

LOGARITHMIC FORMS ON AFFINE ARRANGEMENTS

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1. Introduction

Let V be an affine of dimension l over some field \mathbf{K} . An *arrangement* \mathcal{A} is a finite collection of affine hyperplanes in V . We call \mathcal{A} an l -arrangement when we want to emphasize the dimension of V . We use [6] as a general reference. Choose an arbitrary point of V and fix it throughout this paper. We will use it as the origin. If \mathcal{A} is *central*, i.e., $U = \bigcap_{H \in \mathcal{A}} H \neq \emptyset$, then it is convenient to choose the origin from U . Let S be the coordinate ring of V and identify S with the polynomial ring $\mathbf{K}[x_1, \dots, x_l]$, where x_1, \dots, x_l are affine coordinates of V . For an integer p , $0 \leq p \leq l$, consider the module of *logarithmic differential p -forms* $\Omega^p(\mathcal{A})$ (see 2.1 for definition). We usually simply write $\Omega^p = \Omega^p(\mathcal{A})$ in this paper. It is a finitely generated S -module. Put $\Omega^p = 0$ if $p < 0$ or $p > l$. Now fix a homogeneous polynomial $f \in S$ of a positive degree and define a coboundary operator $df \wedge$ on $\Omega^* = \bigoplus_p \Omega^p$ by taking the exterior product with df . Then we have a cochain complex, which is also denoted by $\Omega^* : \Omega^* = (\Omega^*, df \wedge)$. In this paper, we study its cohomology groups $H^p(\Omega^*)$.

Let L be the set of all nonempty intersections of hyperplanes from \mathcal{A} . Introduce a partial order to L by reverse inclusion. The poset L is called the *intersection poset of \mathcal{A}* . Let μ denote the Möbius function on L . The *characteristic polynomial* of \mathcal{A} is

$$(1.1) \quad \chi(\mathcal{A}, t) = \sum_{X \in L} \mu(V, X) t^{\dim X}.$$

Our main results are the Theorems 1-5 below.

THEOREM 1 (Theorem 7.1). *Suppose $f \in S$, $\deg f = r > 0$, satisfies a certain genericity condition with respect to \mathcal{A} . Then*

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- (1) the cohomology groups $H^p(\Omega^*)$ are finite dimensional for all p ,
 (2) the Euler characteristic (= the alternating sum of dimensions of cohomology groups) of the complex Ω^* is equal to $\chi(\mathcal{A}, 1 - r)$.

A motivation to study these cohomology groups comes from the study of twisted de Rham complex $\Omega_{\nabla}^* = (\Omega^*, \nabla_f)$, where the connection map ∇_f is defined by

$$\nabla_f = d + \sum_{j=1}^n \lambda_j \frac{d\alpha_j}{\alpha_j} \wedge + df \wedge,$$

where d is the usual exterior differentiation and $\lambda_j \in \mathbf{K}$. In the case of complex field the complex Ω_{∇}^* is closely related to the (irregular singular) hypergeometric integrals

$$\int e^{f(u)} U(u, \lambda, x) du_1 \wedge \cdots \wedge du_n$$

studied in [1] [2]. Theorems 1 and 2 can be regarded as a generalization of results in [4, Proposition 5.10, Theorem 10.3], in which arrangements are assumed to be in general position.

THEOREM 2 (Theorem 7.9). *Suppose f , $\deg f = r > 0$, satisfies a certain genericity condition with respect to \mathcal{A} . Then*

- (1) the cohomology groups $H^p(\Omega_{\nabla}^*)$ are finite dimensional for all p ,
 (2) the Euler characteristic of the complex Ω_{∇}^* is equal to $\chi(\mathcal{A}, 1 - r)$.

The outline of the paper is as follows. In order to prove Theorem 2 out of Theorem 1, we compare the two complexes Ω^* and Ω_{∇}^* . In Section 2, for this purpose, we introduce an increasing filtration $\Omega_{\leq q}^p$ ($q \in \mathbf{Z}$) to each Ω^p so that the E_1 -terms of the respective spectral sequences of the filtered complexes $\{\Omega_{\leq q}^*\}$ and $\{(\Omega_{\nabla}^*)_{\leq q}\}$ coincide.

In Section 3, we define the cone $c\mathcal{A}$ of \mathcal{A} . Note that $c\mathcal{A}$ is a central arrangement which is easier to handle than an affine arrangement. We introduce an important short exact sequence (3.4) involving $\Omega^p(c\mathcal{A})$.

In Section 4, we introduce the other important short exact sequence (4.1) involving $\mathrm{Gr}_q \Omega^p(\mathcal{A}) = \Omega_{\leq q}^p / \Omega_{\leq q-1}^p$, which enables us, together with the short exact sequence in Section 3, to compare $\Omega^*(c\mathcal{A})$ and $\Omega^*(\mathcal{A})$.

We prove, as a by-product, in Section 5, the following affine generalization of Solomon-Terao's formula for $\chi(\mathcal{A}, t)$ [9] [6, 4.136].

THEOREM 3 (Theorem 5.3).

$$\lim_{x \rightarrow 1} \sum_{p, q} \dim(\text{Gr}_q \Omega^p(\mathcal{A})) x^{q-p} \{t(x-1) - 1\}^p = \chi(\mathcal{A}, t).$$

In Section 6, we compare the cohomology groups $H^p(\Omega^*(c\mathcal{A}))$ and $H^p(\text{Gr}\Omega^*(\mathcal{A}))$ by using the short exact sequences introduced in Sections 3 and 4. We prove

THEOREM 4 (Theorem 6.9.) *Suppose $f \in S$, $\deg f > 0$, satisfies a certain genericity condition with respect to \mathcal{A} . Then the following conditions are equivalent for every p ;*

- (i) $H^{p+1}(\Omega^*(c\mathcal{A})) = 0$,
- (ii) $H^p(\text{Gr}\Omega^*(\mathcal{A})) = 0$.

In Section 7 we prove Theorem 1 using Theorem 3. Then, we “approximate” the cohomology groups $H^p(\Omega^*_\nabla)$ by $H^p(\Omega^*)$ by using a spectral sequence argument and obtain Theorem 2. Theorems 2 and 4 imply immediately

THEOREM 5 (Theorem 7.10). *If $H^p(\Omega^*(c\mathcal{A})) = 0$ ($p \neq l + 1$), then $H^p(\Omega^*_\nabla) = 0$ ($p \neq l$) and $\dim H^l(\Omega^*_\nabla) = |\chi(\mathcal{A}, 1 - r)|$ for every f of degree $r > 0$ which satisfies a certain genericity condition with respect to \mathcal{A} .*

This theorem can be applied to (1) arrangements in general position (Corollary 7.11), (2) arrangements whose cones are free (Corollary 7.12), and (3) 2-arrangements (Corollary 7.13).

2. The module Ω^p of logarithmic forms and its filtration

Fix an affine arrangement \mathcal{A} in an affine l -space V over a field \mathbf{K} . Let S be the affine coordinate ring of V and p an integer. We first define the module of logarithmic differential p -forms $\Omega^p = \Omega^p(\mathcal{A})$. For each $H \in \mathcal{A}$ choose $\alpha_H \in S$ so that $H = \ker(\alpha_H)$. Let

$$\mathcal{Q} = \mathcal{Q}(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

a defining polynomial for \mathcal{A} . Let $\Omega^p[V]$ denote the S -module of all global regular (= polynomial) p -forms on V and $\Omega^p(V)$ the module of all global rational p -forms on V .

