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MÖBIUS GEOMETRY FOR HYPERSURFACES IN S^4

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§0. Introduction

Our purpose in this paper is to study Möbius geometry for those hypersurfaces in S^4 which have different principal curvatures at each point. We will give a complete local Möbius invariant system for such hypersurface in S^4 which determines the hypersurface up to Möbius transformations. And we will classify the so-called Möbius homogeneous hypersurfaces in S^4.

Our main results are following. Let $x : M \to S^4$ be an immersed hypersurface with different principal curvatures λ, μ and ν at each point. A well-known Möbius invariant is the so-called Möbius curvature $W = \dfrac{\nu - \mu}{\lambda - \mu}$. Let $\{t_1, t_2, t_3\}$ be the unit principal vector fields on M corresponding to λ, μ and ν respectively. We denote by $\{\omega^1, \omega^2, \omega^3\}$ the dual basis for $\{t_1, t_2, t_3\}$. It is not difficult to show that the following 1-forms

$$\theta^1 = (\mu - \nu)\omega^1, \quad \theta^2 = (\lambda - \nu)\omega^2, \quad \theta^3 = (\lambda - \mu)\omega^3 \tag{0.1}$$

are also Möbius invariants. We can prove that

THEOREM 1. *$\{\theta^1, \theta^2, \theta^3, W\}$ forms a complete Möbius invariant system which determines the hypersurface x up to Möbius transformations.*

A hypersurface M in S^4 is said to be Möbius homogeneous if for any two point p, q in M there exists a Möbius transformation σ taking M to M and p to q. The 1-parameter-family isoparametric hypersurfaces $x_\theta : M \to S^4$ with different principal curvatures are examples of Möbius homogeneous hypersurfaces (the universal covering of M is S^3). Another example of 1-parameter-family Möbius homogeneous hypersurfaces in S^4 can be obtained by the following way. Let $T_w \subset$

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$S^3 \subset \mathbf{R}^4$ be the 1-parameter-family isoparametric tori. Let C_w be the cone in $\mathbf{R}^4$ spanned by $0 \in \mathbf{R}^4$ and T_w. Using the stereographic projection π from S^4 to $\mathbf{R}^4$ we get 1-parameter-family hypersurfaces $x_w = \pi^{-1}(C_w) : T_w \times \mathbf{R} \to S^4$. One can show that x_w are Möbius homogeneous. We can show that

THEOREM 2. *Let $x : M \to S^4$ be a Möbius homogeneous hypersurface with different principal curvatures, then up to a Möbius transformation $x(M)$ is either a part of some x_θ or a part of some x_w described above.*

In fact, we can prove a stronger theorem (cf. Theorem 4.1), from which we obtain

THEOREM 3. *Let $x : M \to S^4$ be a Dupin hypersurface with different principal curvatures. If the Möbius curvature W is constant, then up to a Möbius transformation $x(M)$ is either a part of some x_θ or a part of some x_w described above.*

This paper is organized as follows. In Section 1 we study the Möbius invariants and the relations among them. In Section 2 we define the adjoint Möbius frame in $\mathbf{R}^7$ for hypersurface in S^4, which allow us to write the structure equations. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorems 2 and 3.

§1. Möbius invariants for hypersurface in S^4

The Möbius group G_4 is the conformal transformation group of the unit sphere S^4 in $\mathbf{R}^5$, which is generated by the inversions of S^4. Since Möbius transformations act nonlinearly on S^4, it is more difficult to find local invariant in Möbius geometry than in other geometry. Fortunately we have the following classical method to linearize the Möbius group.

Let $O(5,1)$ be the orthogonal group with one negative index defined by

$$O(5,1) = \{A \in GL(\mathbf{R}^6) \mid {}^tAI_1A = I_1\}, \tag{1.1}$$

where $I_1 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \in GL(\mathbf{R}^6)$. For any $A = \begin{pmatrix} B & u \\ v & w \end{pmatrix} \in O(5,1)$ with $w \in \mathbf{R}$ we can define a mapping $\sigma(A)\,;\, S^4 \to \mathbf{R}^5$ by

$$\sigma(A)(x) = \frac{Bx + u}{vx + w}, \quad x = {}^t(x_1, x_2, \cdots, x_5) \in S^4. \tag{1.2}$$

One can easily verify that $\sigma(A) : S^4 \to S^4$ and $\sigma(A)$ is a Möbius transformation. In fact, $\sigma : O(5,1) \to G_4$ is a group isomorphism (cf. Wang [11]).

1.1 Definition. Two hypersurfaces $x : M \to S^4$ and $x' : N \to S^4$ are said to be Möbius equivalent if there is a diffeomorphism $\tau : M \to N$ and $A \in O(5,1)$ such that $x' \circ \tau = \sigma(A) \circ x$. Such (τ, A) (or simply A) is called a Möbius equivalence. Briefly, x and x' are Möbius equivalent if their images in S^4 differ only by a Möbius transformation.

In the rest of the paper we will always assume that $x : M \to S^4$ is an immersion with different principal curvatures at each point of M and that M is simply connected.

Let $x : M \to S^4$ be a hypersurface with principal curvatures λ, μ and ν. Let $n : M \to S^4$ be the unit normal for x. The mappings a, b and $c : \mathrm{M} \to \mathbf{R}^7$ defined by

$$a = \begin{pmatrix} \lambda x + n \\ \lambda \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} \mu x + n \\ \mu \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} \nu x + n \\ \nu \\ 1 \end{pmatrix} \tag{1.3}$$

are called the curvature spheres for x. Since a and b are linearly independent, we can write

$$c = Wa + (1 - W)b, \quad W = \frac{\nu - \mu}{\lambda - \mu}. \tag{1.4}$$

It is known that W is a Möbius invariant. Let $\langle\, , \rangle$ be the inner product in $\mathbf{R}^7$ defined by

$$\langle u, u \rangle = \sum_{i=1}^{5} u_i^2 - u_6^2 - u_7^2, \quad u = {}^t(u_1, u_2, \cdots, u_7) \in \mathbf{R}^7. \tag{1.5}$$

We denote by $O(5,2)$ the orthogonal group of $\mathbf{R}^7$ preserving $\langle\, , \rangle$. Thus we can identify $O(5,1)$ with the subgroup $\mathbf{O}(5,1) = \{\mathbf{A} \mid \mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, A \in O(5,1)\}$ of $O(5,2)$. The following theorems are essentially classical, so we state them here without giving proofs.

