K. Cho and K. Matsumoto
Nagoya Math. J.
Vol. 139 (1995), 67–86

INTERSECTION THEORY FOR TWISTED COHOMOLOGIES AND TWISTED RIEMANN'S PERIOD RELATIONS I

KOJI CHO AND KEIJI MATSUMOTO

To the memory of Professor Michitake Kita

Introduction

The beta function $B(\alpha, \beta)$ is defined by the following integral

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,$$

where $\arg t = \arg(1-t) = 0$, $\Re\alpha$, $\Re\beta > 0$, and the gamma function $\Gamma(\alpha)$ by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,$$

where $\arg t = 0$, $\Re\alpha > 0$. By the use of the well known formulae

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha},$$

we get the following formula:

$$B(\alpha, \beta)B(-\alpha, -\beta) = 2\pi i\Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\Big(-\frac{\exp(2\pi i(\alpha+\beta)) - 1}{(\exp(2\pi i\alpha) - 1)(\exp(2\pi i\beta) - 1)}\Big).$$

If we regard the interval (0,1) of integration as a twisted cycle defined by the multi-valued function $t^\alpha(1-t)^\beta$, the factor

$$-\frac{\exp(2\pi i(\alpha+\beta)) - 1}{(\exp(2\pi i\alpha) - 1)(\exp(2\pi i\beta) - 1)}$$

is nothing but the twisted self-intersection number ([KY1]) of the cycle (0,1). It is quite natural to think that the factor

Received May 23, 1994.

$$2\pi i\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$$

should be the "twisted self-intersection number" of the 1-form

$$\frac{dt}{t}+\frac{dt}{1-t},$$

so that the above formula should be thought of a twisted version of Riemann's period relation.

This paper establishes the intersection theory for twisted cocycles and the twisted Riemann's period relation connecting the intersection theories for twisted cycles [KY1] and for twisted cocycles.

In the following we explain the results of this paper using as plain language as possible; the notion and notation used are rigorously fixed in the text. Let $x_0, \ldots, x_n$ be $n+1$ distinct points on $\boldsymbol{P}^1$, and

$$\omega=\sum_{j=0}^{n}\alpha_j\frac{dt}{t-x_j}, \quad \left(\sum_{j=0}^{n}\alpha_j=0,\ \alpha_j\notin \mathbf{N}-\{0\}\right)$$

a connection form. The first twisted cohomology group

$$H^1(U, L)\simeq \mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D), \nabla)),\ U:=\boldsymbol{P}^1-D$$

with respect to the connection $\nabla=d+\omega\wedge$ is known to be isomorphic to

$$\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega, \quad D:=x_0+\cdots+x_n,$$

where

$$L:=\ker(\nabla|_U:\mathcal{O}_U\to\Omega^1_{\boldsymbol{P}^1}(\log D)|_U)$$

is a local system on U defined by ∇.

The dual of the cohomology group $H^1(U, L)$ is given by the cohomology group with compact support $H^1_c(U, L^\vee)$, where $L^\vee$ is the local system defined by the connection $\nabla^\vee:=d-\omega\wedge$ dual to ∇. We show that the dual cohomology group is isomorphic to $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$. Since there is a natural dual pairing between the two cohomology groups $H^1(U, L)$ and $H^1_c(U, L^\vee)$, there should exist the induced bilinear form on the spaces $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega$ and $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$. By using elements

$$\varphi_j=\frac{dt}{t-x_j}-\frac{dt}{t-x_{j+1}}\in\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)), \quad 1\le j\le n-1,$$

we give bases for the spaces above by

$$\varphi_j^+ \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega,\ \varphi_j^- \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega),\ 1 \leq j \leq n-1,$$

where φ_j^+ and φ_j^- are the images of φ_j by the natural projections from $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$. Our first main theorem gives explicitly the bilinear form, which turns out to be symmetric and will be called the *intersection form*:

$$\langle \varphi_j^+, \varphi_j^- \rangle = 2\pi i\left(\frac{1}{\alpha_j} + \frac{1}{\alpha_{j+1}}\right),$$

$$\langle \varphi_j^+, \varphi_{j+1}^- \rangle = \langle \varphi_{j+1}^+, \varphi_j^- \rangle = -\frac{2\pi i}{\alpha_{j+1}},$$

$$\langle \varphi_j^+, \varphi_k^- \rangle = 0 \quad \text{if } |j-k| \geq 2.$$

Our second main theorem states the relation between the three pairings: the intersection form for twisted cohomologies, that for twisted homologies, and the pairing of twisted homologies and twisted cohomologies, i.e. integrals. Let

$$\gamma_j^+ \in H_1(U, L^\vee), \quad \delta_j^- \in H_1(U, L), \quad j = 1, \ldots, n-1$$

be any bases of twisted cycles (the notation is slightly different from that in [KY1]) and

$$\xi_j^+ \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega, \quad j = 1, \ldots, n-1,$$
$$\eta_j^- \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega), \quad j = 1, \ldots, n-1,$$

be any bases of twisted cocycles; let I_h and I_{ch} be the intersection matrices:

$$I_h = \begin{pmatrix} \langle \gamma_1^+, \delta_1^- \rangle & \cdots & \langle \gamma_1^+, \delta_{n-1}^- \rangle \\ \vdots & & \vdots \\ \langle \gamma_{n-1}^+, \delta_1^- \rangle & \cdots & \langle \gamma_{n-1}^+, \delta_{n-1}^- \rangle \end{pmatrix}, \quad I_{ch} = \begin{pmatrix} \langle \xi_1^+, \eta_1^- \rangle & \cdots & \langle \xi_1^+, \eta_{n-1}^- \rangle \\ \vdots & & \vdots \\ \langle \xi_{n-1}^+, \eta_1^- \rangle & \cdots & \langle \xi_{n-1}^+, \eta_{n-1}^- \rangle \end{pmatrix}.$$

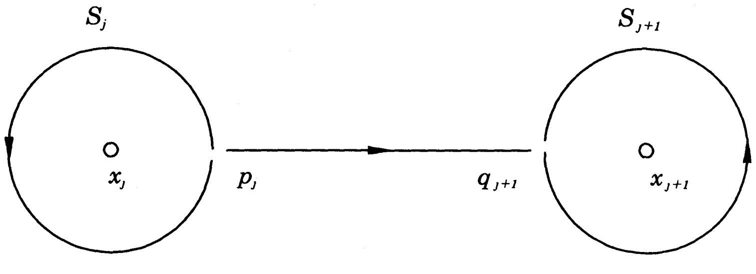
The intersection matrix I_h can be explicitly computed [KY1]; take for instance bases γ_j^+ and $\delta_j^- := \varphi_j^-$ as follows: let us assume for simplicity that the x_j's are all real and are arranged as $x_0 < x_1 < \cdots < x_n$, and u_0 a branch of the multi-valued function $u = \Pi(t - x_j)^{\alpha_j}$ defined on the lower half t-plane. We define special cycles by

$$\gamma_j^+ = \overrightarrow{(p_j, q_{j+1})} \otimes u_0 + \frac{1}{c_j - 1} S_j \otimes u_0 - \frac{1}{c_{j+1} - 1} S_{j+1} \otimes u_0,$$

$$\gamma_j^- = \overrightarrow{(p_j, q_{j+1})} \otimes u_0^{-1} - \frac{c_j}{c_j - 1} S_j \otimes u_0^{-1} + \frac{c_{j+1}}{c_{j+1} - 1} S_{j+1} \otimes u_0^{-1},\ c_j = \exp 2\pi i\alpha_j,$$

where S_k is a positively oriented circle with center x_k and with starting point p_k

or q_k; see Figure.