DEFINITION 2.1. The module $\Omega^p = \Omega^p(\mathcal{A})$ of logarithmic p -forms with poles along \mathcal{A} is defined as

$$\Omega^p = \Omega^p(\mathcal{A}) = \{\omega \in \Omega^p(V) \mid Q\omega \in \Omega^p[V] \text{ and } Q(d\omega) \in \Omega^{p+1}[V]\}.$$

Put $\Omega^p = 0$ if $p < 0$ or $p > l$.

Remark. The module of logarithmic p -forms was first introduced by K. Saito [8] for an arbitrary hypersurface germ in \mathbf{C}^l . It has been used for the study of central arrangements of hyperplanes [10] [13] [11].

In [6], $\Omega^p(\mathcal{A})$ is defined only for central arrangements. However, many results in there remain true for affine arrangements. In fact, in many cases, one can give practically the same proofs. The following two Propositions, which are two of such examples, will be used later.

PROPOSITION 2.2. For each $\omega \in \Omega^p(V)$, the following two conditions are equivalent:

- (1) $\omega \in \Omega^p$,
- (2) $Q\omega \in \Omega^p[V]$ and $Q(d\alpha_H/\alpha_H) \wedge \omega \in \Omega^{p+1}[V]$ for all $H \in \mathcal{A}$.

Proof. Same as Proposition 4.69 in [6]. □

PROPOSITION 2.3. Let $\Omega^* = \bigoplus_p \Omega^p$. The S -module Ω^* is closed under exterior product:

$$\Omega^p \times \Omega^q \rightarrow \Omega^{p+q}.$$

Proof. Same as Proposition 4.79 in [6]. □

It is easy to see that Ω^p is finitely generated S -module containing $\Omega^p[V]$. Also $\Omega^0(\mathcal{A}) = S$ and $\Omega^l(\mathcal{A}) = (1/Q)\Omega^l[V]$. When \mathcal{A} is not central, the S -modules Ω^p are not naturally graded but we can introduce a natural increasing filtration on them. Let $\beta \in \Omega^p[V]$. If each coefficient of β is a polynomial of degree at most $q - p$ then we say that the total degree of β is $\leq q$ and write $\text{tdeg}\beta \leq q$. Let $\omega \in \Omega^p$. It follows from the definition that ω can be written in the form $\omega = \beta/Q$ where $\beta \in \Omega^p[V]$. Let $n = \deg Q = |\mathcal{A}|$. We may formally consider the degree of $1/Q$ as $-n$ and say that the total degree $\text{tdeg}\omega \leq q$ if $\text{tdeg}\beta \leq q + n$. For example, if $l = 1$, $Q = x(x - 1)$, and $\omega = dx/x(x - 1)$, then $\text{tdeg}\omega \leq q$ for $q \geq -1$.

DEFINITION 2.4. Total degree introduces on increasing filtration on Ω^p for $q \in \mathbf{Z}$ by

$$\Omega_{\leq q}^p = \Omega_{\leq q}^p(\mathcal{A}) = \{\omega \in \Omega^p \mid \text{tdeg}\omega \leq q\}.$$

Define \mathbf{K} -vector spaces for $q \in \mathbf{Z}$ by

$$\text{Gr}_q \Omega^p = \text{Gr}_q \Omega^p(\mathcal{A}) = \Omega_{\leq q}^p / \Omega_{\leq q-1}^p.$$

If \mathcal{A} is central, then Ω^p is a graded by the total degree, i.e., there exists a decomposition

$$\Omega^p = \bigoplus_{q \in \mathbf{Z}} \Omega_q^p$$

with

$$\Omega_{\leq q}^p = \bigoplus_{q' \leq q} \Omega_{q'}^p.$$

The space $\Omega_q^p = \Omega_q^p(\mathcal{A})$ is called the *homogeneous part* of Ω^p of tdegree q .

DEFINITION 2.5. Let $f \in \mathcal{S}$. Define maps

$$df \wedge : \Omega^p \rightarrow \Omega^{p+1}$$

by sending $\omega \in \Omega^p$ to $df \wedge \omega$. This map is well-defined because of Proposition 2.3. We obtain the cochain complex

$$\Omega^* = \Omega^*(\mathcal{A}) = (\Omega^*, df \wedge) : \cdots \xrightarrow{df \wedge} \Omega^p \xrightarrow{df \wedge} \Omega^{p+1} \xrightarrow{df \wedge} \cdots$$

DEFINITION 2.6. Suppose $f \in \mathcal{S}$ is a polynomial of degree $r (> 0)$. Let $q \in \mathbf{Z}$. The cochain complex Ω^* has a subcomplex $\Omega_{\leq q}^* = \Omega_{\leq q}^*(\mathcal{A})$ which is defined by

$$\Omega_{\leq q}^* = \Omega_{\leq q}^*(\mathcal{A}) : \cdots \xrightarrow{df \wedge} \Omega_{\leq q+(p-l)r}^p \xrightarrow{df \wedge} \Omega_{\leq q+(p-l+1)r}^{p+1} \xrightarrow{df \wedge} \cdots \xrightarrow{df \wedge} \Omega_{\leq q}^l \rightarrow 0.$$

Then this provides an increasing filtration of the cochain complex Ω^* . For each $q \in \mathbf{Z}$, define the complex

$$\text{Gr}_q \Omega^* = \text{Gr}_q \Omega^*(\mathcal{A}) = \Omega_{\leq q}^* / \Omega_{\leq q-1}^*.$$

If \mathcal{A} is central and f is a homogeneous polynomial of degree $r (> 0)$, then the cochain complex Ω^* has a subcomplex $\Omega_q^* = \Omega_q^*(\mathcal{A})$ which is defined by

$$\Omega_q^* = \Omega_q^*(\mathcal{A}) : \cdots \xrightarrow{df \wedge} \Omega_{r+(p-l)r}^p \xrightarrow{df \wedge} \Omega_{r+(p-l+1)r}^{p+1} \xrightarrow{df \wedge} \cdots \xrightarrow{df \wedge} \Omega_q^l \rightarrow 0.$$

Then

$$\Omega_{\leq q}^* = \bigoplus_{q' \leq q} \Omega_{q'}^*.$$

Next consider a slightly different cochain complex Ω_{∇}^* .

DEFINITION 2.7. Let $\lambda_H \in \mathbf{K}$ for each $H \in \mathcal{A}$. Define maps $\nabla_f: \Omega^p \rightarrow \Omega^{p+1}$

$$\nabla_f(\omega) = d\omega + \sum_{H \in \mathcal{A}} \lambda_H \frac{d\alpha_H}{\alpha_H} \wedge \omega + df \wedge \omega, \quad \omega \in \Omega^p,$$

where d is the exterior differentiation. This map is well-defined because Ω^* is closed with respect to exterior product and exterior differential. We obtain the cochain complex

$$\Omega_{\nabla}^* = \Omega_{\nabla}^*(\mathcal{A}) : \cdots \xrightarrow{\nabla_f} \Omega^p \xrightarrow{\nabla_f} \Omega^{p+1} \xrightarrow{\nabla_f} \cdots.$$

Observe that while $df \wedge$ is an S -linear operator, ∇_f is only \mathbf{K} -linear.