1.2 Theorem. *Let E_1, E_2 and E_3 be never zero principal vector fields of x corresponding to λ, μ and ν respectively. If x' is Möbius equivalent by (τ, A) to x, then $\tau_*(E_1)$, $\tau_*(E_2)$ and $\tau_*(E_3)$ are principal vector fields for x'.*

1.3 THEOREM. *Two hypersurfaces x and x' are equivalent by the Möbius equivalence (τ, A) if and only if we can arrange the order of the curvature spheres $\{a', b', c'\}$ of x' such that*

$$a' \circ \tau = \mathbf{A}a, \quad b' \circ \tau = \mathbf{A}b, \quad c' \circ \tau = \mathbf{A}c, \quad \mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.6}$$

It is easy to see that E_1, E_2 and E_3 in Theorem 1.2 are characterized by the properties that $E_1(a) \parallel (a-b)$, $E_2(b) \parallel (a-b)$ and $E_3(c) \parallel (a-b)$. By (1.3) we have

$$E_2(a) = E_2(\lambda)\begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} + (\lambda - \mu)\begin{pmatrix} x_*(E_2) \\ 0 \\ 0 \end{pmatrix}$$

Since $x \perp x_*(TM)$, we have $\langle E_2(a), E_2(a)\rangle = (\lambda-\mu)^2 \,|\, x_*(E_2)\,|^2 > 0$. Similarly $\langle E_3(b), E_3(b)\rangle > 0$ and $\langle E_1(c), E_1(c)\rangle > 0$.

1.4 DEFINITION. (E_1, E_2, E_3) are called the Möbius vector fields corresponding to the curvature spheres (a, b, c) of x if they are principal vector fields corresponding to (λ, μ, ν) and

$$\langle E_2(a), E_2(a)\rangle = \langle E_3(b), E_3(b)\rangle = \langle E_1(c), E_1(c)\rangle = 1. \tag{1.7}$$

It is clear that (E_1, E_2, E_3) are determined by the hypersurface x up to signs. By Theorems 1.2 and 1.3 we have immediately

1.5 PROPOSITION. *E_1, E_2 and E_3 are Möbius invariants.*

In the rest of this paper we will always assume that (E_1, E_2, E_3) are the Möbius vector fields, and for simplicity we will denote by f_i the partial derivative $E_i(f)$ for $f \in C^\infty(M)$. By (1.3) we have

$$\langle a, a\rangle = \langle a, b\rangle = \langle b, b\rangle = 0\,; \tag{1.8}$$

$$a_1 = R(a-b), \; b_2 = S(a-b), \; c_3 = T(a-b), \tag{1.9}$$

where

$$R = \frac{\lambda_1}{\lambda-\mu}, \; S = \frac{\mu_2}{\lambda-\mu}, \; T = \frac{\nu_3}{\lambda-\mu}. \tag{1.10}$$

By Theorem 1.3 and Proposition 1.5 we know that R, S, T and W defined by

(1.9) and (1.4) are Möbius invariants. Since $\{E_1, E_2, E_3\}$ is a basis for TM, we can find $C_{ij}^k \in C^\infty(M)$, $1 \leq i, j, k \leq 3$, such that

$$(1.11)\quad [E_i, E_j] = -\sum_k C_{ij}^k E_k,\ C_{ij}^k = -C_{ji}^k,\ \text{i.e.,}\ f_{ij} - f_{ji} = \sum_k C_{ij}^k f_k,\ \forall f \in C^\infty(M).$$

It is clear that all C_{ij}^k are Möbius invariants. We can define two other Möbius invariants by

$$(1.12)\qquad \Phi = \langle a_{22}, a_{22}\rangle,\ \Psi = (1-W)^2\langle c_{11}, c_{11}\rangle.$$

By (1.11) we know that C_{ij}^k are determined by the Möbius vector fields (E_1, E_2, E_3). In the rest of this section we show that the Möbius invariants R, S and T are determined by (E_1, E_2, E_3, W).

By (1.4) and (1.9) we obtain

$$(1.13)\qquad b_1 = -(1-W)^{-1}(RW + W_1)(a-b) + (1-W)^{-1}c_1;$$

$$(1.14)\qquad c_2 = (W_2 + S - SW)(a-b) + Wa_2;$$

$$(1.15)\qquad a_3 = W^{-1}(T - W_3)(a-b) - W^{-1}(1-W)b_3.$$

Thus by (1.4), (1.7) and (1.8) we have

$$(1.16)\quad \langle b_1, b_1\rangle = (1-W)^{-2},\ \langle c_2, c_2\rangle = W^2,\ \langle a_3, a_3\rangle = W^{-2}(1-W)^2.$$

It follows from (1.13), (1.14) and (1.15) that

1.6 Proposition. *For any $k, k' \in \{a, b, c\}$ we have*

$$(1.17)\qquad \langle k_i, k'\rangle = 0,\ i = 1,2,3;$$

1.7 Proposition. *We have the product table*

(1.18)

$\langle\ ,\ \rangle$	a	b	a_2	b_3	c_1
a	0	0	0	0	0
b	0	0	0	0	0
a_2	0	0	1	0	0
b_3	0	0	0	1	0
c_1	0	0	0	0	1.

Proof. We have to prove that $\langle a_2, b_3\rangle = \langle b_3, c_1\rangle = \langle c_1, a_2\rangle = 0$. By (1.11) we have $b_{32} = b_{23} + C_{32}^1 b_1 + C_{32}^2 b_2 + C_{32}^3 b_3$, thus $\langle a_2, b_3\rangle = -\langle a, b_{32}\rangle = -\langle a, b_{23}\rangle = -\langle a, S_3(a-b) + S(a_3 - b_3)\rangle = 0$ (cf. (1.9)). Similarly we have

$\langle b_3, c_1\rangle = \langle c_1, a_2\rangle = 0.$ Q.E.D.

1.8 Corollary. *For any k, $k' \in \{a, b, c\}$ we have*

$$\langle k_i, k'_j\rangle = 0,\ i \neq j,\ 1 \leq i, j \leq 3. \tag{1.19}$$

1.9 Proposition. *Let $F = C^1_{23}$, then we have*

$$\begin{array}{lll} C^1_{12} = (1-W)^{-1}W_2 + S, & C^1_{13} = (1-W)^{-1}T, & C^1_{23} = F; \\ C^2_{12} = R, & C^2_{13} = -W^{-2}F, & C^2_{23} = W^{-1}(W_3 - T); \\ C^3_{12} = (1-W)^{-2}F, & C^3_{13} = W^{-1}(1-W)^{-1}(W_1 + WR), & C^3_{23} = -W^{-1}S. \end{array} \tag{1.20}$$

Proof. By (1.11), (1.9) and (1.17)~(1.19) we have

$$(\langle a_i, a_i\rangle)_1 = 2\langle a_{i1}, a_i\rangle = 2\langle a_{1i} + C^i_{i1}a_i, a_i\rangle = 2\langle R(a_i - b_i), a_i\rangle + 2C^i_{i1}\langle a_i, a_i\rangle.$$

Since $\langle a_2, b_2\rangle = 0$ and $\langle a_3, b_3\rangle = -W^{-1}(1-W)$ (cf.(1.15)), we obtain $C^2_{12} = R$ and $C^3_{13} = W^{-1}(1-W)^{-1}(W_1 + WR)$. Similarly, by calculating $(\langle b_i, b_i\rangle)_2$ for $i = 1,3$ and $(\langle c_i, c_i\rangle)_3$ for $i = 1,2$ we obtain $C^1_{12} = (1-W)^{-1}W_2 + S$, $C^3_{23} = -W^{-1}S$, $C^1_{13} = (1-W)^{-1}T$, $C^2_{23} = W^{-1}(W_3 - T)$. Furthermore, by (1.11), (1.9) and (1.16)~(1.19) we have

$$\begin{aligned} C^3_{12} &= \langle b_{12} - b_{21}, b_3\rangle = \langle b_{12}, b_3\rangle = -\langle b_1, b_{32}\rangle = -\langle b_1, b_{23} - C^1_{23}b_1\rangle \\ &= C^1_{23}\langle b_1, b_1\rangle = (1-W)^{-2}F; \\ C^1_{23} &= \langle c_{23}, c_1\rangle = -\langle c_2, c_{13}\rangle = -\langle c_2, c_{31} + C^2_{13}c_2\rangle = -W^2C^2_{13}. \end{aligned}$$

Q.E.D.