Figure

Then the intersection matrix for these special bases turns out to be

$$I_h = \begin{pmatrix} -d_{12}/d_1d_2 & 1/d_2 & 0 & \cdots & 0 & 0 \\ c_2/d_2 & -d_{23}/d_2d_3 & \cdots & & 0 & 0 \\ 0 & \vdots & & & & \vdots \\ \vdots & & & & \vdots & 0 \\ 0 & 0 & & \cdots & -d_{n-2,n-1}/d_{n-2}d_{n-1} & 1/d_{n-1} \\ 0 & 0 & \cdots & 0 & c_{n-1}/d_{n-1} & -d_{n-1,n}/d_{n-1}d_n \end{pmatrix},$$

where $d_j = c_j - 1$, $d_{jk} = c_jc_k - 1$. It is easy to see that

$${}^tI_h^{-1} = \frac{-1}{d_{1\cdots n}} \begin{pmatrix} d_1d_{2\cdots n} & d_1c_2d_{3\cdots n} & d_1c_{23}d_{4\cdots n} & \cdots & d_1c_{2\cdots n-1}d_n \\ d_1d_{3\cdots n} & d_{12}d_{3\cdots n} & d_{12}c_3d_{4\cdots n} & \cdots & d_{12}c_{3\cdots n-1}d_n \\ d_1d_{4\cdots n} & d_{12}d_{4\cdots n} & d_{123}d_{4\cdots n} & \cdots & d_{123}c_{4\cdots n-1}d_n \\ \vdots & \vdots & \vdots & & \vdots \\ d_1d_n & d_{12}d_n & d_{123}d_n & \cdots & d_{1\cdots n-1}d_n \end{pmatrix},$$

where $c_{jk\cdots} = c_jc_k\cdots$, $d_{jk\cdots} = c_jc_k\cdots - 1$. Let us arrange the integrals (periods) as follows:

$$P^+ = \begin{pmatrix} \int_{\gamma_1^+} \xi_1^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_1^+ \\ \vdots & & \vdots \\ \int_{\gamma_1^+} \xi_{n-1}^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_{n-1}^+ \end{pmatrix}, \quad P^- = \begin{pmatrix} \int_{\delta_1^-} \eta_1^- & \cdots & \int_{\delta_{n-1}^-} \eta_1^- \\ \vdots & & \vdots \\ \int_{\delta_1^-} \eta_{n-1}^- & \cdots & \int_{\delta_{n-1}^-} \eta_{n-1}^- \end{pmatrix}.$$

Here the integral $\int_{\gamma_1^+} \xi^+$ (resp. $\int_{\delta^-} \eta^-$) of a twisted cocycle ξ^+ (resp. η^-) over a twisted cycle $\gamma^+ \in H_1(U, L^\vee)$ (resp. $\delta^- \in H_1(U, L)$) is defined as follows: for a twisted cocycle ξ^+ (resp. η^-) take a representing form ξ (resp. η) of $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$ and for a twisted cycle γ^+ (resp. δ^-) take a representing twisted chain $\sum_i g_i \otimes u_i$ (resp. $\sum_i g_i' \otimes u_i^{-1}$), where g_i (resp. g_i') is a topological chain and u_i (resp. u^{-1}) is a branch of the multi-valued function

$$u = \prod_{j=0}^{n} (t - x_j)^{\alpha_j} \quad (\text{resp. } u^{-1})$$

along g_i (resp. g_i'); then

$$\int_{\gamma^+} \xi^+ := \sum_i \int_{g_i} u_i \xi, \quad \int_{\delta^-} \eta^- := \sum_i \int_{g_i'} u_i^{-1} \eta,$$

which are independent of the choice of representatives. Our theorem reads

$$P^+ \, {}^t I_h^{-1} \, {}^t P^- = I_{ch}, \quad \text{i.e.} \quad {}^t P^- \, I_{ch}^{-1} \, p^+ = {}^t I_h.$$

We would like to call these identities twisted Riemann's period relations because it resembles Riemann's period relation for a basis of holomorphic 1-forms $\omega_1, \ldots, \omega_g$ and a $\boldsymbol{Z}$-basis of cycles $\gamma_1, \ldots, \gamma_{2g}$ on a compact Riemann surface of genus g. The period matrix P and the intersection matrix I_h of cycles are

$$P = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}, \quad I_h = \begin{pmatrix} \langle \gamma_1, \gamma_1 \rangle & \cdots & \langle \gamma_1, \gamma_{2g} \rangle \\ \vdots & & \vdots \\ \langle \gamma_{2g}, \gamma_1 \rangle & \cdots & \langle \gamma_{2g}, \gamma_{2g} \rangle \end{pmatrix};$$

then Riemann's period relations are given as follows:

$$\begin{pmatrix} P \\ \bar{P} \end{pmatrix} {}^t I_h^{-1} \, ({}^t P \, {}^t \bar{P}) = \begin{pmatrix} \int \omega_j \wedge \omega_k & \int \omega_j \wedge \bar{\omega}_k \\ \int \bar{\omega}_j \wedge \omega_k & \int \bar{\omega}_j \wedge \bar{\omega}_k \end{pmatrix} = i \begin{pmatrix} 0 & H \\ -\bar{H} & 0 \end{pmatrix},$$

where H is positive definite. We remarked it not only because of the resemblance but also because we shall in [Cho1] establish a theory including both Riemann's period relations.

The simplest case, i.e. $n = 2$ is nothing but the formulae for $B(\alpha, \beta)B(-\alpha, -\beta)$ given in the beginning; the next simplest case, i.e. $n = 3$ yields (§4 Example 1) the famous formula

$$\begin{aligned} &F(\alpha, \beta, \gamma; x) F(1-\alpha, 1-\beta, 2-\gamma; x) \\ &\qquad = F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; x) F(\gamma-\alpha, \gamma-\beta, \gamma; x), \end{aligned}$$

where

$$F(\alpha, \beta, \gamma; x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} x^n \quad (\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1).$$

We cordially thank Professors K. Mimachi and M. Yoshida for their constant encouragement and stimulating discussions.

