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TOTALLY REAL ORBITS IN AFFINE QUOTIENTS OF REDUCTIVE GROUPS

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Let K be a compact connected Lie group and L a closed subgroup of K. In [8] M. Lassalle proves that if K is semisimple and L is a symmetric subgroup of K then the holomorphy hull of any K-invariant domain in $K^{\mathbf{C}}/L^{\mathbf{C}}$ contains K/L. In [1] there is a similar result if L contains a maximal torus of K. The main group theoretic ingredient there was the characterization of K/L as the unique totally real K-orbit in $K^{\mathbf{C}}/L^{\mathbf{C}}$. On the other hand, Patrizio and Wong construct in [9] special exhaustion functions on the complexification of symmetric spaces K/L of rank 1 and find that the minimum value of their exhaustions is always achieved on K/L. By a lemma of Harvey and Wells [6] one knows that the set where a strictly plurisubharmonic (briefly s.p.s.h) function achieves its minimum is totally real. There is a related result in [2, Lemma 1.3] which states that if ϕ is any differentiable function on a complex manifold M then the form $dd^{\mathbf{C}}\phi$ vanishes identically on any real submanifold N contained in the critical set of ϕ ; in particular if φ is s.p.s.h then N must be totally real. In view of these results we give in this note a description of all totally real K-orbits in the affine quotients $K^{\mathbf{C}}/L^{\mathbf{C}}$ of K/L. Our main result is as follows:

PROPOSITION. *Let $G = K^{\mathbf{C}}$, $H = L^{\mathbf{C}}$. The group L has finitely many totally real orbits in G/H if and only if $N(H^{\circ})/H^{\circ}$ is finite, H° being the connected component of H and $N(H^{\circ})$ its normalizer in G, and in this case there is a unique totally real K-orbit in G/H.*

This proposition has the following consequence.

COROLLARY. *If $N(H^{\circ})/H^{\circ}$ is finite then any K-invariant s.p.s.h. function on G/H is proper and achieves its minimum value on K/L. Moreover, the holomorphy hull of any K-invariant domain in G/H meets K/L.*

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This Corollary follows from the Proposition, using the main results of [2] and repeating the arguments of Corollary 2 of [1].

The appendix verifies that the assumptions of the Corollary hold if K is semi-simple and K/L is a symmetric space, thereby recovering results of Lassalle and Patrizio-Wong mentioned above.

From the Corollary we see that if $N(H^\circ)/H^\circ$ is finite then any K-invariant s.p.s.h. function on G/H is a canonical exhaustion function in the sense of [9]. In Eliashberg-Gromov [5, Th. 1.4 A] it is stated that any two exhausting strictly plurisubharmonic functions on a Stein manifold M give the same symplectic structure. This is not quite true as e.g. the function $\| z \|^2$ and $\log(1 + \| z \|^2)$ $(z \in \mathbf{C}^n)$ are exhausting strictly plurisubharmonic functions, but their associated forms $dd^{\mathbf{C}} \| z \|^2$ and $dd^{\mathbf{C}} \log(1 + \| z \|^2)$ are not symplectically equivalent, as they give, infinite and finite volume to $\mathbf{C}^n$. However, by imposing an extra geometric condition, namely that the metric be complete and Ricci flat, it is in some cases possible to pin down a unique s.p.s.h. exhaustion function. This happens in the case of complex symmetric varieties of rank 1, where the existence of a complete Ricci-flat metric is assured by a theorem of Bando-Kobayashi [3]. Their conditions can be verified by using the description of invariant Chern forms given in Borel-Hirzebruch [4]. It would be interesting to understand, in the spirit of [9], the geometric significance of these distinguished metrics.

Our notation is standard. In particular, if H is a subgroup of G, then $N_G(H)$ and $Z_G(H)$ denote the normalizer and centralizer of H in G. Also, H° denotes the connected component of identity of H and x_y the conjugate xyx^{-1}.

Proof of the Proposition. Let G and K be as in the statement of the Proposition. We note that totally real K-orbits in G/H are precisely those of half the dimension of G/H. Moreover, if $\pi : G/H^\circ \to G/H$ is the natural map and Ω is a totally real K-orbit in G/H then $\pi^{-1}(\Omega)$ is a union of totally real K-orbits which are permuted by the right action of the finite group H/H° on G/H°. Therefore, we may assume that H is connected. Let $G = KP$ be the Cartan decomposition of G [7]. We give the proof in several steps, some of which are of independent interest.