1.10 Corollary. *The Möbius invariants R, S and T are determined by (E_1, E_2, E_3, W).*

§2. Adjoint Möbius frame in $\mathbf{R}^7$ for hypersurface in S^4

In order to write the structure equations and establish the fundamental theorem for the hypersurface $x: M \to S^4$ under the Möbius group we need an adjoint frame $\mathbf{U}: M \to GL(\mathbf{R}^7)$ along x which is invariant under the "Möbius group" $\mathbf{O}(5,1)$ in $\mathbf{R}^7$.

To construct the adjoint frame $\mathbf{U}$ we know from (1.18) that at each point of M $\{a, b, a_2, b_3, c_1\}$ is a subbasis for $\mathbf{R}^7$. Since $\langle a_{22}, a\rangle = -1$, $\langle a_{22}, b\rangle = 0$ and $\langle b_{11}, b\rangle = -(1-W)^{-2}$, we know that $\{a, b, a_2, b_3, c_1, a_{22}, b_{11}\}$ forms a Möbius invariant moving frame in $\mathbf{R}^7$ along M. In order to simplify the products among them we modify a_{22} and b_{11} to $d, e: M \to \mathbf{R}^7$,

$$d = a_{22} + A_1 a + A_2 c_1, \tag{2.1}$$

$$e = (1 - W)c_{11} + B_1 a + B_2 b + B_3 b_3, \tag{2.2}$$

where A_α, $B_\beta \in C^\infty(M)$ are to be determined. Our goal is to choose A_α, B_β in (2.1) and (2.2) such that we have the product table

(2.3)

$\langle\, ,\rangle$	a	b	a_2	b_3	c_1	d	e
a	0	0	0	0	0	-1	0
b	0	0	0	0	0	0	-1
a_2	0	0	1	0	0	0	0
b_3	0	0	0	1	0	0	0
c_1	0	0	0	0	1	0	0
d	-1	0	0	0	0	0	0
e	0	-1	0	0	0	0	0

2.1 Proposition. *We have*

$$\begin{aligned} a_{21} = {} & [R_2 - 2RS - (1 - W)^{-1}W_2R - W^{-1}(1 - W)^{-2}(T - W_3)F](a - b) \\ & + W^{-1}(1 - W)^{-1}Fb_3; \end{aligned} \tag{2.4}$$

$$\begin{aligned} a_{23} = {} & [- W^{-1}(W_2 + S - SW)_3 - 2W^{-2}(W_2 + S - SW)(T - W_3) \\ & + W^{-1}T_2 - W^{-2}(1 + W)ST](a - b) \\ & + W^{-2}(W_2 + S - SW)b_3 + W^{-1}Fc_1; \end{aligned} \tag{2.5}$$

$$\begin{aligned} b_{31} = {} & W(1 - W)^{-1}[(W^{-1}(T - W_3))_1 + W^{-2}(1 - W)^{-1}(1 + W) \\ & (T - W_3)(RW + W_1) - R_3 + (1 - W)^{-1}RT](a - b) \\ & - W^{-1}(1 - W)^{-1}Fa_2 - (1 - W)^{-2}(T - W_3)c_1; \end{aligned} \tag{2.6}$$

$$\begin{aligned} b_{32} = {} & [S_3 + 2W^{-1}(T - W_3)S + (1 - W)^{-1}(RW + W_1)F](a - b) \\ & - (1 - W)^{-1}Fc_1; \end{aligned} \tag{2.7}$$

$$\begin{aligned} c_{12} = {} & [(1 - W)S_1 + (2 - 3W)RS - W_1S + (RW + W_1)_2](a - b) \\ & + (RW + W_1)a_2 + (1 - W)^{-1}Fb_3; \end{aligned} \tag{2.8}$$

$$\begin{aligned} c_{13} = {} & [W^{-1}(1 - W)^{-1}(1 + W)(RW + W_1)T - W^{-2}(W_2 + S - SW)F \\ & + T_1 + RT](a - b) - W^{-1}Fa_2. \end{aligned} \tag{2.9}$$

Proof. The idea is to use (1.9), (1.13), (1.14) and (1.15) to reduce the order of derivatives. We calculate here only a_{21} and a_{23}. By (1.11), (1.20), (1.9) and (1.15) we have

$$
\begin{aligned}
a_{21} &= a_{12} - C_{12}^1 a_1 - C_{12}^2 a_2 - C_{12}^3 a_3 \\
&= (R(a-b))_2 - [(1-W)^{-1}W_2 + S]R(a-b) - Ra_2 - (1-W)^{-2}Fa_3 \\
&= [R_2 - 2RS - (1-W)^{-1}W_2 R - W^{-1}(1-W)^{-2}(T - W_3)F](a-b) \\
&\quad + W^{-1}(1-W)^{-1}Fb_3.
\end{aligned}
$$

By (1.14) we have

$$c_{23} = (W_2 + S - SW)_3(a-b) + (W_2 + S - SW)(a_3 - b_3) + W_3 a_2 + W a_{23}.$$

On the other hand we get from (1.11), (1.9) and (1.20) that

$$
\begin{aligned}
c_{23} &= c_{32} + C_{23}^1 c_1 + C_{23}^2 c_2 + C_{23}^3 c_3 \\
&= T_2(a-b) + T(a_2 - b_2) + Fc_1 + W^{-1}(W_3 - T)c_2 - W^{-1}ST(a-b).
\end{aligned}
$$

From these two formulas, (1.14) and (1.15) we get (2.5). Q.E.D.

2.2 Proposition. *We have*

$$\text{(2.10)} \quad \langle a_{22}, c_{11}\rangle = -(R_1 W + 2RW_1 + W_{11} + R^2 W) + 2W^{-1}(1-W)^{-2}F^2;$$

$$
\begin{aligned}
\text{(2.11)} \quad \langle a_{22}, b_{33}\rangle = &-[W^{-2}(W_2 + S - SW)]_2 + W^{-3}S(W_2 + S - SW) \\
&- 2W^{-1}(1-W)^{-1}F^2;
\end{aligned}
$$

$$
\begin{aligned}
\text{(2.12)} \quad \langle b_{33}, c_{11}\rangle = &[(1-W)^{-2}(T - W_3)]_3 - (1-W)^{-3}T(T - W_3) \\
&+ 2W^{-2}(1-W)^{-1}F^2.
\end{aligned}
$$