§1. Preliminaries

In the following, notation is so chosen that generalizations to Riemann surfaces of higher genus [Cho1] and to varieties of higher dimension [Cho2] would be smooth. Let $x_0, \ldots, x_n$ be $n+1$ distinct points on $\boldsymbol{P}^1$; put

$$D := x_0 + \cdots + x_n, \quad U := \boldsymbol{P}^1 - D, \quad j : U \hookrightarrow \boldsymbol{P}^1.$$

Let ω be a logarithmic 1-form on $\boldsymbol{P}^1$ with poles at D with residue α_j at x_j; note that

$$\sum_{j=0}^{n} \alpha_j = 0.$$

Consider the connection ∇ with connection form ω:

$$\nabla = d + \omega\wedge : \mathcal{O}_{\boldsymbol{P}^1} \to \Omega^1_{\boldsymbol{P}^1}(\log D),$$

where $\mathcal{O}_{\boldsymbol{P}^1}$ is the sheaf of holomorphic functions on $\boldsymbol{P}^1$, $\Omega^1_{\boldsymbol{P}^1}$ the sheaf of holomorphic 1-forms on $\boldsymbol{P}^1$ and $\Omega^1_{\boldsymbol{P}^1}(\log D)$ the sheaf of meromorphic 1-forms with logarithmic singularities only on D. Let L be a local system on U defined by

$$L := \ker(\nabla|_U : \mathcal{O}_U \to \Omega^1_{\boldsymbol{P}^1}(\log D)|_U),$$

where $\mathcal{O}_U$ is the sheaf of holomorphic functions on U.

We are going to present several isomorphisms for two hypercohomologies; they shall be made explicit in the next section; the definition of hypercohomology shall be also given in §2.2. If $\alpha_j \notin \mathbf{N} - \{0\}$ then the following quasi-isomorphism [Del1] holds

$$\begin{aligned} \boldsymbol{R}j_*L &\underset{qis}{\simeq} (\Omega^{\cdot}(\log D), \nabla) \\ &:= \cdots 0 \to \mathcal{O}_{\boldsymbol{P}^1} \xrightarrow{\nabla} \Omega^1_{\boldsymbol{P}^1}(\log D) \to 0 \cdots, \end{aligned}$$

which leads to

$$\begin{aligned} H^1(U, L) &\simeq \mathbf{H}^1(\boldsymbol{P}^1, (\Omega^\cdot(\log D), \nabla)) \\ &\simeq \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega, \end{aligned}$$

where the last isomorphism is derived by the (Hodge-to-logarithmic de Rham) spectral sequence:

$$E_1^{pq} \simeq H^q(\boldsymbol{P}^1, \Omega^p(\log D)) \Rightarrow \mathbf{H}^{p+q}(\boldsymbol{P}^1, (\Omega^\cdot(\log D), \nabla)),$$

and $E_1^{pq} = 0$ if $q > 0$.

On the other hand by the Poincaré-Verdier duality [EV1], (i.e. by performing $\boldsymbol{R}\mathscr{H}om(\cdot, \mathbf{C}_{\boldsymbol{P}^1})$) we have:

$$\begin{aligned} j_!L^\vee &\underset{qis}{\simeq} (\Omega^\cdot(\log D)(-D), \nabla^\vee) \\ &:= \cdots 0 \to \mathcal{O}_{\boldsymbol{P}^1}(-D) \xrightarrow{\nabla^\vee} \Omega^1_{\boldsymbol{P}^1}(\log D)(-D) \simeq \Omega^1_{\boldsymbol{P}^1} \to 0 \cdots, \end{aligned}$$

where $\nabla^\vee := d - \omega$, and $!$ means the zero-extension; this leads to

$$\begin{aligned} H^1_c(U, L^\vee) &\simeq \mathbf{H}^1(\boldsymbol{P}^1, (\Omega^\cdot(\log D)(-D), \nabla^\vee)) \\ &\simeq \ker(\nabla^\vee : H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \to H^1(P^1, \Omega^1_{\boldsymbol{P}^1})) \\ &= \ker(-\omega : H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \to H^1(P^1, \Omega^1_{\boldsymbol{P}^1})), \end{aligned}$$

where H_c means cohomology with compact support, and the second isomorphism is derived by the spectral sequence:

$$E_1^{pq} = H^q(\boldsymbol{P}^1, \Omega^p(\log D)(-D)) \Rightarrow \mathbf{H}^{p+q}(\boldsymbol{P}^1, (\Omega^\cdot(\log D)(-D), \nabla^\vee)),$$

and $E_1^{pq} = 0$ if $q = 0$. Notice that the duality between $(\Omega^\cdot(\log D), \nabla)$ and $(\Omega^\cdot(\log D)(-D), \nabla^\vee)$ holds without any condition for α_j [EV2]. Notice also that the duality above between $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)/\mathbf{C}\cdot\omega$ and $\ker(-\omega : H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \to H^1(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1}))$ is induced by the Serre duality. We denote by φ^+ (resp. φ^-) the image of $\varphi \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$ under the natural projection to $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)/\mathbf{C}\cdot\omega$ (resp. $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)/\mathbf{C}\cdot(-\omega))$.

§2. Intersection theory for twisted cocycles

Consider the following exact sequence of complexes, which will be referred to as the *basic sequence*:

$$0 \to (\Omega^\cdot(\log D)(-D), \nabla^\vee) \xrightarrow{\iota} (\Omega^\cdot(\log D), \nabla^\vee) \to (\oplus_{j=0}^n \mathbf{C}_{x_j} \xrightarrow{\times\mathrm{res}} \oplus_{j=0}^n \mathbf{C}_{x_j}) \to 0;$$

that is

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(-D) \arrow[r, "\iota"] \arrow[d, "\nabla^{\vee}"'] & \mathcal{O} \arrow[r] \arrow[d, "\nabla^{\vee}"'] & \bigoplus_{j=0}^{n} \mathbf{C}_{x_j} \arrow[r] \arrow[d, "\times \mathrm{res}"] & 0 \\
0 \arrow[r] & \Omega^1 \arrow[r, "\iota"] \arrow[d] & \Omega^1(\log D) \arrow[r, "\mathrm{Res}"] \arrow[d] & \bigoplus_{j=0}^{n} \mathbf{C}_{x_j} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{1}
$$

where

$$\times\ \mathrm{res} : (c_0, \dots, c_n) \to (-\alpha_0 c_0, \dots, -\alpha_n c_n).$$

If $\alpha_j \neq 0$ then $\times$ res is isomorphic, so we have the following isomorphism

$$\iota : \mathbf{H}^{\cdot}(P^1, (\Omega^{\cdot}(\log D)(-D), \nabla^{\vee})) \xrightarrow{\sim} \mathbf{H}^{\cdot}(P^1, (\Omega^{\cdot}(\log D), \nabla^{\vee})),$$

in particular,

$$\iota : \ker(-\omega : H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \to H^1(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1})) \xrightarrow{\sim} \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega).$$

We shall explicitly give the inverse of the isomorphism ι. We first define a homomorphism: $\tau : \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega) \to \ker(-\omega : H^1(\mathcal{O}(-D)) \to H^1(\Omega^1))$ and secondly prove that this gives the inverse of the natural isomorphism ι.

