^{n+1} by

$$\int f d\mu = \int \int_{-2}^{-1} f((-s)^{-1/2\alpha} y,\ s) ds\, d\nu(y)$$

for every continuous function f on R^{n+1}. By Lemma 8, $W^{(\alpha)} * \mu$ is bounded on R^{n+1}. Hence $\mathrm{cap}^{(\alpha)}(T^{(\alpha)}(E,1,2)) > 0$. By Proposition 6, O is a regular boundary point of $\Omega^{(\alpha)}(E)$ for $L^{(\alpha)}$. Conversely suppose that O is regular for $L^{(\alpha)}$. By Proposition 6, we have $\mathrm{cap}^{(\alpha)}(T^{(\alpha)}(E,\ 3)) > 0$. We may assume that E is compact and that $0 \notin E$ if $0 < \alpha < 1/2$. For $s < 0$, we denote by μ_s the equilibrium measure of $T^{(\alpha)}(E,\ s)$ with respect to $W^{(\alpha)}$. We remark that for any $s > 0$,

$$\mu_3 |_{R^n \times (-s, 0)} \neq 0.$$

For $\lambda > 0$, since $\tau_\lambda^{(\alpha)}(T^{(\alpha)}(E,\ s)) = T^{(\alpha)}(E,\ \lambda^{2\alpha} s)$, Proposition 5 shows $\mu_{\lambda^{2\alpha} s} = \lambda^n \cdot \tau_\lambda^{(\alpha)} \mu_s$. On the other hand, for $0 < s_1 < s_2$, $\mu_{s_2} \leq \mu_{s_1}$ on $T^{(\alpha)}(E,\ s_1)$ (see [N, Lemma 2.14]), so that for $0 < \lambda < 1$,

$$\mu_3 \leq \tau_\lambda^{(\alpha)} \mu_3 \quad \text{on} \quad T^{(\alpha)}(E,\ 3\lambda^{2\alpha}).$$

By Lemma 10, there exists a positive Radon measure $\nu \neq 0$ on R^n such that $\mathrm{supp}(\nu) \subset E$ and that

$$\int \int_{-2}^{-1} f((-s)^{-1/2\alpha} y,\ s) ds\, d\nu(y) \leq \int f d\mu_3$$

for every continuous function $f \geq 0$ on R^{n+1}. Therefore Lemma 8 gives

$$\int K_{2\alpha}(x,\, y)d\nu(y) \leq M\, W^{(\alpha)} * \mu_3(x,\, -1)$$
$$\leq M,$$

which implies $C_{2\alpha}(E) > 0$. This completes the proof.

Proof of Corollary 3. First we remark that $\bar{E}$ has a positive 2α-Riesz capacity if and only if $1/2 < \alpha \leq 1$. Then in the case $O \in H$, the assertion of Corollary 3 follows from Lemma 7. Next we assume $O \notin H$. By Theorem 1, we have only to show $C_{2\alpha}(\bar{E}) > 0$ in the case $0 < \alpha < 1/2$. We may assume that $\bar{E}$ is compact. Let ν be the restriction to $\bar{E}$ of the $(n-1)$-dimensional Lebesgue measure on H. Then we can show that $\int K_{2\alpha}(x,\, y)d\nu(y)$ is bounded on R^n. In fact, if $x \in H$ or if x and O are in the same component of $R^n \setminus H$, then $\theta(y,\, x-y) \geq C_0$ for some constant $C_0 > 0$, so that we have

$$\int K_{2\alpha}(x,\, y)d\nu(y) \approx \int_{B_H(x_0,1)} |\, x-y\,|^{2-n-2\alpha} d\nu(y)$$
$$\leq \int_{B_H(x_0,1)} |\, x_0 - y\,|^{2-n-2\alpha} d\nu(y)$$
$$\approx 1$$

for $x \in R^n$ near $\bar{E}$, where x_0 is the nearest point in H from x, and where $B_H(x_0,\, r)$ denotes the ball in H with radius r and center x_0. If x and O are not in the same component of $R^n \setminus H$, denote by x_0 the constant multiplication of x belonging to H. Then