DEFINITION 2.8. Let $f \in S$ have degree $r (> 0)$ and let $q \in \mathbf{Z}$. Then we can define an increasing filtration of the cochain complex Ω_{∇}^* in the same manner as in the case of Ω^* in Definition 2.6. In other words, subcomplexes $(\Omega_{\nabla}^*)_{\leq q} = (\Omega_{\nabla}^*)_{\leq q}(\mathcal{A})$ are defined by

$$(\Omega_{\nabla}^*)_{\leq q} : \cdots \xrightarrow{\nabla_f} \Omega_{\leq q+(p-1)r}^p \xrightarrow{\nabla_f} \Omega_{\leq q+(p-1)r}^{p+1} \xrightarrow{\nabla_f} \cdots \xrightarrow{\nabla_f} \Omega_{\leq q}^l \rightarrow 0.$$

For each $q \in \mathbf{Z}$, define the complex

$$\mathrm{Gr}_q \Omega_{\nabla}^* = \mathrm{Gr}_q \Omega_{\nabla}^*(\mathcal{A}) = (\Omega_{\nabla}^*)_{\leq q} / (\Omega_{\nabla}^*)_{\leq q-1}.$$

The following observation is easy but important:

LEMMA 2.9.

$$\mathrm{Gr}_q \Omega_{\nabla}^* = \mathrm{Gr}_q \Omega^*, \quad q \in \mathbf{Z}.$$

Proof. Note that $df \wedge$ is the part of the highest degree in the definition of ∇_f because $f > 0$. The result is now obvious from definition. \square

Remark. This shows that the spectral sequences of these two filtered complexes share the same E_1 -term. If the both spectral sequences degenerate at

E_1 -term, then we have $H^p(\Omega^*) \simeq H^p(\Omega^*_V)$.

3. Splitting of Ω^* for central arrangements

Only in this section, let \mathcal{A} denote a central $(l + 1)$ -arrangement in V . Let x_0, x_1, \dots, x_l be linear coordinates for V .

First we need to recall the S -module $D(\mathcal{A})$ dual to $\Omega^1(\mathcal{A})$ (see [6, 4.1]). Let $\text{Der}_{\mathbf{K}}(S)$ be the (free) S -module of all derivations of S over \mathbf{K} . Then

$$D(\mathcal{A}) = \{\theta \in \text{Der}_{\mathbf{K}}(S) \mid \theta(Q) \in QS\}.$$

While $D(\mathcal{A})$ can be defined in the same way for all arrangements in the case of central arrangement $D(\mathcal{A})$ has a very important element θ_E , called the *Euler derivation* and defined by

$$\theta_E = \sum_{i=1}^l x_i \partial / \partial x_i.$$

Let $1 \leq p \leq l$. There exists an S -bilinear map (called *interior product*)

$$\langle \cdot, \cdot \rangle : D(\mathcal{A}) \times \Omega^p(\mathcal{A}) \rightarrow \Omega^{p-1}(\mathcal{A})$$

with the following properties (see [6, 4.73, 4.74])

- (1) $\langle \theta, \omega_1 \wedge \omega_2 \rangle = \langle \theta, \omega_1 \rangle \wedge \omega_2 + (-1)^p \omega_1 \wedge \langle \theta, \omega_2 \rangle$,
- (2) $\langle \theta, df \rangle = \theta(f)$

for all $\theta \in D(\mathcal{A})$, $f \in S$, $\omega_1 \in \Omega^p(\mathcal{A})$, and $\omega_2 \in \Omega^q(\mathcal{A})$. We will often use the properties (1) and (2) without reference.

Now assume that \mathcal{A} contains the hyperplane $H_0 = \ker(x_0)$ and let p be an integer. Define two S -module homomorphisms

$$\phi_p : \Omega^p \rightarrow \Omega^{p-1}, \text{ and } \delta_p : \Omega^p \rightarrow \Omega^{p+1}$$

via $\phi_p(\omega) = (-1)^{p-1} \langle \theta_E, \omega \rangle$ and $\delta_p(\omega) = (-1)^p \frac{dx_0}{x_0} \wedge \omega$. We will often omit the indexes from the notation.

The following equalities can be checked easily ((3.2) is sometimes called H. Cartan's homotopy formula):

$$(3.1) \quad \phi^2 = 0, \delta^2 = 0,$$

$$(3.2) \quad \phi\delta + \delta\phi = \text{id}_{\Omega^p}.$$

The equalities (3.1) and (3.2) imply in particular that the following sequence is split exact

$$(3.3) \quad 0 \rightarrow K^p \rightarrow \Omega^p \xrightarrow{\delta} K^{p+1} \rightarrow 0$$

where $K^p = \ker(\delta_p)$ and the splitting can be given by the restriction of ϕ_{p+1} to K^{p+1} .

Clearly both mappings ϕ and δ preserve the (total degree) grading $\Omega^p = \bigoplus_q \Omega_q^p$ where Ω_q^p is the homogeneous part of tdegree q of $\Omega^p(\mathcal{A})$. Thus $K^p = \bigoplus_q K_q^p$ is a homogeneous submodule of Ω^p and (3.3) produces for every q the exact sequence of linear spaces

$$(3.4) \quad 0 \rightarrow K_q^p \rightarrow \Omega_q^p \xrightarrow{\delta} K_q^{p+1} \rightarrow 0.$$

Now fix a homogeneous polynomial $f \in S$ of degree r ($r > 0$) and consider the cochain complex $\Omega^* = (\Omega^*, df \wedge)$ defined in the previous section. Clearly δ commutes with $df \wedge$ whence $K^* = (K^*, df \wedge)$ is a subcomplex of $\Omega^* = (\Omega^*, df \wedge)$ and (3.3) generates the exact sequence

$$(3.5) \quad \cdots \rightarrow H^p(K^*) \rightarrow H^p(K^*) \rightarrow H^p(\Omega^*) \rightarrow H^{p+1}(K^*) \rightarrow H^{p+1}(K^*) \rightarrow \cdots.$$

PROPOSITION 3.1. *For every p the connecting homomorphism $\gamma: H^p(K^*) \rightarrow H^p(K^*)$ in (3.5) is the action of $\bar{f} = (-1)^{p-1}rf$ on the S -module $H^p(K^*)$.*