§2.1. Definition of τ

The corresponding long exact sequences of (1) read

$$
\begin{tikzcd}
{} \arrow[r] & H^0(\mathcal{O}) \arrow[r] & \bigoplus_{j=0}^{n} \mathbf{C}_{x_j} \arrow[r, "\delta"] \arrow[d, "\times \mathrm{res}"] & H^1(\mathcal{O}(-D)) \\
{} \arrow[r] & H^0(\Omega^1(\log D)) \arrow[r, "\mathrm{Res}"] & \bigoplus_{j=0}^{n} \mathbf{C}_{x_j} \arrow[r] & H^1(\Omega^1)
\end{tikzcd}
$$

where δ is the connecting homomorphism. Tracing the above commutative diagram, we have

$$\delta \circ (\times\ \mathrm{res})^{-1} \circ \mathrm{Res} : H^0(\Omega^1(\log D)) \to H^1(\mathcal{O}(-D))\ ;$$

it is immediate that this induces the isomorphism

$$\tau : \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega) \to \ker(-\omega : H^1(\mathcal{O}(-D)) \to H^1(\Omega^1)).$$

§2.2. Naturality of τ

LEMMA. $\tau = \iota^{-1}$.

Proof. Let us honestly see the homomorphism ι, i.e. following the definition of hypercohomologies. A fine resolution of the complex $(\Omega^{\cdot}(\log D)(-D), \nabla^{\vee})$ is given by

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\nabla^{\vee}} & \Omega^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}^{00}(-D) & \xrightarrow{\partial-\omega} & \mathcal{E}^{10} & \longrightarrow & 0 \\
 & & \bar{\partial}\downarrow & & \downarrow\bar{\partial} & & \\
0 & \longrightarrow & \mathcal{E}^{01}(-D) & \xrightarrow{\partial-\omega} & \mathcal{E}^{11} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{E}^{pq}$ stands for the sheaf of smooth (p, q)-forms on $\boldsymbol{P}^1$ and $\mathcal{E}^{pq}(-D)$ the sheaf of (p, q)-forms g on $\boldsymbol{P}^1$ such that g/t_j is smooth for a local parameter t_j around x_j. The associated single complex is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\nabla^{\vee}} & \Omega^1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}^{00}(-D) & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{01}(-D) \oplus \mathcal{E}^{10} & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{11} & \longrightarrow & 0.
\end{array}$$

Thus we have

$$\mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D)(-D), \nabla^{\vee})) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}(-D)) \oplus \Gamma(\mathcal{E}^{10}) \to \Gamma(\mathcal{E}^{11})\}}{\nabla^{\vee}\Gamma(\mathcal{E}^{00}(-D))};$$

for $\eta \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$, we denote by $\eta^{\vee}$ the image of $\eta^{-} \in \Gamma(\Omega^1(\log D))/\mathbf{C} \cdot (-\omega)$ under τ. Since the Dolbeault resolution implies

$$H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \simeq \frac{\Gamma(\mathcal{E}^{01}(-D))}{\bar{\partial}\Gamma(\mathcal{E}^{00}(-D))}, \quad H^1(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1})) \simeq \frac{\Gamma(\mathcal{E}^{11})}{\bar{\partial}\Gamma(\mathcal{E}^{10})},$$

$\nabla^{\vee} = d - \omega$ annihilates $\eta^{\vee}$ means that there exists $\mu \in \Gamma(\mathcal{E}^{10})$ such that

$$(d-\omega)\eta^{\vee} = \bar{\partial}\mu,$$

namely,

$$\nabla^{\vee}(\eta^{\vee} + \mu) = 0.$$

This gives an explicit expression of the isomorphism

$$\ker(\nabla^{\vee} : H^1(\mathcal{O}(-D)) \to H^1(\Omega^1)) \xrightarrow{\sim} \mathbf{H}^1((\Omega^{\cdot}(\log D)(-D),\ \nabla^{\vee})) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}(-D)) \oplus \Gamma(\mathcal{E}^{10}) \to \Gamma(\mathcal{E}^{11})\}}{\nabla^{\vee}\Gamma(\mathcal{E}^{00}(-D))}.$$

Similarly a single fine resolution of $(\Omega^{\cdot}(\log D),\ \nabla^{\vee})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \xrightarrow{\nabla^{\vee}} & \Omega^1(\log D) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00} & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{01} \oplus \mathcal{E}^{10}(\log D) & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{11}(\log D) & \longrightarrow & 0 \end{array}$$

gives

$$\mathbf{H}^1(\boldsymbol{P}^1,\ (\Omega^{\cdot}(\log D),\ \nabla^{\vee})) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}) \oplus \Gamma(\mathcal{E}^{10}(\log D)) \to \Gamma(\mathcal{E}^{11}(\log D))\}}{\nabla^{\vee}\Gamma(\mathcal{E}^{00})}.$$

An explicit expression of the isomorphism

$$\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega) \xrightarrow{\sim} \mathbf{H}^1(\boldsymbol{P}^1,\ (\Omega^{\cdot}(\log D),\ \nabla^{\vee})) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}) \oplus \Gamma(\mathcal{E}^{10}(\log D)) \to \Gamma(\mathcal{E}^{11}(\log D))\}}{\nabla^{\vee}\Gamma(\mathcal{E}^{00})}$$

is given by

$$\eta^{-} \mapsto 0 \oplus \eta.$$

Summing up, a fine resolution of the basic sequence is given as follows (pay attention that rows and columns are reversed):

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00}(-D) & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{01}(-D) \oplus \mathcal{E}^{10} & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{11} & \longrightarrow & 0 \\ & & \downarrow & & \iota\downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00} & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{01} \oplus \mathcal{E}^{10}(\log D) & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{11}(\log D) & \longrightarrow & 0 \\ & & \text{restr}\downarrow & & (\text{restr},\downarrow \text{Res}) & & \downarrow & & \\ 0 & \longrightarrow & \oplus\,\mathbf{C}_{x_j} & \xrightarrow[(0,\times\text{res})]{} & \oplus\mathbf{C}_{x_j} \oplus \oplus\mathbf{C}_{x_j} & \xrightarrow[(\times\text{res})\circ pr_1]{} & \oplus\,\mathbf{C}_{x_j} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Now we are going to trace back ι. Let us give $\eta \in \Gamma(\Omega^1(\log D))$. We change the represetative η to

$$\eta' = \eta + \nabla^\vee h$$

so that $(\mathrm{restr}, \mathrm{Res})\eta' = 0$; this can be achieved by taking $h \in \Gamma(\mathscr{E}^{00})$ so that

$$(0, \times \mathrm{res}) \circ \mathrm{restr}\, h = (\mathrm{restr}, \mathrm{Res})\eta.$$