Proof. We calculate here only $\langle a_{22}, c_{11}\rangle$. By (2.4) we have $\langle a_2, c_{11}\rangle = -\langle a_{21}, c_1\rangle = 0$. Using (2.4), (2.8) and (2.9) we obtain

$$
\begin{aligned}
\langle a_{22}, c_{11}\rangle &= -\langle a_2, c_{112}\rangle = -\langle a_2, c_{121} + C_{12}^1 c_{11} + C_{12}^2 c_{12} + C_{12}^3 c_{13}\rangle \\
&= -(\langle a_2, c_{12}\rangle)_1 + \langle a_{21}, c_{12}\rangle - C_{12}^2\langle a_2, c_{12}\rangle - C_{12}^3\langle a_2, c_{13}\rangle \\
&= -(R_1 W + 2RW_1 + W_{11} + R^2 W) + 2W^{-1}(1-W)^{-2}F^2. \qquad \text{Q.E.D.}
\end{aligned}
$$

Now we come to determine A_α, B_β in (2.1) and (2.2). By (1.18) and (2.1) we have $\langle d, a\rangle = -1$, $\langle d, b\rangle = -\langle a_2, b_2\rangle = 0$ (cf. (1.9)) and $\langle d, a_2\rangle = 0$. By (2.7) we have $\langle d, b_3\rangle = \langle a_{22}, b_3\rangle = -\langle a_2, b_{32}\rangle = 0$. In order that $\langle d, c_1\rangle = 0$ we need

$$\text{(2.13)} \qquad A_2 = -\langle a_{22}, c_1\rangle = \langle a_2, c_{12}\rangle = RW + W_1 \quad \text{(cf. (2.8))}$$

In order that $\langle d, d\rangle = 0$ we choose

$$A_1 = \frac{1}{2}(\Phi - A_2^2) = \frac{1}{2}[\Phi - (W_1 + WR)^2], \tag{2.14}$$

where Φ is defined by (1.12). By (1.18) and (2.2) we have $\langle e, a\rangle = 0$ and $\langle e, b\rangle = -1$. By (2.4) we have $\langle e, a_2\rangle = (1-W)\langle c_{11}, a_2\rangle = -(1-W)\langle c_1, a_{21}\rangle = 0$. In order that $\langle e, b_3\rangle = 0$ we need

$$B_3 = -(1-W)\langle c_{11}, b_3\rangle = (1-W)\langle c_1, b_{31}\rangle = -(1-W)^{-1}(T - W_3). \tag{2.15}$$

By (2.2) we have $\langle e, c_1\rangle = 0$. In order that $\langle e, e\rangle = 0$ we need

$$(1-W)^2\langle c_{11}, c_{11}\rangle + B_3^2 + 2(1-W)B_2\langle c_{11}, b\rangle + 2(1-W)B_3\langle c_{11}, b_3\rangle = 0.$$

Since $\langle c_{11}, b\rangle = -(1-W)^{-1}$, $\langle c_{11}, b_3\rangle = -\langle c_1, b_{31}\rangle = 0$, we get

$$B_2 = \frac{1}{2}(\Psi - B_3^2) = \frac{1}{2}[\Psi - (1-W)^{-2}(T - W_3)^2], \tag{2.16}$$

where Ψ is defined by (1.12). Finally we choose B_1 such that $\langle d, e\rangle = 0$, that is,

$$(1-W)\langle a_{22}, c_{11}\rangle - B_1 + B_3\langle a_{22}, b_3\rangle = 0.$$

By (2.7) we have $\langle a_{22}, b_3\rangle = -\langle a_2, b_{32}\rangle = 0$, so by (2.10) we obtain

$$\begin{aligned} B_1 &= (1-W)\langle a_{22}, c_{11}\rangle \\ &= -(1-W)(R_1 W + 2RW_1 + W_{11} + R^2 W) + 2W^{-1}(1-W)^{-1}F^2. \end{aligned} \tag{2.17}$$

Thus we obtain a Möbius invariant moving frame $\{a, b, a_2, b_3, c_1, d, e\}$ in $\mathbf{R}^7$ along M with the product matrix J given by (2.3). We denote by $O^*(5,2)$ the subset of $GL(\mathbf{R}^7)$,

$$O^*(5,2) = \{A \in GL(\mathbf{R}^7) \mid {}^t A I_2 A = J\},\ I_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.18}$$

Then we have $U = (a, b, a_2, b_3, c_1, d, e) : M \to O^*(5,2)$. We call it the adjoint Möbius frame in $\mathbf{R}^7$ for x.

In the rest of this section we show that

2.3 Proposition. *The Möbius invariants Φ and Ψ are determined by* (E_1, E_2, E_3, W).

We define

$$V = \left(\frac{1}{\sqrt{2}}(a-d), \frac{1}{\sqrt{2}}(b-e), a_2, b_3, c_1, \frac{1}{\sqrt{2}}(a+d), \frac{1}{\sqrt{2}}(b+e)\right).$$

By (2.3) we know that $V: M \to O(5,2)$. We denote by $p: \mathbf{R}^7 \to \mathbf{R}$ the projection

$$p \circ {}^t(u_1, u_2, \cdots, u_7) = u_7. \tag{2.19}$$

Then by (1.3), (2.1) and (2.2) we have

$$\gamma := (p(a), p(b), p(a_2), p(b_3), p(c_1), p(d), p(e)) = (1,1,0,0,0,A_1, B_1 + B_2), \tag{2.20}$$

which is the last column of U. Thus the last column γ^* of V is given by

$$\gamma^* = \left(\frac{1}{\sqrt{2}}(1 - A_1), \frac{1}{\sqrt{2}}(1 - B_1 - B_2), 0,0,0, \frac{1}{\sqrt{2}}(1 + A_1), \frac{1}{\sqrt{2}}(1 + B_1 + B_2)\right).$$

Since $V: M \to O(5,2)$, we have $\langle {}^t\gamma^*, {}^t\gamma^* \rangle = -1$, i.e., $2A_1 + 2B_1 + 2B_2 = 1$. It follows from (2.14), (2.16) and (2.17) that

$$\begin{aligned} \Phi + \Psi = {} & (W_1 + WR)^2 + (1 - W)^{-2}(T - W_3)^2 \\ & + 2(1 - W)(R_1 W + 2RW_1 + W_{11} + R^2 W) - 4W^{-1}(1 - W)^{-1}F^2 + 1. \end{aligned} \tag{2.21}$$

On the other hand we get from (2.3) and (1.15) that

$$b_{33} = -\langle b_{33}, d\rangle a - \langle b_{33}, e\rangle b - \langle b_3, a_{23}\rangle a_2 - \langle b_3, c_{13}\rangle c_1 - W^{-1}(1 - W)d + e. \tag{2.22}$$

Since $p(b_{33}) = 0$, (2.20) and (2.22) imply

$$0 = -\langle b_{33}, d\rangle - \langle b_{33}, e\rangle - W^{-1}(1 - W)A_1 + B_1 + B_2 = 0. \tag{2.23}$$

By (2.1), (2.2) and Proposition 2.1 we have

$$\begin{aligned} \langle b_{33}, d\rangle &= \langle b_{33}, a_{22}\rangle + W^{-1}(1 - W)A_1, \\ \langle b_{33}, e\rangle &= (1 - W)\langle b_{33}, c_{11}\rangle + W^{-1}(1 - W)B_1 - B_2. \end{aligned} \tag{2.24}$$

Thus Proposition 2.2 and (2.23) imply

$$\begin{aligned} & -W^{-1}(1 - W)\Phi + \Psi \\ & = (1 - W)[(1 - W)^{-2}(T - W_3)]_3 - [W^{-2}(W_2 + S - SW)]_2 \\ & \quad + 4W^{-2}(1 - W)^{-1}(1 - 2W)F^2 + W_3(1 - W)^{-2}(W_3 - T) \\ & \quad + W^{-1}(1 - W)[(2W - 1)(R_1 + 2RW_1 + W_{11} + R^2 W) - (W_1 + WR)^2] \\ & \quad + W^{-3}S(W_2 + S - SW). \end{aligned} \tag{2.25}$$

Proposition 2.3 follows from (2.21), (2.25) and Corollary 1.10.