Proof. Let $\omega \in K^p$, $df \wedge \omega = 0$, and $[\omega]$ is the class of ω in $H^p(K^*)$. Since ϕ splits the homomorphism $\delta: \Omega^{p-1} \rightarrow K^p$ one can view $\gamma([\omega])$ as the class of $df \wedge \phi(\omega)$. Now we have

$$\begin{aligned} df \wedge \phi(\omega) &= (-1)^{p-1} df \wedge \langle \theta_E, \omega \rangle = (-1)^{p-1} (\langle \theta_E, df \wedge \omega \rangle + \langle \theta_E, df \rangle \omega) \\ &= (-1)^{p-1} \theta_E(f) \omega = \bar{f} \omega \end{aligned}$$

which completes the proof. \square

Finally let $f \in S$ be a homogeneous polynomial of degree r . Recall that $df \wedge$ is homogeneous of degree $+r$ with respect to the grading by total degree and that the complex Ω^* is the direct sum of the complexes Ω_q^* where Ω_q^* consists of the linear spaces $\Omega_{q+(p-1)r}^*$. Similarly $K^* = \bigoplus_q K_q^*$ and (3.5) generates for every q the analogous exact sequence of the cohomology spaces of complexes Ω_q^* and K_q^* :

$$(3.6) \quad \cdots \rightarrow H^p(K_{q-1}^*) \rightarrow H^p(K_q^*) \rightarrow H^p(\Omega_q^*) \rightarrow H^{p+1}(K_q^*) \rightarrow H^{p+1}(K_{q+1}^*) \rightarrow \cdots.$$

4. Restriction of forms to affine arrangement

In the rest of the paper \mathcal{A} is an affine l -arrangement in an affine space V over a field \mathbf{K} . Let x_1, \dots, x_l be affine coordinates. Let cV be an affine $(l + 1)$ -space with coordinates x_0, x_1, \dots, x_l . Regard V as an affine hyperplane of cV defined by $x_0 = 1$. Let $S^c = \mathbf{K}[x_0, x_1, \dots, x_l]$ be the coordinate ring of cV .

For each $g \in S$ define the homogenization of g as

$$g^h = x_0^{\deg g} g(x_1/x_0, \dots, x_l/x_0)$$

where $\deg g$ is the degree of g . Clearly $g^h \in S^c$ and is homogeneous of the degree g . If $s \in S^c$ then we put $\varepsilon(s) = s(x_0 = 1)$. Clearly $\varepsilon: S^c \rightarrow S$ is a ring homomorphism and $\varepsilon(g^h) = g$ for every $g \in S$. There is a unique ε -homomorphism of the S^c -module $\Omega^p(cV)$ to the S -module $\Omega^p(V)$ that maps $dx_{i_1} \wedge \dots \wedge dx_{i_p}$ to itself if none of the indexes i_j is 0 and to 0 otherwise. We denote this homomorphism also by ε .

DEFINITION 4.1. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ and $Q = \prod_{i=1}^n \alpha_i$ be a defining polynomial for \mathcal{A} . The cone $c\mathcal{A}$ of \mathcal{A} is defined as the central arrangement in cV given by the polynomial $Q^c = x_0 Q^h = x_0 \prod_{i=1}^n \alpha_i^h$.

The goal of this section is to relate the modules of logarithmic forms on $c\mathcal{A}$ and \mathcal{A} . For that we put $\Omega^* = \Omega^*(c\mathcal{A})$ and use the notation of the previous section for the differentials and submodules of Ω^* .

DEFINITION 4.2. Let $\omega \in K^{p+1}$. Call the restriction of ω to \mathcal{A} the form

$$\rho(\omega) = \varepsilon\phi(\omega).$$

- PROPOSITION 4.3. (i) $\rho(\omega) \in \Omega^p(\mathcal{A})$ for every $\omega \in K^{p+1}$.
 (ii) $\rho: K^{p+1} \rightarrow \Omega^p(\mathcal{A})$ is an ε -homomorphism of an S^c -module to an S -module.
 (iii) The homomorphism ρ is surjective.
 (iv) $\ker \rho = (x_0 - 1)K^{p+1}$.

Proof. (i) Let $\omega \in K^{p+1}$. Clearly $Q\rho(\omega) \in \Omega^p[V]$. Now let $i \in \{1, \dots, n\}$. Since $\phi(\omega) \in \Omega^p(c\mathcal{A})$ we know, by Proposition 2.2, that α_i^h divides $d\alpha_i^h \wedge Q^c\phi(\omega)$. Applying ε to the coefficients of the latter form we see that α_i divides $d\alpha_i \wedge Q\rho(\omega)$ which proves the statement again by Proposition 2.2.

(ii) Clear by definition of ρ .

(iii) Let

$$\eta = \sum_{1 \leq i_1 < \dots < i_p \leq l} \frac{f_{i_1, \dots, i_p}}{Q} dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \Omega^p(\mathcal{A}).$$

Denote by m the maximal degree of the polynomials f_I where $I = (i_1, \dots, i_p)$ and define \tilde{f}_I as the homogeneous polynomial from S^c of degree m such that $\varepsilon(\tilde{f}_I) = f_I$. Now put

$$\omega = \frac{dx_0}{x_0} \wedge \sum_I \frac{\tilde{f}_I}{Q^h} dx_I$$

where $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$.

Our goal is to prove that $\omega \in \Omega^{p+1}(c\mathcal{A})$. Clearly $Q^c \omega \in \Omega^{p+1}[cV]$ and $dx_0 \wedge \omega = 0$. Let $i \in \{1, \dots, n\}$. We know, by Proposition 2.2, that α_i divides the form $d\alpha_i \wedge Q\eta$, i.e., α_i divides every coefficient g_j of $dx_j = dx_{j_1} \wedge \dots \wedge dx_{j_{p+1}}$ in this form. This implies that α_i^h divides g_j^h . Notice that g_j^h is a linear combination over \mathbf{K} of the coefficients f_I of $Q\eta$. Let us compare g_j^h with the respective coefficient q_j of $dx_0 \wedge dx_{j_1} \wedge \dots \wedge dx_{j_{p+1}}$ in the form $d\alpha_i^h \wedge Q^c \omega$. Notice that both g_j^h and q_j are homogeneous polynomials having the same images under ε and $\deg(q_j) \geq \deg(g_j^h)$. Thus $q_j = x_0^s g_j^h$ for some non-negative integer s . This implies that α_i^h divides $d\alpha_i^h \wedge Q^c \omega$ and $\omega \in \Omega^{p+1}(c\mathcal{A})$ by Proposition 2.2.

To complete the proof one should notice that $\delta(\omega) = 0$ by construction and check that $\rho(\omega) = \pm \eta$.

(iv) Clearly $(x_0 - 1)K^{p+1} \subset \ker(\rho)$. Suppose that $\eta \in K^{p+1}$ and $\rho(\eta) = 0$. Represent $\phi(\eta) = dx_0 \wedge \eta_1 + \eta_2$ where η_1 and η_2 are forms without dx_0 and such that $Q^c \eta_i \in \Omega^*[cV]$ ($i = 1, 2$). Since $\rho(\eta) = 0$ we have by definition of ρ that $\eta_2 = (x_0 - 1)\eta_3$ where again $Q^c \eta_3 \in \Omega^*[cV]$. On the other hand we have $\eta = \delta\phi(\eta) + \phi\delta(\eta) = \delta\phi(\eta) = \delta\eta_2 = (x_0 - 1)\delta\eta_3$. One checks easily that $\delta\eta_3 \in K^{p+1}$ which completes the proof. \square

Remark. 4.4. For any $\eta \in \Omega^p(\mathcal{A})$ represent it as in the proof of (iii). Let $q = m - n + p$. Then we have $\text{tdeg}\eta \leq q$ and the proof of (iii) shows that there exists $\omega \in K_q^{p+1}$ such that $\rho(\omega) = \eta$.