$$\int_{B_H(x_0,|x-x_0|,1)} K_{2\alpha}(x,\, y)\; d\nu(y)$$
$$\approx \int_{B_H(x_0,|x-x_0|,1)} |\, x,\, y\,|^{2-n-2\alpha}\; d\nu(y)$$
$$\approx 1,$$
$$\int_{B_{H'}(x_0,|x-x_0|^{1-2\alpha},|x-x_0|)} K_{2\alpha}(x,\, y)\; d\nu(y)$$
$$\approx \int_{B_{H'}(x_0,|x-x_0|^{1-2\alpha},|x-x^0|)} |\, x - x_0|\, |\, y - x_0\,|^{1-n-2\alpha}\; d\nu(y)$$
$$\approx 1,$$

where $B_H(x_0,\, r_1,\, r_2) = B_H(x_0,\, r_2) \setminus B_H(x_0,\, r_1)$. Moreover

$$\begin{aligned}
&\int_{B_H(x_0,|x-x_0|^{1-2\alpha})} K_{2\alpha}(x, y)\, d\nu(y) \\
&\approx \int_{B_H(x_0,|x-x_0|^{1-2\alpha})} |x - x_0|^{(1-n)/2\alpha}\, d\nu(y) \\
&\approx 1.
\end{aligned}$$

Therefore we have $C_{2\alpha}(\bar{E}) > 0$, which shows Corollary 3.

§4. Capacity $C_{2\alpha}$ and the Riesz capacity

For $0 < \alpha < 1/2$, we discuss a relation between the capacity $C_{2\alpha}$ and the Riesz capacities. Since

$$|x - y|^{2-2\alpha-n} \leq K_{2\alpha}(x, y) \leq |x - y|^{(1-n)/2\alpha}$$

for $y \neq 0$ and $x \in R^n$ such that $|x - y|$ is sufficiently small, $C_{2\alpha}(E) > 0$ implies $C^{(n)}_{2-2\alpha}(E) > 0$ and $C_{2\alpha}(E) = 0$ implies $C^{(n)}_{(1-(1-2\alpha)n)/2\alpha}(E) = 0$, where $C^{(n)}_{\beta}(\cdot)$ denotes the β-Riesz capacity of $(\cdot)$ in R^n. Corollary 3 gives an example of E such that $C_{2\alpha}(E) > 0$ and $C^{(n)}_1(E) = 0$ since $C^{(n)}_1(H) = 0$ for every hyperplane H in R^n. It would be a question to be answered whether for every $\varepsilon > 0$, there exists a compact set E in $R^n \setminus \{0\}$ such that $C_{2\alpha}(E) > 0$ and $C^{(n)}_{2-2\alpha-\varepsilon}(E) = 0$. Conversely what is the infimum of β satisfying the following condition?

There exists a compact set E in $R^n \setminus \{0\}$ such that $C_{2\alpha}(E) = 0$ and $C^{(n)}_{\beta}(E) > 0$.

Concerning the latter question, we have the following example.

Example. Let H be a hyperplane in $R^n (n \geq 2)$ with $0 \notin H$. For every compact set $K \subset H$, we put

$$E_K = \{sy \in R^n ; y \in K,\ 1 \leq s \leq 2\}.$$

Then $C_{2\alpha}(E_K) > 0$ if and only if $C^{(n-1)}_{2\alpha}(K) > 0$. Choosing a generalized Cantor set K whose Hausdorff dimension is equal to $n - 2\alpha - 1$, we have $C_{2\alpha}(E_K) = 0$ and for any $\varepsilon > 0$, $C^{(n)}_{2\alpha+\varepsilon}(E_K) > 0$. From this observation, the infimum of β of the latter question is less than or equal to 2α.

References

[EF] E. Effros and J. Kazdan, On the Dirichlet problem for the heat equation, Indiana Univ. Math. J., **20** (1971), 683–693.

[EG] L. Evans and R. Gariepy, Wiener's criterion for the heat equation, Arch. Rational

Mech. Anal., **78** (1982), 293–314.

[IN] M. Itô and M. Nishio, Poincaré type conditions of the regularity for the parabolic operator of order α, Nagoya Math. J., **115** (1989), 1–22.

[N] M. Nishio, The Wiener criterion of regular points for the parabolic operator of order α, Nagoya Math. J., **116** (1989), 163–179.

[W] N. A. Watson, Thermal capacity, Proc. London Math. Soc., **37** (1978), 342–362.

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