§3. Fundamental theorem for hypersurfaces in S^4

In this section we will show that (E_1, E_2, E_3, W) is a complete Möbius invariant system for $x : M \to S^4$.

Let $\mathbf{U} = (a, b, a_2, b_3, c_1, d, e) : M \to O^*(5,2)$ be the adjoint Möbius frame in $\mathbf{R}^7$ for x. We have the mappings $X, Y, Z : M \to o^*(5,2)$ (= Lie algebra of $O^*(5,2)$) defined by

$$E_1(\mathbf{U}) = \mathbf{U}X,\ E_2(\mathbf{U}) = \mathbf{U}Y,\ E_3(\mathbf{U}) = \mathbf{U}Z. \tag{3.1}$$

3.1 PROPOSITION. *All elements of the* 7×7 *matrices* X, Y *and* Z *are Möbius invariants determined by* (E_1, E_2, E_3, W).

Proof. By (2.3) we have for any mapping $u : M \to \mathbf{R}^7$ the formula

$$\begin{aligned} u = & -\langle u, d\rangle a - \langle u, e\rangle b + \langle u, a_2\rangle a_2 + \langle u, b_3\rangle b_3 + \langle u, c_1\rangle c_1 \\ & - \langle u, a\rangle d - \langle u, b\rangle e. \end{aligned} \tag{3.2}$$

We denote by $\mathfrak{R}(M)$ the set of all mappings $u : M \to \mathbf{R}^7$ such that all coefficients of u in (3.2) with respect to $\{a, b, a_2, b_3, c_1, d, e\}$ are Möbius invariants determined by (E_1, E_2, E_3, W). To prove Propposition 3.1 it suffices to show that the partial derivatives of $a, b, a_2, b_3, c_1, d, e$ in the directions of E_1, E_2, E_3 are elements in $\mathfrak{R}(M)$. We prove this fact in several steps.

Step I. $E_1(a), E_2(a), E_3(a), E_1(b), E_2(b), E_3(b) \in \mathfrak{R}(M)$.

It follows immediately from (1.9), (1.13) and (1.15).

Step II. $E_1(a_2), E_3(a_2), E_1(b_3), E_2(b_3), E_2(c_1), E_3(c_1) \in \mathfrak{R}(M)$.

It follows immediately from Proposition 2.1.

Step III. $E_2(a_2), E_3(b_3), E_1(c_1) \in \mathfrak{R}(M)$.

It follows immediately from (2.1), (2.2), (2.22), (2.24) and Proposition 2.2.

Step IV. $E_1(d), E_3(d), E_2(e), E_3(e) \in \mathfrak{R}(M)$.

By (2.1), (2.2), Steps I, II and III it suffices to show that $a_{221}, a_{223}, c_{112}\, c_{113} \in \mathfrak{R}(M)$. Since A_1, A_2, B_1, B_2, B_3 defined by (2.13)-(2.17) and their partial derivatives in the directions of E_1, E_2, E_3 are Möbius invariants determined by (E_1, E_2, E_3, W), by (1.11) we need only to show that $a_{121}, a_{232}, c_{121}\, c_{131} \in \mathfrak{R}(M)$, which follows from Proposition 2.1, Steps I, II and III.

Step V. $E_2(d),\ E_1(e) \in \mathfrak{R}(M)$.

By (3.2) and (2.3) we have

$$E_2(d) = \langle d, e_2\rangle b - \langle d, a_{22}\rangle a_2 - \langle d, b_{32}\rangle b_3 - \langle d, c_{12}\rangle c_1 + \langle d, a_2\rangle d + \langle d, b_2\rangle e.$$

Since $e_2, a_{22}, b_{32}, c_{12}, a_2, b_2 \in \mathfrak{R}(M)$, we obtain $E_2(d) \in \mathfrak{R}(M)$. Similarly we can prove that $E_1(e) \in \mathfrak{R}(M)$. Q.E.D.

Now we can prove the following fundamental theorems for hypersurfaces in S^4:

3.2 THEOREM. *(E_1, E_2, E_3, W) forms a complete Möbius invariant system for hypersurfaces in S^4 with different principal curvatures, which determines the hypersurface up to Möbius transformations.*

Proof. We have to show that if (E_1', E_2', E_3', W') is the Möbius invariant system for another hypersurface $x': N \to S^4$ and there exists a diffemorphism $\tau: M \to N$ such that

$$E_i' = \varepsilon_i \tau_* E_i,\ \varepsilon_i = \pm 1,\ i = 1,2,3\ ;\ W = W' \circ \tau, \tag{3.3}$$

then there is $A \in O(5,1)$ such that (τ, A) is a Möbius equivalence for x and x'.

By taking $(\varepsilon_1 E_1, \varepsilon_2 E_2, \varepsilon_3 E_3)$ as the Möbius vector fields we may assume that in (3.3) $\varepsilon_i = 1$. Take a point $q \in M$ we have $\mathbf{U}_o := \mathbf{U}(q) \in O^*(5,2)$ and $\mathbf{U}_o' := \mathbf{U}' \circ \tau(q) \in O^*(5,2)$. We define $\mathbf{A} = \mathbf{U}_o' \circ \mathbf{U}_o^{-1}$. By (2.18) we know that $\mathbf{A} \in O(5,2)$. By (2.20) and (3.3) we have $\gamma_o := \gamma(q) = \gamma' \circ \tau(q)$. Since $\mathbf{A}\,\mathbf{U}_o = \mathbf{U}_o'$ and $\gamma(q)$ (resp. $\gamma' \circ \tau(q)$) is the last column of $\mathbf{U}_o$ (resp. $\mathbf{U}_o'$), if we write $\mathbf{A} = \begin{pmatrix} A & u \\ v & w \end{pmatrix}$ for $w \in \mathbf{R}$ and $\mathbf{U}_o = \begin{pmatrix} B \\ \gamma_o \end{pmatrix}$, then we have $\gamma_o = vB + w\gamma_o$, i.e., $(v, w-1)\mathbf{U}_o = 0$. Since $\det(\mathbf{U}_o) \neq 0$, we get $v = 0$ and $w = 1$. Thus $\mathbf{A} \in O(5,2)$ implies $u = 0$, i.e., $\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. By (3.3) and Proposition 3.1 we know that $X = X' \circ \tau$, $Y = Y' \circ \tau$ and $Z = Z' \circ \tau$. Thus both $\mathbf{A}\,\mathbf{U}$ and $\mathbf{U}' \circ \tau$ are solutions for the linear PDE system (3.1) with the same initial value $\mathbf{A}\,\mathbf{U}_o = \mathbf{U}_o'$. By the uniqueness theorem we obtain $\mathbf{A}\,\mathbf{U} \equiv \mathbf{U}' \circ \tau$ on M. In particular, $\mathbf{A}\,a = a' \circ \tau$ and $\mathbf{A}\,b = b' \circ \tau$, and by (1.4) and (3.3) $\mathbf{A}\,c = c' \circ \tau$. Thus Theorem 1.2 implies that (τ, A) is a Möbius equivalence for x and x'. Q.E.D.