Then there is a form $\tilde{\eta} + \mu \in \Gamma(\mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10})$ which maps under ι to η'; it can be readily checked that $\tilde{\eta} + \mu$ represents an element of

$$\mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D)(-D), \nabla^\vee)) \simeq \frac{\ker\{\Gamma(\mathscr{E}^{01}(-D)) \oplus \Gamma(\mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11})\}}{\nabla^\vee \Gamma(\mathscr{E}^{00}(-D))}.$$

Recall the connecting homomorphism $\delta : \oplus \mathbf{C}_{x_j} \to H^1(\mathscr{O}(-D))$ used when defining τ; it is exactly the same as tracing part of the above diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & * & \longrightarrow & \mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10} & \longrightarrow & * & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle\iota}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}^{00} & \xrightarrow{\nabla^\vee} & \mathscr{E}^{01} \oplus \mathscr{E}^{10}(\log D) & \longrightarrow & * & \longrightarrow & 0 \\
 & & {\scriptstyle\mathrm{restr}}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus \mathbf{C}_{x_j} & \longrightarrow & * & \longrightarrow & * & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Therefore we proved that in cohomology level

$$\tilde{\eta} = \eta^\vee \quad \text{in } H^1(\mathscr{O}(-D));$$

and so (it will be the key in §3),

$$(2) \qquad \iota(\eta^\vee + \mu) = \eta + \nabla^\vee h, \quad \mu \in \Gamma(\mathscr{E}^{10}), \quad h \in \Gamma(\mathscr{E}^{00}).$$

§2.3. Intersection form for cocycles

We assume $\alpha_j \neq 0$. Let us fix an isomorphism

$$\int : H^1(\Omega^1) \to \mathbf{C}$$

by

$$H^1(\Omega^1) \simeq H^1_{\mathrm{Dol}}(\Omega^1) := \Gamma(\mathscr{E}^{11}) / \bar{\partial}\Gamma(\mathscr{E}^{10}) \ni \zeta \mapsto \int_{\boldsymbol{P}^1} \zeta \in \mathbf{C}.$$

For cocycles ξ^+ and η^- represented by $\xi, \eta \in \Gamma(\Omega^1(\log D))$, we now define the intersection form by the natural bilinear form $\langle *, * \rangle$:

$$\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot\omega \times \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega) \to \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot\omega \times H^1_{\mathrm{Dol}}(\mathcal{O}(-D))$$
$$\xrightarrow{\text{Serre duality}} H^1_{\mathrm{Dol}}(\Omega^1) \xrightarrow{\int} \mathbf{C}$$
$$(\xi^+, \eta^-) \mapsto (\xi^+, \eta^\vee) \mapsto \eta^\vee \wedge \xi \mapsto \int \eta^\vee \wedge \xi.$$

Since $\eta^\vee \in \ker(-\omega : H^1_{\mathrm{Dol}}(\mathcal{O}(-D)) \to H^1_{\mathrm{Dol}}(\Omega^1))$ and $\omega^\vee \sim 0$, it is well defined, and is non-degenerate thanks to non-degeneracy of the Serre duality. We compute the intersection numbers for the following forms:

$$\omega_{ij} = \left(\frac{1}{t-x_i} - \frac{1}{t-x_j}\right)dt \in \Gamma(\Omega^1(\log D)),\ 0 \le i \ne j \le n.$$

Let us first explicitly write the image $\omega_{ij}^\vee \in H^1(\mathcal{O}(-D))$ under τ of ω_{ij}^- in terms of the Čech cohomology $\check{H}^1(\mathcal{U}, \mathcal{O}(-D))$ with respect to the covering $\mathcal{U} = \{U_j\}$

$$U_j := U \cup \{x_j\},\ j = 0, \dots, n.$$

CLAIM. *Let $(\omega_{ij}^\vee)_{\mathrm{Cech}}$ be the expression of $\omega_{ij}^\vee$ in the Čech cohomology, then we have*

$$(\omega_{ij}^\vee)_{\mathrm{Cech}} = \begin{cases} 1/\alpha_i + 1/\alpha_j & \text{on } U_{ij} \\ 1/\alpha_i & \text{on } U_{ik}\ (k \ne i, j) \\ -1/\alpha_j & \text{on } U_{jk}\ (k \ne i, j) \\ 0 & \text{on } U_{kl}\ (k, l \ne i, j), \end{cases}$$

where $U_{ij} := U_i \cap U_j$.

Here we use the convention $s_{ij} = -s_{ji}$ for $\{s_{ij}\} \in C^1(\mathcal{O}(-D))$, where $s_{ij} \in \Gamma(U_{ij}, O(-D))$, $s_{ji} \in \Gamma(U_{ji}, O(-D))$.

Proof. It is easy to see that

$$\omega_{ij} \overset{\mathrm{Res}}{\mapsto} (1 \in \mathbf{C}_{x_i}, -1 \in \mathbf{C}_{x_j}, 0 \in \mathbf{C}_{x_k}\ k \ne i, j)$$
$$\overset{\times\mathrm{Res}}{\mapsto} (-1/\alpha_i \in \mathbf{C}_{x_i}, 1/\alpha_j \in \mathbf{C}_{x_j}, 0 \in \mathbf{C}_{x_k}).$$

The connecting map δ is given by tracing the following commutative diagram from the right-top to the left-bottom:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(\mathcal{O}(-D)) & \longrightarrow & C^0(\mathcal{O}) & \longrightarrow & C^0(\oplus_{j=0}^n \mathbf{C}_{x_j}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1(\mathcal{O}(-D)) & \longrightarrow & C^1(\mathcal{O}) & \longrightarrow & C^1(\oplus_{j=0}^n \mathbf{C}_{x_j}) & \longrightarrow & 0, \end{array}$$

where $C^{\bullet}$ denotes a space of cochains. Thus the claim follows. □

THEOREM 1. *The intersection numbers for the twisted forms are*

$$\langle \omega_{pq}^{+}, \omega_{ij}^{-} \rangle = 2\pi i \left(\frac{1}{\alpha_i}(\delta_{ip} - \delta_{iq}) - \frac{1}{\alpha_j}(\delta_{jp} - \delta_{jq}) \right),$$

where δ_{ip} is the Kronecker delta. As a result, the intersection form is symmetric.