COROLLARY 4.5. Consider the structure of S^c -module on S given by ε . Then for every p , Proposition 4.3 defines the isomorphism of the following S -modules

$$\Omega^p(\mathcal{A}) \simeq K^{p+1} \otimes_{S^c} S. \quad \square$$

We will also need a graded version of Proposition 4.3. Recall the increasing filtration in 2.4 $\cdots \subset \Omega_{\leq q}^p(\mathcal{A}) \subset \Omega_{\leq q+1}^p(\mathcal{A}) \subset \cdots \subset \Omega^p(\mathcal{A})$ where $\Omega_{\leq q}^p(\mathcal{A}) = \{\omega \in \Omega^p(\mathcal{A}) \mid \text{tdeg}(\omega) \leq q\}$. Also recall the associated graded module $\text{Gr}\Omega^p(\mathcal{A}) = \bigoplus_q \text{Gr}_q\Omega^p(\mathcal{A})$ where $\text{Gr}_q\Omega^p(\mathcal{A}) = \Omega_{\leq q}^p(\mathcal{A})/\Omega_{\leq q-1}^p(\mathcal{A})$.

Now fix $p \in \{0, 1, \dots, l\}$ and $q \in \mathbf{Z}$ and denote by π the standard projection $\Omega_{\leq q}^p(\mathcal{A}) \rightarrow \text{Gr}_q\Omega^p(\mathcal{A})$. Clearly the restriction of ρ defines a linear map $K_q^{p+1} \rightarrow \Omega_{\leq q}^p(\mathcal{A})$ that we still denote by ρ . Put $\sigma = \pi\rho : K_q^{p+1} \rightarrow \text{Gr}_q\Omega^p(\mathcal{A})$.

PROPOSITION 4.6. *The following sequence is exact*

$$(4.1) \quad 0 \rightarrow K_{q-1}^{p+1} \xrightarrow{x_0} K_q^{p+1} \xrightarrow{\sigma} \text{Gr}_q\Omega^p(\mathcal{A}) \rightarrow 0.$$

Proof. The surjectivity of σ follows from Remark 4.4. It suffices to prove that $\ker \sigma = x_0 K_{q-1}^{p+1}$. Clearly $x_0 K_{q-1}^{p+1} \subset \ker \sigma$. Let $\eta \in \ker \sigma$, i.e., $\rho(\eta) \in \Omega_{\leq q-1}^p(\mathcal{A})$. Then by the surjectivity of σ and Proposition 4.3 (iv) there exist $q' < q$, $\eta' \in K_{q'}^{p+1}$, and $\eta'' \in K^{p+1}$ such that $\eta - \eta' = (x_0 - 1)\eta''$. If η_1 is the homogeneous component of η'' of maximal degree then $\eta = x_0\eta_1$ which completes the proof. □

5. A formula for the characteristic polynomials of affine arrangements

Use the same notation as in the previous section. Let p and q be integers. By (3.4), we have

$$(5.1) \quad \dim \Omega_q^p(c\mathcal{A}) = \dim K_q^{p+1} + \dim K_q^p.$$

Also, by (4.1), we have

$$(5.2) \quad \dim \text{Gr}_q\Omega^p(\mathcal{A}) = \dim K_q^{p+1} - \dim K_{q-1}^{p+1}.$$

Since $\Omega^*(c\mathcal{A})$ is doubly graded, we can naturally define the Poincaré series $\text{Poin}(\Omega^*(c\mathcal{A}); x, y)$ in two variables by

$$\text{Poin}(\Omega^*(c\mathcal{A}); x, y) = \sum_{p,q} (\dim \Omega_q^p(c\mathcal{A})) x^q y^p.$$

Similarly we can define $\text{Poin}(K^*; x, y)$ and $\text{Poin}(\text{Gr}\Omega^*(\mathcal{A}); x, y)$;

$$(5.3) \quad \begin{aligned} \text{Poin}(K^*; x, y) &= \sum_{p,q} (\dim K_q^p) x^q y^p, \\ \text{Poin}(\text{Gr}\Omega^*(\mathcal{A}); x, y) &= \sum_{p,q} (\dim \text{Gr}_q\Omega^p(\mathcal{A})) x^q y^p. \end{aligned}$$

PROPOSITION 5.1.

$$\text{Poin}(\text{Gr}\Omega^*(\mathcal{A}) ; x, y) = (1 - x)(1 + y)^{-1}\text{Poin}(\Omega^*(c\mathcal{A}) ; x, y).$$

Proof. By (5.1) we have

$$\text{Poin}(\Omega^*(c\mathcal{A}) ; x, y) = (1 + y^{-1})\text{Poin}(K^* ; x, y).$$

By (5.2) we have

$$\text{Poin}(\text{Gr}\Omega^*(\mathcal{A}) ; x, y) = (y^{-1} - xy^{-1})\text{Poin}(K^* ; x, y). \quad \square$$

The following result is essentially due to Solomon-Terao [9] [6, Proposition 4.133, Theorem 4.136]:

THEOREM 5.2. (1) $\Psi(c\mathcal{A} ; x, t) = \text{Poin}\left(\Omega^*(c\mathcal{A}) ; x, \frac{t(1-x)-1}{x}\right)$ has no pole at $x = 1$,
 (2) $\Psi(c\mathcal{A} ; 1, t) = \chi(c\mathcal{A}, t)$. \square

Remark. In [6], the results are stated in terms of pdegree instead of tdegree. This is why they appear slightly different.

The following result is the affine generalization of Theorem 5.2. It contains Theorem 3 in the Introduction.

THEOREM 5.3. (1) $\Psi(\mathcal{A} ; x, t) := \text{Poin}\left(\text{Gr}\Omega^*(\mathcal{A}) ; x, \frac{t(1-x)-1}{x}\right)$ has no pole at $x = 1$,
 (2) $\Psi(\mathcal{A} ; 1, t) = \chi(\mathcal{A}, t)$.

Proof. Set $y = x^{-1}(t(1-x) - 1)$ in Proposition 5.1. Then we have

$$\begin{aligned} \Psi(\mathcal{A} ; x, t) &= \text{Poin}(\text{Gr}\Omega^*(\mathcal{A}) ; x, x^{-1}(t(1-x) - 1)) \\ &= \frac{x}{t-1} \text{Poin}(\Omega^*(c\mathcal{A}) ; x, x^{-1}(t(1-x) - 1)) \\ &= x(t-1)^{-1}\Psi(c\mathcal{A} ; x, t). \end{aligned}$$

Since $\chi(c\mathcal{A}, t) = (t-1)\chi(\mathcal{A}, t)$ (e.g., [6, Proposition 2.51, Definition 2.52]) Theorem 5.2 completes the proof. \square

6. Relations between cohomology of affine arrangement and its cone

Use the same notation as the previous section. Fix a polynomial $f \in S$ of degree $r > 0$. Let $f^h \in S^c$ be the homogenization of f . As in (3.3), the exterior multiplication by df^h defines a complexes $(K^*, df^h \wedge) = K^*$ for the cone $c\mathcal{A}$. Also the exterior multiplication by df defines another complex $(\Omega^*(\mathcal{A}), df \wedge) = \Omega^*(\mathcal{A})$ as in 2.5. The goal of this section is to relate the cohomology of these complexes.