Step 1. *If $p = e^x \in P$ centralizes $y \in \mathrm{Lie}(G)$ then the* 1-*parameter subgroup* $\{e^{rX} : r \in \mathbf{R}\}$ *also centralizes* y.

Proof. There is a faithful representation of G in $GL(n, \mathbf{C})$ in which K is represented by unitary matrices and P by Hermitian matrices. Now if x is a Hermitian matrix and $e^{nx}(n \in \mathbf{Z}, n > 0)$ commutes with a matrix y then, taking into account that e^x has positive eigenvalues, one sees readily that e^x also commutes with y and therefore so does $e^{rx}(r \in \mathbf{R})$.

Step 2. *If L is connected and $n \in N_G(H)$ then n factorizes as $n = kpx$, where $k \in K \cap N(H)$, $p \in P \cap Z_G(H)$ and $x \in H$ (recall that $H = L^{\mathbf{C}}$ and $G = K^{\mathbf{C}}$).*

Proof. Let $n \in N(H)$. Now L is a maximal compact subgroup of H so, by conjugacy of maximal compact subgroups, we have ${}^{xn}L = L$ for some $x \in H$. Let $xn = kp$ be the Cartan decomposition of xn with $k \in K$ and $p \in P$. Taking into account that K normalizes P we see that if $k, k_1 \in K$ and $p \in P$ with $pkp^{-1} = k_1$, then $k = k_1$. Therefore ${}^pL = {}^{k^{-1}}L \subset K$ shows that p centralizes L and therefore H too, and k normalizes H. Hence $n = x^{-1}kp = k(k^{-1}x^{-1}k)p = kp(k^{-1}x^{-1}k) = kpx'$ where $x' = k^{-1}x^{-1}k \in H$, $k \in K \cap N(H)$ and $p \in P \cap Z_G(H)$.

Step 3. *If L is connected and $N = N(H)/H$ is finite, then N has representatives in K.*

Proof. Let $n \in N(H)$ and $n = kpx$ be the factorization of n given by step 2. Let $p = e^x$. The 1-parameter subgroup $Z = \{e^{rx} : r \in \mathbf{R}\}$ is, by step 1, in $Z_G(H)$. Since ZH/H is in the finite group $N(H)/H$, we must have $Z \subset H$. Therefore $N(H)/H$ has representatives in K.

Step 4. *For a connected group L, the orbits of $N_K(H)H/H$ on $N(H)/H$ parametrize the totally real K-orbits in G/H.*

Proof. A K-orbit Ω in G/H is totally real if and only if $\dim(\Omega) = \dim(K/L)$. Let $\xi_\circ = eH$ and let $Kx\xi_\circ$ be totally real in G/H. So $\dim(K \cap xHx^{-1}) = \dim(L)$ and therefore $\dim({}^{x^{-1}}K \cap H) = \dim(L)$. By conjugacy of maximal compact subgroups, the group $({}^{x^{-1}}K \cap H)^\circ$ is conjugate in $H = L^{\mathbf{C}}$ to L and therefore $(K \cap xHx^{-1})^\circ$ is conjugate in $G = K^{\mathbf{C}}$ to L, say by an element kp, where $k \in K$ and $p \in P$. As in step 2, the element p centralizes L and so $(K \cap xHx^{-1})^\circ = kLk^{-1}$. Hence $kLk^{-1} \subset xHx^{-1}$ and $k^{-1}x \in N(H)$, so $x = kn$ for some $n \in N(H)$. Conversely if $x = kn$ with $n \in N(H)$, then $K \cap xHx^{-1} = K \cap kHk^{-1} \cong K \cap H = L$. Therefore, a K-orbit in G/H is totally real if and only if a representative of the K-orbit can be chosen in $N(H)$. Finally, if $n_1, n_2 \in N(H)$ and

$kn_1(H) = n_2(H)$ with $k \in K$ then $k \in N_K(H)$. Therefore the orbits of the compact group $N_K(H)H/H \cong N_K(H)/L$ on $N(H)/H$ parametrize the totally real K-orbits in G/H.

Step 5. *Conclusion of the proof.* Suppose K has finitely many totally real orbits in G/H. Then it also has finitely many such orbits in G/H°, so we may assume that H is connected. By step 4, the compact group $N_K(H)H/H$ has finitely many orbits on the Stein manifold $N(H)/H$. Therefore, $N(H)/H$ must be finite and by step 3 it must have representatives in K. Hence there is a unique totally real K-orbit in G/H if and only if $N(H^{\circ})/H^{\circ}$ is finite. This completes the proof of the proposition.