Proof. In terms of the Čech cohomology the isomorphism $\int : H^1(\Omega^1) \xrightarrow{\sim} \mathbf{C}$ is given as follows: for

$$(\zeta)_{\text{Cech}} = (\zeta_{pq}) \in \Omega^1(U_{pq}) \in \check{H}^1(\mathcal{U}, \Omega^1), \quad \zeta \in H^1_{\text{Dol}}(\Omega^1),$$

find meromorphic 1-forms η_p on U_p such that

$$\eta_q - \eta_p = \zeta_{pq} \quad \text{on} \quad U_{pq}$$

($\{\eta_p\}$ is called a Mittag-Leffler distribution for $(\zeta)_{\text{Cech}}$), then [For] implies

$$\int \zeta = 2\pi i \sum_{x \in \boldsymbol{P}^1} \operatorname{Res}_x\{\eta_p\}. \tag{3}$$

Since

$$(\overset{\vee}{\omega}_{ij})_{\text{Cech}} \in \check{H}^1(\mathcal{O}(-D)), \ \omega_{pq} \in \Gamma(\Omega^1(\log D))$$

and $U_a \cap U_b = U \ (a \neq b)$, we have

$$(\overset{\vee}{\omega}_{ij})_{\text{Cech}} \cdot \omega_{pq} \in \check{H}^1(\mathcal{U}, \Omega^1).$$

Notice that

$$H^1_{\text{Dol}}(\Omega^1) \ni \overset{\vee}{\omega}_{ij} \wedge \omega_{pq} \leftrightarrow -\overset{\vee}{\omega}_{ij} \cdot \omega_{pq} \in \check{H}^1(\mathcal{U}, \Omega^1).$$

If we define $\xi = \{\xi_l\}$ by

$$\begin{array}{ll} \xi_i := \omega_{pq}/\alpha_i & \text{a meromorphic 1-form on } U_i \\ \xi_j := -\omega_{pq}/\alpha_j & \text{a meromorphic 1-form on } U_j \\ \xi_k := 0 & \text{on } U_k \quad \text{if } k \neq i, j \end{array}$$

it forms a Mittag-Leffler distribution for $-(\omega_{ij}^{\vee})_{\mathrm{Cech}}\cdot\omega_{pq}$. Hence using the formula (3), we get

$$\langle\omega_{pq}^{+},\omega_{ij}^{-}\rangle=2\pi i\sum_{x\in\boldsymbol{P}^1}\mathrm{Res}_x\,\xi,$$

which completes the proof. □

By using forms

$$\varphi_j=\frac{dt}{t-x_j}-\frac{dt}{t-x_{j+1}}\in\Gamma(\boldsymbol{P}^1,\Omega^1(\log D)),\quad 1\leq j\leq n-1,$$

we give bases for the spaces $\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot\omega$ and $\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$ by

$$\varphi_j^{+}\in\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot\omega,\quad\varphi_j^{-}\in\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot(-\omega),\quad 1\leq j\leq n-1.$$

COROLLARY. *For the bases above, the intersection numbers are given as follows:*

$$\langle\varphi_j^{+},\varphi_j^{-}\rangle=2\pi i\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right),$$
$$\langle\varphi_j^{+},\varphi_{j+1}^{-}\rangle=\langle\varphi_{j+1}^{+},\varphi_j^{-}\rangle=-\frac{2\pi i}{\alpha_{j+1}},$$
$$\langle\varphi_j^{+},\varphi_k^{-}\rangle=0\quad \textit{if}\ |j-k|\geq 2.$$

§3. Twisted Riemann's period relations

In this section we assume $\alpha_j\notin\boldsymbol{Z}$. Let ξ_j (resp. η_j) $1\leq j\leq n-1$ be elements of $\Gamma(\Omega^1(\log D))$ such that ξ_j^{+} (resp η_j^{-}) forms a basis of $\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot\omega$ (resp. $\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$). Recall the de Rham expression:

$$H_c^1(L^{\vee})\simeq\frac{\ker\{\nabla^{\vee}:\Gamma_c(U,\mathscr{E}^1)\to\Gamma_c(U,\mathscr{E}^2)\}}{\nabla^{\vee}\Gamma_c(U,\mathscr{E}^0)};$$

the natural inclusion

$$\ker\{\nabla^{\vee}:\Gamma_c(U,\mathscr{E}^1)\to\Gamma_c(U,\mathscr{E}^2)\}\hookrightarrow\ker\{\Gamma(\mathscr{E}^{01}(-D))\oplus\Gamma(\mathscr{E}^{10})\to\Gamma(\mathscr{E}^{11})\}$$

induces the isomorphism (here the assumption $\alpha_j\notin\mathbf{N}-\{0\}$ is used)

$$(H_c^1(L^{\vee})\simeq)\ \frac{\ker\{\nabla^{\vee}:\Gamma_c(U,\mathscr{E}^1)\to\Gamma_c(U,\mathscr{E}^2)\}}{\nabla^{\vee}\Gamma_c(U,\mathscr{E}^0)}$$

$$\xrightarrow{\sim} \frac{\ker\{\Gamma(\mathscr{E}^{01}(-D)) \oplus \Gamma(\mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11})\}}{\nabla^{\vee}\Gamma(\mathscr{E}^{00}(-D))} \Big(\underset{\iota}{\overset{\tau}{\rightleftarrows}} \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega) \Big).$$

For each η_j there exist (see §2 (2)) $\mu_j \in \Gamma(\mathscr{E}^{10})$ and $h_j \in \Gamma(\mathscr{E}^{00})$ such that

$$\eta_j^{\vee} + \mu_j = \eta_j + \nabla^{\vee} h_j\,;$$

moreover by the isomorphism above there exist $f_j \in \Gamma(\mathscr{E}^{00}(-D))$ such that

$$\eta_j^{\mathrm{c}} := \eta_j^{\vee} + \mu_j + \nabla^{\vee} f_j \in \Gamma_c(U, \mathscr{E}^1),$$

which form a basis of $\Gamma_c(U, \mathscr{E}^1)$. Let

$$\gamma_j^+ \in H_1(L^{\vee}), \quad \delta_j^- \in H_1(L)$$

be bases of the twisted cycles. We use the following isomorphism called the Poincaré duality (without any condition):

$$\theta_c : H_1(U, L^{\vee}) \xrightarrow{\sim} H^1(\Gamma_c(U, \mathscr{E}^{\cdot}), \nabla^{\vee}).$$

Let us define the intersection matrices and the period matrices as follows:

$$I_h = \begin{pmatrix} \langle\gamma_1^+, \delta_1^-\rangle & \cdots & \langle\gamma_1^+, \delta_{n-1}^-\rangle \\ \vdots & & \vdots \\ \langle\gamma_{n-1}^+, \delta_1^-\rangle & \cdots & \langle\gamma_{n-1}^+, \delta_{n-1}^-\rangle \end{pmatrix}, \; I_{ch} = \begin{pmatrix} \langle\xi_1^+, \eta_1^-\rangle & \cdots & \langle\xi_1^+, \eta_{n-1}^-\rangle \\ \vdots & & \vdots \\ \langle\xi_{n-1}^+, \eta_1^-\rangle & \cdots & \langle\xi_{n-1}^+, \eta_{n-1}^-\rangle \end{pmatrix}.$$

$$P^+ = \begin{pmatrix} \int_{\gamma_1^+} \xi_1^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_1^+ \\ \vdots & & \vdots \\ \int_{\gamma_1^+} \xi_{n-1}^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_{n-1}^+ \end{pmatrix}, \; P^- = \begin{pmatrix} \int_{\delta_1^-} \eta_1^- & \cdots & \int_{\delta_{n-1}^-} \eta_1^- \\ \vdots & & \vdots \\ \int_{\delta_1^-} \eta_{n-1}^- & \cdots & \int_{\delta_{n-1}^-} \eta_{n-1}^- \end{pmatrix},$$

where the intersection for twisted cycles are defined by

$$\langle\gamma^+, \delta^-\rangle := \int_{\delta^-} \theta_c(\gamma^+), \quad \gamma^+ \in H_1(L^{\vee}), \quad \delta^- \in H_1(L).$$