3.3 *Remark.* Let $\{t_1, t_2, t_3\}$ be the unit principal vector fields for x corresponding to λ, μ and ν respectively. Let $\{\omega^1, \omega^2, \omega^3\}$ be its dual basis. Then we have

$$E_1 = \frac{t_1}{\lambda - \nu}, \; E_2 = \frac{t_2}{\mu - \lambda}, \; E_3 = \frac{t_3}{\nu - \mu}, \; W = \frac{\nu - \mu}{\lambda - \mu}. \tag{3.4}$$

The dual basis for $\{E_1, E_2, E_3\}$ is $\{(\lambda - \nu)\omega^1, (\mu - \lambda)\omega^2, (\nu - \mu)\omega^3\}$. Thus the Möbius invariant system $(\theta^1, \theta^2, \theta^3, W)$ in Theorem 1 is equivalent to (E_1, E_2, E_3, W).

§4. Möbius homogeneous hypersurfaces in S^4

Our goal in this section is to prove the following theorem:

4.1 THEOREM. *Let $x : M \to S^4$ be a hypersurface with constant Möbius invariants R, S, T defined by* (1.10) *and constant Möbius curvature W, then up to a Möbius transformation x is either a part of some x_θ or a part of some x_w described in* §0.

Since for any Möbius homogeneous hypersurface the Möbius invariants R, S, T and W are constant, we have Theorem 2 as a consequence of Theorem 4.1. As for Dupin hypersurfaces we have $R = S = T = 0$ (cf. (1.9) and Pinkall [10]), we get also Theorem 3.

The proof of Theorem 4.1 bases on the relations among the Möbius invariants. Let $x : M \to S^4$ be hypersurface with constant R, S, T and W. By (1.11) and (1.20) we have

$$\begin{aligned}
[E_1, E_2] &= -SE_1 - RE_2 - (1 - W)^{-2}FE_3; \\
[E_1, E_3] &= -(1 - W)^{-1}TE_1 + W^{-2}FE_2 - (1 - W)^{-1}RE_3; \\
[E_2, E_3] &= -FE_1 + W^{-1}TE_2 + W^{-1}SE_3.
\end{aligned} \tag{4.1}$$

The Jacobi identity $[[E_1, E_2], E_3] + [[E_2, E_3], E_1] + [[E_3, E_1], E_2] = 0$ implies that

$$F_1 = -(2 - W)(1 - W)^{-1}(W^{-1}ST + RF); \tag{4.2}$$

$$F_2 = (1 + W)(W(1 - W)^{-1}RT + W^{-1}SF); \tag{4.3}$$

$$F_3 = -(2W - 1)W^{-1}[(1 - W)RS - (1 - W)^{-1}TF]. \tag{4.4}$$

4.2 PROPOSITION. *The Möbius invariants F, Φ and Ψ are also constant.*

Proof. By (4.3) and (4.4) we have $F_{23} = -(1+W)W^{-1}SF_3$ and $F_{32} = (1-2W)W^{-1}(1-W)^{-1}TF_2$. Using (4.1)~(4.4) we get

$$\begin{aligned} 0 &= F_{23} - F_{32} + FF_1 - W^{-1}TF_2 - W^{-1}SF_3 \\ &= -(2-W)(1-W)^{-1}RF^2 + \text{lower terms in } F. \end{aligned}$$

Similarly we have the quadratic equations of F with constant coefficients:

$$(1+W)SF^2 + \text{lower terms in } F = 0\,;\ (1-2W)TF^2 + \text{lower terms in } F = 0.$$

If one of $\{(2-W)R,\ (1+W)S,\ (1-2W)T\}$ is nonzero, we get $F =$ constant. But if all of them are zero, we get from (4.2)~(4.4) that $F_1 = F_2 = F_3 = 0$. Thus F is constant. It follows from (2.21) and (2.25) that Φ, Ψ are constant. Q.E.D.

4.3 COROLLARY. *It follows from* (2.21), (2.25), (2.1) *and* (2.2) *that*

(4.5) $\Phi = W + WR^2 + W(1-W)^{-2}T^2 - W^{-2}(1-W)S^2 - 4W^{-1}F^2$;

(4.6) $\Psi = 1 - W + W(1-W)R^2 + (1-W)^{-1}T^2 + W^{-2}(1-W)S^2 - 4(1-W)^{-1}F^2$;

(4.7) $d = a_{22} + \frac{1}{2}(\Phi - W^2R^2)a + RWc_1$;

(4.8) $e = (1-W)c_{11} + [2W^{-1}(1-W)^{-1}F^2 - R^2W(1-W)]a + \frac{1}{2}[\Psi - (1-W)^{-2}T^2]b - (1-W)^{-1}Tb_3.$

Since three curvature spheres a, b and c are colinear in $\mathbf{R}^7$, we can arrange the order such that c lies between a and b. Thus by (1.4) we have $0 < W < 1$. Since W is assumed to be constant, we can arrange a, b such that $0 < W \leq \frac{1}{2}$. Moreover, by changing E_i to $-E_i$ if necessary we may assume that $R \geq 0$, $S \geq 0$ and $T \geq 0$ (cf. (1.9)).

4.4 PROPOSITION. *We have only the following* 6 *possibilities*:

(I) $W = \frac{1}{2}$, $T = F = 0$; (II) $W = \frac{1}{2}$, $R = 2S$, $T = -F \neq 0$; (III) $0 < W < \frac{1}{2}$, $R = S = F = 0$; (IV) $0 < W < \frac{1}{2}$, $R = T = F = 0$; (V) $0 < W < \frac{1}{2}$, $S = T = F = 0$; (VI) $R = S = T = 0$.

Proof. Since F is constant, we get from (4.2), (4.3) and (4.4) that

(4.9) $ST = -WRF\,;\ RT = -W^{-2}(1-W)SF\,;\ (2W-1)[RS-(1-W)^{-2}TF] = 0.$

It follows that

$$\begin{aligned}&(2W-1)F[WR^2 + W^{-2}(1-W)S^2 + 2(1-W)^{-2}T^2]\\&= (2W-1)(-RST - RST + 2RST) = 0.\end{aligned}$$

Thus either (i) $W = \frac{1}{2}$; or (ii) $W \neq \frac{1}{2}$ and $F = 0$; or (iii) $R = S = T = 0$. From (i) and (4.9) we get the cases (I) and (II). From (ii) and (4.9) we get the cases (III), (IV) and (V). (VI) follows from (iii). Q.E.D.