PROPOSITION 6.1. *The homomorphisms ρ define a homomorphism of the complexes $K^* \rightarrow \Omega^*(\mathcal{A})$ that decreases dimension by 1.*

Proof. It suffices to prove that ρ commutes with the differentials. Let $\omega \in K^{p+1}$. Notice that $\varepsilon(\omega) = 0$. Using this we have

$$\begin{aligned} \rho(df^c \wedge \omega) &= \varepsilon\phi(df^c \wedge \omega) = \varepsilon((-1)^{p+1}\theta_E(f^c)\omega + df^c \wedge \phi(\omega)) \\ &= \varepsilon(df^c) \wedge \rho(\omega) = df \wedge \rho(\omega) \end{aligned}$$

that proves the statement. □

COROLLARY 6.2. *We have the short exact sequence of complexes*

$$(6.1) \quad 0 \rightarrow K^* \xrightarrow{x_0^{-1}} K^* \xrightarrow{\rho} \Omega^*(\mathcal{A}) \rightarrow 0. \quad \square$$

COROLLARY 6.3. *We have the following exact sequence of modules*

$$(6.2) \quad \begin{aligned} \dots \rightarrow H^{p+1}(K^*) \xrightarrow{x_0^{-1}} H^{p+1}(K^*) \rightarrow H^p(\Omega^*(\mathcal{A})) \\ \rightarrow H^{p+2}(K^*) \xrightarrow{x_0^{-1}} H^{p+2}(K^*) \rightarrow \dots. \end{aligned} \quad \square$$

COROLLARY 6.4. *Consider the complex $\Omega^*(c\mathcal{A}) = (\Omega^*(c\mathcal{A}), df^h \wedge)$ and suppose that $H^{p+1}(\Omega^*(c\mathcal{A})) = 0$ for some p . Then $H^p(\Omega^*(\mathcal{A})) = 0$.*

Proof. Put $H^i = H^i(K^*)$ for every i . It follows from (3.5) and Proposition 3.1 that the action of f^h on H^{p+1} is surjective. Since H^{p+1} is a graded module and f^h is homogeneous of positive degree this implies that $H^{p+1} = 0$. Since H^{p+2} is also a graded module and x_0 has a positive degree the action of $x_0 - 1$ on H^{p+2} is injective. Now the result follows from the exact sequence (6.2). □

The rest of the section concerns cohomology of a complex containing more subtle information about $\Omega^*(\mathcal{A})$.

Fix an integer q and recall the complex $\text{Gr}_q^*(\mathcal{A}) = \text{Gr}_q^*$ from Definition 2.6. Denote by Gr^* the direct sum of the complexes Gr_q^* for all q . Then it follows from Proposition 6.1 that the differentials commute with σ whence (4.1) defines the short exact sequence of complexes

$$(6.4) \quad 0 \rightarrow K^* \xrightarrow{x_0} K^* \xrightarrow{\sigma} \text{Gr}^* \rightarrow 0.$$

The sequence (6.4) induces the exact sequence of cohomology

$$(6.5) \quad \cdots H^{p+1}(K^*) \xrightarrow{x_0} H^{p+1}(K^*) \rightarrow H^p(\text{Gr}^*) \rightarrow H^{p+2}(K^*) \xrightarrow{x_0} H^{p+2}(K^*) \rightarrow \cdots.$$

Now we need certain information about the annihilator $\text{Ann}(H^p)$ of the S^c -module $H^p = H^p(K^*)$. Define

$$D_0(c\mathcal{A}) = \{\theta \in D(c\mathcal{A}) \mid \theta(x_0) = 0\}.$$

PROPOSITION 6.5. *Let $\theta \in D_0(c\mathcal{A})$. Then $\theta(f^h) \in \text{Ann}(H^p)$ for every p .*

Proof. Let $\omega \in K^p$ and $df^h \wedge \omega = 0$. Then we have

$$0 = \langle \theta, df^h \wedge \omega \rangle = \theta(f^h)\omega - df^h \wedge \langle \theta, \omega \rangle.$$

Since $\theta \in D(c\mathcal{A})$ then $\eta = \langle \theta, \omega \rangle \in \Omega^{p-1}(c\mathcal{A})$. To prove the result it suffices to check that $\eta \in K^{p-1}$, i.e., $dx_0 \wedge \eta = 0$. We have

$$dx_0 \wedge \langle \theta, \omega \rangle = \theta(x_0)\omega - \langle \theta, dx_0 \wedge \omega \rangle$$

that is 0 since $\theta(x_0) = 0$ and $\omega \in K^p$. □

It suffices for our purposes to use only a particular case of Proposition 6.5. For any $X \in L(c\mathcal{A}) \setminus \{0\}$ put $\pi_X = \prod_{H \in \mathcal{A}_X} \alpha_H$. Besides for any nonzero vector $a \in cV$ denote by ∂_a the derivation of S^c that is the differentiation in the direction of a .

COROLLARY 6.6. *For any nonzero $X \in L(c\mathcal{A})$ and nonzero vector $a \in X \cap H_0$ we have $\pi_X \partial_a f^h \in \text{Ann } H^p$.*

Proof. Due to Proposition 6.5 it suffices to check that $\pi_X \partial_a \in D_0(c\mathcal{A})$. The check is straightforward. □

DEFINITION 6.7. A polynomial f is \mathcal{A} -transverse if for every $X \in L(c\mathcal{A})$ the restriction of its homogenization f^h to X has no critical points outside the origin.

Note that this is a generic condition for f with a fixed positive degree.

LEMMA 6.8. Suppose that f is \mathcal{A} -transverse. Let $D_0(c\mathcal{A})f^h = \{\theta(f^h) \mid \theta \in D_0(c\mathcal{A})\}$ and $S_+^c = \{g \in S^h \mid g(0, \dots, 0) = 0\}$.

(1) Let I_f be the ideal of S^c generated by f^c and $D_0(c\mathcal{A})f^h$. Then the radical of I_f is equal to S_+^c .

(2) Let I_x be the ideal of S^c generated by x_0 and $D_0(c\mathcal{A})f^h$. Then the radical of I_x is equal to S_+^c .

Proof. (1): We can assume that the field \mathbf{K} is algebraically closed by passing to its algebraic closure if necessary. Let $I = I_f$ and $V(I)$ the set of zeros of I . By the Nullstellensatz it is enough to show that $V(I)$ is contained in $\{0\}$. Take arbitrary $v \in V(I)$ and find the maximal $X \in L(c\mathcal{A})$ such that $v \in X$. Suppose $X \neq 0$. Then due to Corollary 6.6, $\pi_x \partial_a f^h \in I$ for every vector $a \in X \cap H_0$. Since $\pi_x(v) \neq 0$ we have $\partial_a f^h(v) = 0$. Since f is \mathcal{A} -transverse, the restriction of f^h to X cannot have a critical point at v . This is a contradiction.

(2) The proof is similar. □

Now we are ready to prove one of the main results of the paper. The following theorem is Theorem 4 in the Introduction.

THEOREM 6.9. Let f be \mathcal{A} -transverse. Then the following conditions are equivalent for every p ;

- (i) $H^{p+1}(\Omega^*(c\mathcal{A})) = 0$,
- (ii) $H^p(\text{Gr}^*) = 0$.