Then we have the twisted Riemann's period relation:

THEOREM 2.

$$P^+\, {}^tI_h^{-1}\, {}^tP^- = I_{ch}, \quad \textit{i.e.}\ {}^tP^-\, {}^tI_{ch}^{-1}\, P^+ = {}^tI_h.$$

Proof. Let $\Theta = (\theta_{ij})$ be the matrix expression of θ_c under the bases above:

$$\theta_c(\gamma_j^+) = \sum_k \theta_{kj}\eta_k{}^c.$$

The intersection numbers for twisted cycles are computed as follows:

$$\begin{aligned}\langle \gamma_j^+, \delta_k^- \rangle &:= \int_{\delta_k^-} \theta_c(\gamma_j^+) = \int_{\delta_k^-} \sum_a \theta_{aj}\eta_a{}^c \\ &= \sum_a \theta_{aj} \int_{\delta_k^-} \eta_a + \nabla^\vee h_a + \nabla^\vee f_a = \sum_a \theta_{aj} \int_{\delta_k^-} \eta_a^-,\end{aligned}$$

that is

$$I_h = {}^t\Theta P^-.$$

The (k, j)-components θ_{kj} of Θ are computed as follows:

$$\begin{aligned}\int_{\gamma_j^+} \xi_a^+ &= \int \theta_c(\gamma_j^+) \wedge \xi_a^+ = \int \sum_k \theta_{kj}\eta_k{}^c \wedge \xi_a \\ &= \sum_k \theta_{kj} \int (\eta_k^\vee + \mu_k + \nabla^\vee f_k) \wedge \xi_a \\ &= \sum_k \theta_{kj} \int \eta_k^\vee \wedge \xi_a = \sum_k \langle \xi_a^+, \eta_k^- \rangle \theta_{kj},\end{aligned}$$

that is

$$P^+ = I_{ch}\Theta.$$

Eliminating Θ from the two equalities above, we get the relation. □

§4. Examples

EXAMPLE 1. Quadric relations for the Gauss hypergeometric functions. For

$$n = 3,\ x_0 = x_4 = \infty,\ x_1 = 0,\ x_2 = 1,\ x_3 = 1/x\ (0 < x < 1),$$
$$\alpha_1 = \alpha,\ \alpha_2 = \gamma - \alpha,\ \alpha_3 = -\beta,\ \alpha_0 = \beta - \gamma,$$

put

$$u = t^\alpha(1-t)^{\gamma-\alpha}(1-xt)^{-\beta},$$

$$\varphi_1 = \left(\frac{dt}{t-x_1} - \frac{dt}{t-x_2}\right) = \frac{dt}{t(1-t)},\ \varphi_3 = \left(\frac{dt}{t-x_3} - \frac{dt}{t-x_4}\right) = \frac{-xdt}{1-xt},$$

$\gamma_1^+, \gamma_3^+ \in H_1(U, L^\vee)$ and $\gamma_1^-, \gamma_3^- \in H_1(U, L)$, (see Figure), then we have

$$P^{+}=\begin{pmatrix}\int_0^1 u\varphi_1 & \int_{1/x}^{\infty} u\,\varphi_1\\ \int_0^1 u\varphi_3 & \int_{1/x}^{\infty} u\,\varphi_3\end{pmatrix},\ P^{-}=\begin{pmatrix}\int_0^1 u^{-1}\varphi_1 & \int_{1/x}^{\infty} u^{-1}\varphi_1\\ \int_0^1 u^{-1}\varphi_3 & \int_{1/x}^{\infty} u^{-1}\varphi_3\end{pmatrix},$$

$$I_h=-\begin{pmatrix}d_{12}/d_1d_2 & 0\\ 0 & d_{30}/d_3d_0\end{pmatrix},\ I_{ch}=2\pi i\begin{pmatrix}1/\alpha+1/(\gamma-\alpha) & 0\\ 0 & -1/\beta+1/(\beta-\gamma)\end{pmatrix}.$$

By the help of the well-known formulae

$$\int_0^1 u\,\varphi_1=B(\alpha,\gamma-\alpha)F(\alpha,\beta,\gamma\,;x),$$

$$\int_{1/x}^{\infty} u\,\varphi_1=-(-1)^{\gamma-\alpha-\beta}x^{1-\gamma}B(\beta-\gamma+1,-\beta+1)$$
$$\times F(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma\,;x),$$

the identity

$$P^{+}\,{}^tI_h^{-1}\,{}^tP^{-}=I_{ch},$$

leads quadratic identities for hypergeometric functions in [SY]: the (1,2)-component yields the formula presented in Introduction

$$F(\alpha,\beta,\gamma\,;x)F(1-\alpha,1-\beta,2-\gamma\,;x)$$
$$=F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma\,;x)F(\gamma-\alpha,\gamma-\beta,\gamma\,;x),$$

and the (1, 1)-component yields

$$F(\alpha,\beta,\gamma\,;x)F(-a,-\beta,-\gamma\,;x)-1$$
$$=\frac{\alpha\beta(\gamma-\alpha)(\gamma-\beta)}{\gamma^2(\gamma+1)(\gamma-1)}F(\beta-\gamma+1,\alpha-\gamma+1,-\gamma+2\,;x)$$
$$\times F(\gamma-\beta+1,\gamma-\alpha+1,\gamma+2\,;x).$$

EXAMPLE 2. Quadric relations for Lauricella's hypergeometric function. Lauricella's hypergeometric function F_D of m-variable is defined by

$$F_D(\alpha,\beta,\gamma\,;z)=\sum_{n_1,n_2,\ldots,n_m=0}^{\infty}\frac{(\alpha)_{n_1+\cdots+n_m}(\beta_1)_{n_1}\cdots(\beta_m)_{n_m}}{(\gamma)_{n_1+\cdots+n_m}(1)_{n_1}\cdots(1)_{n_m}}z_1^{n_1}\cdots z_m^{n_m},$$

where

$$z=(z_1,\ldots,z_m),\quad \beta=(\beta_1,\ldots,\beta_m)\,;$$

the series admits the integral representation

$$F_D(\alpha, \beta, \gamma ; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-z_1 t)^{-\beta_1}\cdots(1-z_m t)^{-\beta_m} dt.$$