4.5 Proposition. (i) $R = S = T = 0$ *implies* $T = 0$; (ii) $R = T = F = 0$ *implies* $S = 0$; (iii) $S = T = F = 0$ *implies* $R = 0$.

Proof. We assume that $R = S = F = 0$. By (4.1), (4.7) and (2.5) we have

$$d = a_{22} + \frac{1}{2}\Phi a\,;\ [E_2, E_3] = W^{-1}TE_2\,;\ a_{23} = 0.$$

Using (2.3) and (1.15) we have

$$\begin{aligned}0 = \langle d_3, d\rangle &= \langle a_{223} + \frac{1}{2}\Phi a_3, d\rangle = \langle a_{223}, d\rangle - \frac{1}{2}W^{-1}\Phi T\\&= \langle a_{232} - W^{-1}Ta_{22}, d\rangle - \frac{1}{2}W^{-1}\Phi T = -W^{-1}\Phi T.\end{aligned}$$

By (4.5) we have $\Phi = W + W(1-W)^{-2}T^2 > 0$, thus $T = 0$. Similarly, if $R = T = F = 0$, we calculate $0 = \langle e_2, e\rangle$ and get $S = 0$; if $S = T = F = 0$, we calculate $0 = \langle d_1, d\rangle$ and get $R = 0$. Q.E.D.

4.6 Proposition. *If* $W = \frac{1}{2}$, $T = F = 0$, *then* $R = S = 0$.

Proof. We assume that $W = \frac{1}{2}$, $T = F = 0$. By (4.7), (4.8), (4.1) and (2.4) we have

$$\begin{aligned}&d = a_{22} + \frac{1}{2}\left(\Phi - \frac{1}{4}R^2\right)a + \frac{1}{2}Rc_1\,; && e = \frac{1}{2}c_{11} - \frac{1}{4}R^2a + \frac{1}{2}\Psi b\,;\\&[E_1, E_2] = -SE_1 - RE_2\,; && a_{21} = -2RS(a-b).\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}0 = \langle d_1, d\rangle &= \left\langle a_{221} + \frac{1}{2}\left(\Phi - \frac{1}{4}R^2\right)R(a-b) + \frac{1}{2}Rc_{11}, d\right\rangle \\ &= \langle a_{221}, d\rangle - \frac{1}{2}R\left(\Phi - \frac{1}{4}R^2\right) + R\left\langle e + \frac{1}{4}R^2 a - \frac{1}{2}\Psi b, d\right\rangle \\ &= \langle a_{212} - Sa_{21} - Ra_{22}, d\rangle - \frac{1}{2}R\left(\Phi - \frac{1}{4}R^2\right) - \frac{1}{4}R^3 \\ &= -\langle 2RS(a_2 - b_2), d\rangle - S\langle a_{21}, d\rangle - R\left(\Phi - \frac{1}{4}R^2\right) - \frac{1}{4}R^3 \\ &= -R(4S^2 + \Phi),\end{aligned}$$

where the relation $\langle b_2, d\rangle = -S$ follows from (1.9) and (2.3). From (4.5) we get $\Phi = \frac{1}{2} + \frac{1}{4}R^2 - 2S^2$, thus $R = 0$. Since $R = T = F = 0$, we get from Proposition 4.5 that $S = 0$. Q.E.D.

4.7 PROPOSITION. *The case that* $W = \frac{1}{2}$, $R = 2S$ *and* $T = -F \neq 0$ *is impossible.*

Proof. We assume that $W = \frac{1}{2}$, $R = 2S$ and $T = -F \neq 0$. By (4.7), (4.8) and (4.1) we have

$$\begin{aligned}(4.10)\quad & d = a_{22} + \frac{1}{2}(\Phi - S^2)a + Sc_1\,;\ e = \frac{1}{2}c_{11} + (8T^2 - S^2)a + \frac{1}{2}(\Psi - 4T^2)b - 2Tb_3\,; \\ & [E_1, E_3] = -2TE_1 - 4TE_2 - 4SE_3\,;\ [E_2, E_3] = TE_1 + 2TE_2 + 2SE_3.\end{aligned}$$

By Proposition 2.1, (1.13) and (1.15) we have

$$(4.11)\quad \begin{aligned} & a_{21} = (-4S^2 + 8T^2)(a-b) - 4Tb_3\,; && a_{23} = -10ST(a-b) + 2Sb_3 - 2Tc_1\,; \\ & b_{31} = 16ST(a-b) + 4Ta_2 - 4Tc_1\,; && b_{32} = 2ST(a-b) + 2Tc_1\,; \\ & c_{12} = S^2(a-b) + Sa_2 - 2Tb_3\,; && c_{13} = 10ST(a-b) + 2Ta_2\,; \\ & a_3 = 2T(a-b) - b_3\,; && b_1 = -2S(a-b) + 2c_1. \end{aligned}$$

One can easily verify that $0 = \langle d_3, d\rangle = T(-44S^2 + 8T^2 - 2\Phi)$. By (4.5) we have $\Phi = \frac{1}{2} - 6T^2$, thus $T \neq 0$ implies

$$(4.12)\qquad 44S^2 - 20T^2 + 1 = 0.$$

On the other hand we have by (4.10) that

$$a_{231} - a_{213} = -2Ta_{21} - 4Ta_{22} - 4Sa_{23}. \tag{4.13}$$

We can easily get from (4.10) and (4.11) that

$$\langle a_{231}, e\rangle = -8S^2T - 2T\Psi + 8T^3;$$
$$\langle -2Ta_{21} - 4Ta_{22} - 4Sa_{23}, e\rangle = 48S^2T - 16T^3.$$

Using (4.10), (4.11), (2.12) and (2.24) we obtain

$$\langle a_{231}, e\rangle = (-4S^2 + 8T^2)\langle a_3, e\rangle - 4T\langle b_{33}, e\rangle = -4S^2T - 40T^3 + 2T\Psi.$$

Thus we get from (4.13) that $52S^2T - 64T^3 + 4T\Psi = 0$. Since $T \neq 0$ and $\Psi = \frac{1}{2} + 3S^2 - 6T^2$ (cf. (4.6)) we obtain

$$32S^2 - 44T^2 + 1 = 0, \tag{4.14}$$

which contradicts to (4.12). Q.E.D.

It follows from Propositions 4.4, 4.5, 4.6 and 4.7 that

4.8 Proposition. $R = S = T = 0$.

4.9 Proposition. $F = 0$ or $F = \pm \dfrac{W(1-W)}{2\sqrt{1 - W + W^2}}$.

Proof. Since $R = S = T = 0$, we get from Propositions 2.1, (4.5), (4.6), (4.7) and (4.8) that

$$[E_1, E_3] = W^{-2}FE_2;\ a_{21} = W^{-1}(1-W)^{-1}Fb_3;\ a_{23} = W^{-1}Fc_1;$$
$$\Phi = W - 4W^{-1}F^2;\ \Psi = 1 - W - 4(1-W)^{-1}F^2;$$
$$\langle a_{22}, e\rangle = 0;\ \langle c_{11}, e\rangle = \frac{1}{2} - 2(1-W)^{-2}F^2.$$

By (2.12), (2.16) and (2.17) we have

$$\langle b_{33}, e\rangle = -\frac{1}{2}(1-W) + 2(1-W)^{-1}F^2 + 4W^{-2}F^2.$$

Since $a_{213} - a_{231} = -W^{-2}Fa_{22}$, we get

$$W^{-1}(1-W)^{-1}F\langle b_{33}, e\rangle - W^{-1}F\langle c_{11}, e\rangle = -W^{-2}F\langle a_{22}, e\rangle,$$

which implies that $F=0$ or $F=\pm\dfrac{W(1-W)}{2\sqrt{1-W+W^2}}$. Q.E.D.