APPENDIX. In this appendix we verify that the assumptions of the proposition hold for the complexification $K^{\mathbf{C}}/L^{\mathbf{C}}$ of a symmetric space K/L, with K being compact.

PROPOSITION. *If $K \supset L$ is a symmetric pair then $K^{\mathbf{C}}/L^{\mathbf{C}}$ has exactly one totally real K-orbit (K-semisimple).*

Proof. We have $\dot{K}^{\mathbf{C}} = \dot{L}^{\mathbf{C}} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is $\mathrm{ad}(L^{\mathbf{C}})$ invariant so $\mathrm{Lie}(N(L^{\mathbf{C}}) = \dot{L}^{\mathbf{C}} \oplus Z_{\dot{L}^{\mathbf{C}}}(\mathfrak{m})$ as vector spaces.

Hence $N(L^{\mathbf{C}})/L^{\mathbf{C}}$ is finite if and only if $Z_{L^{\mathbf{C}}}(\mathfrak{m}) = 0$. The proposition follows from the following lemma.

LEMMA. *If $G \supset H$ is a symmetric pair with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ then adjoint representation of H on $\mathfrak{m}$ is irreducible, provided G is simple.*

Proof. The Killing form is a non-degenerate form. The function $\Theta(x + y) = x - y$ is an automorphism of G. Since the Killing form is invariant under all automorphisms, we see that $(\mathfrak{h}, \mathfrak{m}) = 0$. Hence the Killing form is nondegenerate on both $\mathfrak{h}$ and $\mathfrak{m}$.

Let $\mathfrak{m}_1 \subset \mathfrak{m}$ be an H-invariant subspace and $\mathfrak{m}_2 (= \mathfrak{m}_1^{\perp})$ its orthocomplement. Now $([\mathfrak{m}_1, \mathfrak{m}_2], \mathfrak{m}) = 0$ as $(\mathfrak{m}, \mathfrak{m}) \subset \mathfrak{h}$. Moreover as $([x, y], z) = (x, [y, z])$ we see that $([\mathfrak{m}_1, \mathfrak{m}_2], \mathfrak{h}) = 0$. Hence $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$ by non-degeneracy of the Killing form.

Now $\mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_2]$ is an H-submodule. Moreover

$$[\mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1], \mathfrak{m}] = [\mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1], \mathfrak{m}_1 + \mathfrak{m}_2] \subset [\mathfrak{m}_1, \mathfrak{m}_1] + \mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_2] + [[\mathfrak{m}_1, \mathfrak{m}_1], \mathfrak{m}_2].$$

Since $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$ we have by Jacobi identity $[[\mathfrak{m}_1, \mathfrak{m}_1], \mathfrak{m}_2] = 0$. Hence $[\mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1], \mathfrak{m}] \subset \mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1]$. Thus if G is semisimple then $\mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1]$ is an ideal. In particular if G is simple then $\mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m} \oplus \mathfrak{h}$ shows that $\mathfrak{m}_1 = 0$ or $\mathfrak{m}_1 = \mathfrak{m}$. Hence in this case $\mathfrak{m}$ is an irreducible submodule.

In particular if $K \supset L$ is a symmetric pair and K is simple then in the decomposition of $\dot{K}^{\mathbf{C}} = \dot{L}^{\mathbf{C}} \oplus \mathfrak{m}$ we must have $Z_{L^{\mathbf{C}}}(\mathfrak{m}) = 0$. So in this case there is exactly one totally real K-orbit.

General case. Let $G = G_1 \oplus \cdots \oplus G_r$ be semisimple. If Θ is an involutary automorphism then either $\Theta(G_i) = G_i$ or else the orbit of Θ on G_i is of length 2.

Hence $G = \oplus_{i=1}^{r_0} G_i \oplus_{j=1}^{r_1} (G_j + \Theta G_j)$. In $G_i \oplus \Theta(G_i)$ the fixed point set is $(\xi \oplus \Theta(\xi))$ and (-1) eigen-space is $(\xi - \Theta(\xi))$.

For a symmetric pair $L^{\mathbf{C}} = G_\Theta$ and $\mathfrak{m} = G_{-1}$. Hence $Z_{L^{\mathbf{C}}}(\mathfrak{m})$ is a sum of $Z_{L_i^{\mathbf{C}}}(\mathfrak{m}_i)$. Since each sum is zero we see that for a symmetric pair $K^{\mathbf{C}} \subset L^{\mathbf{C}}$ there is exactly one totally real K-orbit.

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