Proof. Consider the exact sequences (3.5) (taking into consideration Proposition 3.1) and (6.5). Since $H^{p+1}(K^*)$ is a graded S^c -module and both \bar{f}^h and x_0 are homogeneous of positive degrees, each of the conditions (i) and (ii) implies that $H^{p+1}(K^*) = 0$. Put $Z = \bigcup_{I \in \text{Ass}(H)} I$ where $H = H^{p+2}(K^*)$. It suffices to prove that $\bar{f}^h \in Z$ if and only if $x_0 \in Z$.

Suppose $I \in \text{Ass}(H)$ and $\bar{f}^h \in I$. By Proposition 6.5, $D_0(c\mathcal{A})f^h \subseteq I$. Since $f^h \in I$, I is a prime ideal containing I_f in Lemma 6.8 (i). Thus I contains S_+^c and, in particular, x_0 .

Now suppose $I \in \text{Ass}(H)$ and $x_0 \in I$. Using reasoning similar to the above

and applying Lemma 6.8 (ii) we conclude that I contains S_+^c and, in particular, f^h . □

7. Results on $H^p(\Omega^*)$ and $H^p(\Omega^*_\nabla)$

Use the same notation as the previous section.

The following theorem contains Theorem 1 in the Introduction. Recall the definition (1.1) of the characteristic polynomial $\chi(\mathcal{A}, t)$.

THEOREM 7.1. *Suppose $f \in S$, $\deg f = r > 0$, is \mathcal{A} -transverse. Then*

- (1) *the cohomology groups $H^p(\Omega^*)$ and $H^p(\text{Gr}^*)$ are finite dimensional for all p ,*
- (2) *the Euler characteristics (= the alternating sum of dimensions of cohomology groups) of the complexes Ω^* and Gr^* are both equal to $\chi(\mathcal{A}, 1 - r)$.*

Proof. (1): We only need to show that $H^p(\text{Gr}^*)$ is finite dimensional. Thanks to the exact sequence (6.5) it is enough to prove that the kernel and the cokernel of the multiplication map of $x_0 : H^p(K^*) \rightarrow H^p(K^*)$ are both finite dimensional. Recall $H^p(K^*)$ is annihilated by $D_0(c\mathcal{A})f^h$ by Proposition 6.5. Thus both the kernel and the cokernel are annihilated by x_0 and $D_0(c\mathcal{A})f^h$. By Lemma 6.8, the radical of the ideal generated by x_0 and $D_0(c\mathcal{A})f^h$ contains S_+^c . This shows (1).

(2): Let $E_r^{p,q}$ be the spectral sequence associated with the filtered complex $\{\Omega_{\leq q}^*\}$. Then $E_1^{p,q} = H^{p+q}(\text{Gr}_{-p}^*) = 0$ except for finitely many pairs (p, q) by (1). So we have

$$\begin{aligned} \sum_p (-1)^p \dim H^p(\text{Gr}^*) &= \sum_{p,q} (-1)^{p+q} \dim E_1^{p,q} = \sum_{p,q} (-1)^{p+q} \dim E_2^{p,q} \\ &= \dots \\ &= \sum_{p,q} (-1)^{p+q} \dim E_\infty^{p,q} = \sum_p (-1)^p \dim H^p(\Omega^*). \end{aligned}$$

Therefore it is enough to prove the statement for Gr^* . Recall $\text{Poin}(\text{Gr}\Omega^*(\mathcal{A}) : x, y)$ from (5.3) and set $y = -x^{-r}$. Then we get

$$\begin{aligned} \text{Poin}(\text{Gr}\Omega^*(\mathcal{A}) : x, -x^{-r}) &= \sum_{p,q} (\dim \text{Gr}_q \Omega^p) x^q (-x^{-r})^p \\ &= \sum_m x^{m-r} \sum_p (-1)^p (\dim \text{Gr}_{m+(p-1)r} \Omega^p) \\ &= \sum_m x^{m-r} \sum_p (-1)^p \dim H^p(\text{Gr}_m^*). \end{aligned}$$

On the other hand, recall

$$\Psi(\mathcal{A}; x, t) = \text{Poin}\left(\text{Gr}\Omega^*(\mathcal{A}) : x, \frac{t(1-x) - 1}{x}\right)$$

from Theorem 5.3. Set $t = (1 - x^{1-r})(1 - x)^{-1}$ in $\Psi(\mathcal{A}; x, t)$ and take the limit of $x \rightarrow 1$. Then, applying Theorem 5.3, we have

$$\begin{aligned} \chi(\mathcal{A}, 1 - r) &= \lim_{x \rightarrow 1} \Psi(\mathcal{A}; x, (1 - x^{1-r})(1 - x)^{-1}) = \lim_{x \rightarrow 1} \text{Poin}(\text{Gr}\Omega^*(\mathcal{A}) : x, -x^{-r}) \\ &= \sum_p \dim H^p(\text{Gr}^*). \end{aligned} \quad \square$$

The following condition (for central arrangements) was introduced, without name, in [7, Definition 5.4]:

DEFINITION 7.2. Let pd denote the projective dimension over S . If

$$\text{pd } \Omega^p(\mathcal{A}) \leq p \text{ for } 0 \leq p \leq l,$$

then we say that \mathcal{A} is *tame*.

The following result is well-known (e.g., see [7, Theorem 5.8]):

THEOREM 7.3. Let $S = \mathbf{C}[x_1, x_2, \dots, x_l]$ be a polynomial ring. Let

$$\mathcal{C}^{\cdot} : 0 \rightarrow \mathcal{C}^0 \xrightarrow{d_0} \mathcal{C}^1 \xrightarrow{d_1} \dots$$

be a cochain complex of finite S -modules. Suppose that each coboundary map d is S -linear. Assume that each cohomology group $H^q(\mathcal{C}^{\cdot})$ is finite dimensional over \mathbf{C} . If a nonnegative integer q satisfies

$$\text{pd } \mathcal{C}^p < l + p - q \text{ for all } p,$$

then $H^q(\mathcal{C}^{\cdot}) = 0$. □

THEOREM 7.4. If \mathcal{A} is tame, then we have $H^p(\Omega^*) = 0$ ($p \neq l$) and $\dim H^l(\Omega^*) = |\chi(\mathcal{A}, 1 - r)|$ for every \mathcal{A} -transverse polynomial f of degree $r > 0$.

Proof. Since the cohomology groups $H^p(\Omega^*(\mathcal{A}))$ are finite dimensional and the differential $df \wedge$ is S -linear, we can apply Theorem 7.3 to conclude $H^p(\Omega^*) = 0$ ($p \neq l$). Apply Theorem 7.1. □

DEFINITION 7.5. The affine arrangement is called *free* if the S -module $\Omega^1(\mathcal{A})$ is a free S -module.

Remark. In [6], the definition of free arrangement is given only for central arrangements. We can show that an affine arrangement \mathcal{A} is free if and only if its localization $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$ is a (central) free arrangement for all $X \in L(\mathcal{A})$. We can also show that $\Omega^p(\mathcal{A}) = \wedge_S^p \Omega^1(\mathcal{A})$ for all p if \mathcal{A} is free.