To summary we have

4.10 COROLLARY. *Let $x: M \to S^4$ be a hypersurface with Möbius invariant system (E_1, E_2, E_3, W). If the Möbius invariants R, S, T and W are constant, then $R=S=T=0$. Moreover, either (i) $[E_i, E_j]=0$, $1 \le i, j \le 3$; or (ii) $[E_1, E_2] = -(1-W)^{-2}FE_3$; $[E_1, E_3]=W^{-2}FE_2$; $[E_2, E_3]=-FE_1$; where $F=\pm\dfrac{W(1-W)}{2\sqrt{1-W+W^2}}$.*

In order to prove Theorem 4.1 we need the following lemma, which is a direct consequence of Theorem 2.34 in Warner [12], p. 77:

4.11 LEMMA. *Let M and N be two simply connected 3-manifolds. Let (E_1, E_2, E_3) (resp. (E_1', E_2', E_3')) be a basis for TM (resp. TN). If $[E_i, E_j] = -\sum_k C_{ij}^k E_k$ and $[E'_i, E'_j] = -\sum_k C_{ij}^k E'_k$ with the same constant cofficients C_{ij}^k, then there exists a diffeomorphism $\tau ; M \to \tau(M) \subset N$ such that $\tau_*(E_i) = E_i'$, $i=1,2,3$.*

To complete the proof of Theorem 4.1 we look for examples of hypersurfaces in S^4 whose Möbius invariants (E_1', E_2', E_3', W) satisfy (i) or (ii) in Corollary 4.10. Then Lemma 4.11 and Theorem 3.2 will imply that x is Möbius equivalent to one of those examples.

4.12 EXAMPLE. Let $x_w: \mathbf{R}^3 \to S^4$ be the 1-parameter-family hypersurfaces given by

$$x_w(\phi, \Psi, \theta) = \frac{1}{\operatorname{ch}\theta}\,{}^t(\sqrt{1-W}\cos\phi, \sqrt{1-W}\sin\phi, \sqrt{W}\cos\psi, \sqrt{W}\sin\psi, \operatorname{sh}\theta),\ 0 < W \le \frac{1}{2}. \tag{4.15}$$

It is the orbit of the subgroup G of $O(5,1)$ through the point $p = {}^t(\sqrt{1-W}, 0, \sqrt{W}, 0,0) \in S^4$ by the action (1.2), where

$$
(4.16) \qquad G = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi & 0 & 0 \\ 0 & 0 & \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{ch}\,\theta & \mathrm{sh}\,\theta \\ 0 & 0 & 0 & 0 & \mathrm{sh}\,\theta & \mathrm{ch}\,\theta \end{pmatrix}.
$$

Thus x_w are Möbius homogeneous. Using the stereographic projection π from S^4 to $\mathbf{R}^4$ which takes ${}^t(0,0,0,0,1)$ to ${}^t(0,0,0,0)$ we get the hypersurfaces $x_w' = \pi \circ x_w$: $\mathbf{R}^3 \to \mathbf{R}^4$,

$$
(4.17) \quad x_w' = e^{-\theta}\,{}^t(\sqrt{1-W}\cos\phi,\ \sqrt{1-W}\sin\phi,\ \sqrt{W}\cos\psi,\ \sqrt{W}\sin\psi).
$$

They are cones spanned by the isoparametric tori $T_w \subset S^3 \subset \mathbf{R}^4$ and $0 \in \mathbf{R}^4$. One can easily verify that the Möbius invariant system (E_1', E_2', E_3', W') for x_w is given by

$$
(4.18) \quad E_1' = \frac{1}{\sqrt{1-W}}\cdot\frac{\partial}{\partial\psi},\ E_2' = \sqrt{W}\cdot\frac{\partial}{\partial\phi},\ E_3' = \sqrt{\frac{1-W}{W}}\cdot\frac{\partial}{\partial\theta},\ W' = W.
$$

Thus (E_1', E_2', E_3', W) satisfies (i) in Colollary 4.10.

4.13 Example. Let $x_\theta : N \to S^4$ be the 1-parameter-family isoparametric hypersurfaces with three principal curvatures $\lambda = \mathrm{ctg}\,\theta$, $\mu = \mathrm{ctg}\left(\theta + \frac{2}{3}\pi\right)$ and $\nu = \mathrm{ctg}\left(\theta + \frac{1}{3}\pi\right)$ (cf. Cartan [3], Münzner [8]). Cartan pointed out in [3] that x_θ are the orbits of some orthogonal subgroup G of $O(5)$. Since $O(5)$ is naturally a subgroup of the Möbius group on S^4, we have G as a subgroup of Möbius group acting transitively on $x_\theta(N)$. Thus x_θ are Möbius homogeneous. Let W be constant with $0 < W \leq \frac{1}{2}$, we put $\theta = \mathrm{arctg}\frac{\sqrt{3}W}{2-W}$. Let (E_1', E_2', E_3', W') be the Möbius invariant system for x_θ. One can easily verify that $W' = \frac{\nu-\mu}{\lambda-\mu} = W$. Since λ, μ and ν are constant, we know from (1.9) that $R' = S' = T' = 0$. Thus by (4.1) we have

$$
(4.19)\ [E_1', E_2'] = -(1-W)^{-2}F'E_3';\ [E_1', E_3'] = W^{-2}F'E_2';\ [E_2', E_3'] = -FE_1'.
$$

By Proposition 4.9 we know that either $F' = 0$ or $F' = \pm\dfrac{W(1-W)}{2\sqrt{1-W+W^2}}$. If $F' = 0$, the Riemannian metric g on N such that $g(E_i', E_j') = \delta_{ij}$ is flat (cf. (4.19). Thus there is a covering $\pi : \mathbf{R}^3 \to N$, which is impossible because the universal covering of N is S^3. Therefore $F' = \pm \dfrac{W(1-W)}{2\sqrt{1-W+W^2}}$. Here the sign $\pm$ is not essential. E' will change sign if we change E_1' to $-E_1'$. Thus (E_1', E_2', E_3', W) satisfies (ii) in Corollary 4.10. In order that we can use Lemma 4.11 we consider the universal covering $\pi : S^3 \to N$ and the immersion $x_\theta \circ \pi : S^3 \to S^4$ with $x_\theta \circ \pi(S^3) = x_\theta(N)$.

Thus Theorem 4.1 follows from Examples 12, 13, Lemma 4.11 and Theorem 3.2.

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