Put

$$n = m+2,\ x_0 = \infty,\ x_1 = 0,\ x_2 = 1,\ x_{j+2} = 1/z_j\ (1 \le j \le m),$$

$$\alpha_0 = \alpha_{m+3} = \beta_1 + \cdots + \beta_m - \gamma,\ \alpha_1 = \alpha,\ \alpha_2 = \gamma - \alpha,\ \alpha_{j+2} = -\beta_j\ (1 \le j \le m),$$

$$u = t^{\alpha}(1-t)^{\gamma-\alpha}(1-z_1 t)^{-\beta_1}\cdots(1-z_m t)^{-\beta_m},$$

$$\xi_j = \Big(\frac{1}{t-x_1} - \frac{1}{t-x_{j+1}}\Big)dt,\ \eta_j = \Big(\frac{1}{t-x_{j+1}} - \frac{1}{t-x_0}\Big)dt\ \ (1 \le j \le m+1),$$

$$\gamma_j^+,\ H_1(U, L^{\vee}),\ \gamma_j^- \in H_1(U, L)\ (1 \le j \le m),\ \text{(see Figure)}.$$

The (1,1)-component of

$$^{t}P^{-}\, I_{ch}^{-1}\, P^{+} = {}^{t}I_h,$$

reads

$$\Big(\int_0^1 u^{-1}\eta_1, \ldots, \int_0^1 u^{-1}\eta_{m+1}\Big)\, I_{ch}^{-1}\, {}^{t}\Big(\int_0^1 u\,\xi_1, \ldots, \int_0^1 u\xi_{m+1}\Big) = I_h(1,1).$$

Since the (1,1)-component of I_h is $-(e^{2\pi i\gamma}-1)/((e^{2\pi i\alpha}-1)(e^{2\pi i(\gamma-\alpha)}-1))$, and

$$I_{ch}^{-1} = -\frac{1}{2\pi i}\begin{pmatrix} \alpha-\gamma & 0 & 0 & \cdots & 0 \\ 0 & \beta_1 z_1 & 0 & \cdots & 0 \\ 0 & 0 & \beta_2 z_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_m z_m \end{pmatrix},$$

we have the following formula:

$$F_D(\alpha, \beta, \gamma ; z)F_D(1-\alpha, -\beta, -\gamma+1 ; z) - 1$$

$$= \frac{\gamma-\alpha}{\gamma(\gamma-1)} \sum_{j=1}^{m} \beta_j z_j F_D(\alpha, \beta+e_j, \gamma+1 ; z)F_D(-\alpha+1, -\beta+e_j, -\gamma+2 ; z),$$

where

$$e_j = (\ldots, 0, \overset{j\text{-th}}{1}, 0, \ldots).$$

Remark. Once the inversion formula for the beta function is obtained as an example of the twisted Riemann's period relations, the inversion formula for the gamma function can be obtained as a special case of beta's as follows:

$$\Gamma(\alpha)\Gamma(-\alpha)=B(\alpha,-\alpha/2)B(-\alpha,\alpha/2)$$
$$=\frac{-2\pi i}{\alpha}\frac{\exp(\pi i\alpha)}{\exp(2\pi i\alpha)-1}=-\frac{1}{\alpha}\frac{\pi}{\sin\pi\alpha},$$

namely $\Gamma(a)\Gamma(1-\alpha)=\pi/\sin\pi\alpha$. Since the gamma function can be thought of a confluent beta function (see the integral representations of these functions in the beginning of Introduction), this formula suggests a confluent version of our intersection theory.

REFERENCES

[Aom1] K. Aomoto, On vanishing of cohomology attached to certain many valued meromorphic functions, J. Math. Soc. Japan, **27** (1975), 248–255.

[Aom2] ——, On the structure of intergrals of power product of linear functions, Sci. Papers College of General Ed, Univ. of Tokyo, **27** (1977), 49–61.

[Cho1] K. Cho, Intersection theory for twisted cohomologies and twisted Riemann's period relations II-On Riemann srufaces, preprint.

[Cho2] ——, Intersectin theory for twisted cohomologies and twisted Riemann's period relations III-On P^n, preprint.

[CY] K. Cho and M. Yoshida, Comparison of (co)homologies of branched covering spaces and twisted ones of basespaces I, Kyushu J. Math., **48** (1994), 111–122.

[Del1] P. Deligne, Equations différentielles à points singuliers réguliers, Lect. Notes in Math., 163, Springer, 1970.

[Del2] ——, Théorie de Hodge II, Publ. Math., Inst. Hautes Etud. Sci., **40** (1972), 5–57.

[EV1] H. Esnault and E. Viehweg, Logarithmic De Rham complexes and vanishing theorems, Invent. Math., **86** (1986), 161–194.

[EV2] ——, Lectures on Vanishing Theorems, Birkhäuser, 1992.

[ESV] H. Esnault, V. Schechtman and E. Viehweg, Cohomology of local systems on the complement of hyperplanes, Invent, Math., **109** (1992), 557–561.

[For] O. Forster, Lectures on Riemann Surfaces, GTM 81, Springer, 1977.

[GH] P. Griffiths and J. Harirs, Principles of Algebraic Geometry, John Wiley & Sons, Inc., 1978.

[IK1] K. Iwasaki and M. Kita, Exterior power structure of the twisted de Rham cohomology of the complement real Veromese arrangements, to appear in J. Math. Pures et Appl.

[IK2] ——, Twisted homology of the configuration space of n-points with application to hypergeometric functions, preprint UTMS **94–11**, (1944).

[IKSY] K. Iwasaki H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé, Vieweg, 1991.

[Kit] M. Kita, On the hypergeometric functions in several variables II–On the Wronskian of the hypergeometric functions of type $(n+1, m+1)$–, J. Math. Soci. Japan, **45** (1993), 645–669.

[KM] M. Kita and K. Matsumoto, Duality for hypergeometric furctions and inrariant Gauss-Manin systems preprint.

[KN] M. Kita and M. Noumi, On the structure of cohomology groups attached to integrals of certain many valued analytic functions, Japan. J. Math., **9** (1983), 113–157.

[KY1, 2] M. Kita and M. Yoshida, Intersection theory for twisted cycles I, II, Math. Nachrichten, **166** (1994), 287–304, **168** (1994), 171–190.

[SY] T. Sasaki and M. Yoshida, Tensor Products of Linear Differential Equations II – New formulae for the hypergeometric functions –, Funkcialaj Ekvacioj, **33** (1990), 527–549.

[Yos] M. Yoshida, Fuchsian Differential Equations, Vieweg, 1987.

K. Cho
Graduate School of Mathematics
Kyushu University
Hakozaki, Higashi-ku
Fukuoka 812, Japan

K. Matsumoto
Department of Mathematics
Faculty of Science
Hiroshima University
Kagamiyama, Higashi-Hiroshima 739, Japan