The following is corollary from Theorem 7.4:

COROLLARY 7.6. *If an affine l -arrangement \mathcal{A} is free, then we have $H^p(\Omega^*) = 0$ ($p \neq l$) and $H^l(\Omega^*) = |\chi(\mathcal{A}, 1 - r)|$ for every \mathcal{A} -transverse polynomial f of degree $r > 0$. \square*

We also have the vanishing of the three lowest cohomology groups for any affine arrangements:

PROPOSITION 7.7. *Let f be an arbitrary \mathcal{A} -transverse polynomial of a positive degree. Then*

- (1) $H^0(\Omega^*) = 0$ if $l > 0$,
- (2) $H^1(\Omega^*) = 0$ if $l > 1$,
- (3) $H^2(\Omega^*) = 0$ if $l > 2$.

Proof. Same as the proof of Proposition 5.15 in [7]. \square

COROLLARY 7.8. *If \mathcal{A} is an affine l -arrangement with $l \leq 3$, then we have $H^p(\Omega^*) = 0$ ($p \neq l$) and $\dim H^l(\Omega^*) = |\chi(\mathcal{A}, 1 - r)|$ for every \mathcal{A} -transverse polynomial f of degree $r > 0$. \square*

Next we finally get results concerning the dimensions of cohomology groups of the complex Ω_{∇}^* . The following theorem contains Theorem 2 in the Introduction.

THEOREM 7.9. *Suppose $f \in S$, $\deg f = r > 0$, is \mathcal{A} -transverse. Then*

- (1) *the cohomology groups $H^p(\Omega_{\nabla}^*)$ and $H^p(\text{Gr}\Omega_{\nabla}^*)$ are finite dimensional for all p ,*
- (2) *the Euler characteristics (= the alternating sum of dimensions of cohomology groups) of the complexes Ω_{∇}^* and $\text{Gr}\Omega_{\nabla}^*$ are both equal to $\chi(\mathcal{A}, 1 - r)$.*

Proof. Recall $H^p(\text{Gr}_q\Omega^*(\mathcal{A})) = H^p(\text{Gr}_q\Omega_{\nabla}^*(\mathcal{A}))$ from the Remark to 2.9. Thus we can prove this by the same method as in the proof of Theorem 7.1. \square

The following theorem is Theorem 5 in the Introduction.

THEOREM 7.10. *Let f be an \mathcal{A} -transverse polynomial of degree $r > 0$. If the cone $c\mathcal{A}$ of \mathcal{A} satisfies $H^p(\Omega^*(c\mathcal{A})) = 0$ for all $p \neq l + 1$, then $H^p(\Omega_{\nabla}^*) = 0$ for all $p \neq l$ and $\dim H^l(\Omega_{\nabla}^*) = |\chi(\mathcal{A}, 1 - r)|$ for every \mathcal{A} -transverse polynomial f of degree $r > 0$.*

Proof. Since $H^p(\Omega^*(c\mathcal{A})) = 0$ for $p \neq l + 1$, we have $H^p(\text{Gr}_q \Omega^*(\mathcal{A})) = 0$ for $p \neq l, q \in \mathbf{Z}$ by Theorem 6.9. Since $H^p(\text{Gr}_q \Omega^*(\mathcal{A})) = H^p(\text{Gr}_q \Omega_{\nabla}^*(\mathcal{A}))$ from the Remark to 2.9, the spectral sequences of filtered complexes $\{\Omega_{\leq q}^*\}$ and $\{\Omega_{\nabla}^*_{\leq q}\}$ are both E_1 -degenerate. Therefore $\dim H^p(\Omega^*) = \dim H^p(\Omega_{\nabla}^*)$. \square

The following result is one of the main results in [4]:

COROLLARY 7.11. *If \mathcal{A} is in general position (for the definition, e.g., see [6, Definition 5.19]), then $H^p(\Omega_{\nabla}^*) = 0$ ($p \neq l$) and*

$$\dim H^l(\Omega_{\nabla}^*) = |\chi(\mathcal{A}, 1 - r)| = \sum_{i=0}^l \binom{n}{l-i} q^i$$

for every \mathcal{A} -transverse polynomial f of degree $r > 0$.

Proof. When \mathcal{A} is in general position, its cone $c\mathcal{A}$ is a generic arrangement (for the definition, e.g., see [6, Definition 5.22]). We have $H^p(\Omega^*(c\mathcal{A})) = 0$ ($p \neq l + 1$) by Proposition 5.12 in [7]. The characteristic polynomial $\chi(\mathcal{A}, t)$ of \mathcal{A} is directly computed. \square

COROLLARY 7.12. *If \mathcal{A} is an affine l -arrangement such that its cone $c\mathcal{A}$ is free with its exponents (for the definition, e.g., see [6, Definition 4.25]) $\{1, b_1, b_2, \dots, b_l\}$ then $H^p(\Omega_{\nabla}^*) = 0$ ($p \neq l$) and*

$$\dim H^l(\Omega_{\nabla}^*) = |\chi(\mathcal{A}, 1 - r)| = \prod_{i=1}^l (b_i + q)$$

for every \mathcal{A} -transverse polynomial f of degree $r > 0$.

Proof. Since $c\mathcal{A}$ is free, we have $H^p(\Omega^*(c\mathcal{A})) = 0$ ($p \neq l + 1$) by Corollary 7.6. Also its characteristic polynomial $\chi(c\mathcal{A}, t)$ is equal to $(t - 1) \prod_{i=1}^l (t - b_i)$. Therefore $\chi(\mathcal{A}, t) = \prod_{i=1}^l (t - b_i)$ by [6, Proposition 2.51, Definition 2.52]. Apply Theorem 7.10. \square

COROLLARY 7.13. *If \mathcal{A} is a 2-arrangement, then $H^p(\Omega_{\mathcal{V}}^*) = 0$ ($p \neq 2$) and*

$$\dim H^2(\Omega_{\mathcal{V}}^*) = |\chi(\mathcal{A}, 1 - r)|$$

for every \mathcal{A} -transverse polynomial f of degree $r > 0$.

Proof. If $l = 2$, then $c\mathcal{A}$ is a central 3-arrangement. Apply Corollary 7.8 and Theorem 7.10. \square

EXAMPLE 7.14. Let an affine arrangement \mathcal{A} be defined by

$$Q = \prod_{i=1}^l x_i \prod_{i=1}^l (x_i + 1) \prod_{i=1}^l (x_i - 1) \prod_{1 \leq i \leq j \leq l} (x_i + x_j) \prod_{1 \leq i \leq j \leq l} (x_i - x_j).$$

Then $c\mathcal{A}$ is a Coxeter arrangement of type B_{l+1} whose exponents are $\{1, 3, 5, \dots, 2l + 1\}$. Therefore $c\mathcal{A}$ is free (e.g., [6, Theorem 6.60]). By Corollary 7.12, $H^p(\Omega_{\mathcal{V}}^*) = 0$ ($p \neq l$) and

$$\dim H^l(\Omega_{\mathcal{V}}^*) = \prod_{i=1}^l (2i + 1 + q)$$

for every \mathcal{A} -transverse polynomial f of degree $r > 0$.

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