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ON THE ANALYTIC STRUCTURE OF CERTAIN INFINITE DIMENSIONAL TEICMÜLLER SPACES

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Introduction

It is well known since long time that quasiconformally different finite Riemann surfaces give rise to biholomorphically nonequivalent Teichmüller spaces except for a few obvious cases (cf. [R], [E-K]). This is deduced as an application of Royden's theorem asserting that the Teichmüller metric is equal to the Kobayashi metric. For the case of infinite Riemann surfaces, however, it is still unknown whether or not the corresponding result holds, although it has been shown by F. Gardiner [G] that Royden's theorem is also valid for the infinite dimensional Teichmüller spaces. On the other hand, recent activity of several mathematicians shows that the infinite dimensional Teichmüller spaces are interesting objects of complex analytic geometry (cf. [Kru], [T], [N], [E-K-K]). Therefore, based on the generalized form of Royden's theorem, one might well look for further insight into Teichmüller spaces by studying the above mentioned nonequivalence question.

In the present article, we shall restrict our attention to the connected sum of infinitely many finite Riemann surfaces and show that, if their *necks* are sufficiently long at infinity, one can naturally associate to each quasiconformal equivalence class of such surfaces the cofinal equivalence class of a sequence of nonnegative integers which distinguishes the infinitesimal forms of the Teichmüller metric (cf. Proposition 3.3). For that we shall show in section one, that the space of integrable holomorphic quadratic differentials decomposes asymptotically as the surface is pinched along a simple closed geodesic to two hyperbolic surfaces (cf. Theorem 1.5). The proof of this fact relies on the method of solving the $\bar{\partial}$-equation with L^2 estimates as developed by [A-V], [O-1,2] and [D] on complete Kähler manifolds. This, combined with an elementary argument on the quasi-isometric equivalence of normed vector spaces, allows us to distinguish the infinitesimal forms of the Teichmüller metric by the *direct summands at infinity* of the space of integable

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quadratic differentials. As a consequence we shall conclude that there exist uncountably many nonequivalent Teichmüller spaces.

As a related result we must mention a very recent work of C. Earle and F. Gardiner [E-G] that provides the distinction of Teichmüller infinitesimal forms in the case of topologically finite Riemann surfaces.

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§1. Asymptotic decomposition theorem

Let R be a (connected) Riemann surface. In what follows we assume that R is hyperbolic, i.e. it admits the unit disc $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$ as the universal covering space. We shall denote by π_R a covering map $\Delta \to R$. By $d\sigma_R^2$ we denote a Hermitian metric on R uniquely determined by the equation

$$\pi_R^* d\sigma_R^2 = \frac{dzd\bar{z}}{(1-|z|^2)^2}.$$

We shall denote by $d\mu_R$ the volume form of R with respect to $d\sigma_R^2$. Given any section ω of the multi-canonical bundle $K_R^{\otimes m}(m \in \mathbf{Z})$, $|\omega|_R$ will denote the (pointwise) length of ω with respect to the fiber metric of $K_R^{\otimes m}$ induced from $d\sigma_R^2$. For any continuous function $\phi : R \to \mathbf{R}$ and for any $p \geqq 1$, $L_m^p(R, \phi)$ will stand for the Banach space of measurable sections u of K_R^m satisfying

$$\int_R e^{-\phi} |u|_R^p \, d\mu_R < \infty.$$

We put

$$\|u\|_{p,\phi} = \left(\int_R e^{-\phi} |u|_R^p \, d\mu_R\right)^{1/p}$$

for any $u \in L_m^p(R, \phi)$. We shall not refer to ϕ if $\phi = 0$. In the later argument we shall use those ϕ for which $\pi^*\phi(z) - \log(1-|z|^2)$ is a subharmonic function on Δ. By the symbol $\bar{\partial}$ we mean the complex exterior derivative of type (0,1) acting on $L_m^p(R, \phi)$ in the sense of distribution. Our main concern is to know how geometry of R is embodied in the Banach space structure of $A_m^p(R) := L_m^p(R) \cap \operatorname{Ker} \bar{\partial}$. Applications to Teichmüller spaces in mind, we shall only study $A_2^1(R)$ below, although we shall need certain information on $A_2^2(R, \phi)$ by a technical reason.

First let us consider the annuli

$$\Delta(r_1, r_2) := \{z \in \mathbf{C} \mid r_1 < |z| < r_2\} \quad (0 \leqq r_1 < r_2 \leqq \infty).$$

For any $\omega \in A_2^1(\Delta(r_1, r_2))$ we consider its Laurent expansion

$$\omega(z) = (\sum_{k \in Z} c_k z^k) dz^{\otimes 2}$$

and put

$$\omega_+(z) = (\sum_{k>-2} c_k z^k) dz^{\otimes 2}$$
$$\omega_-(z) = (\sum_{k<-2} c_k z^k) dz^{\otimes 2}$$
$$\omega_0(z) = c_{-2} z^{-2} dz^{\otimes 2}$$

Without loosing generality we may assume that $r_1 = r_2^{-1}$, since

$$\Delta(r_1, r_2) \simeq \Delta(\sqrt{r_1/r_2}, \sqrt{r_2/r_1}).$$

LEMMA 1.1. *There exists a constant* C *such that, for any* $\varepsilon \in (0,1)$ *and for any* $r \in \left(0, \frac{1}{2}\right)$ *the following inequalities hold for all* $\omega \in A_2^1(\Delta(r, r^{-1}))$.

$$\int_{\Delta(0, r^{\varepsilon-1})} |\omega_+| \leq Cr^{\varepsilon} \|\omega\|_1 \tag{1.1}$$

$$\int_{\Delta(r^{1-\varepsilon}, \infty)} |\omega_-| \leq Cr^{\varepsilon} \|\omega\|_1 \tag{1.2}$$

$$\int_{\Delta(r, r^{1-\varepsilon}) \cup \Delta(r^{-1+\varepsilon}, r^{-1})} |\omega_0| \leq C\varepsilon \|\omega\|_1 \tag{1.3}$$

Proof. Let $\theta_r : \Delta(r^2, 1) \to \Delta(r, r^{-1})$ be a biholomorpic map given by $\theta_r(z) = z/r$. Then

$$\|\omega\|_1 = \int_{\Delta(r^2, 1)} |\theta_r^* \omega|$$

and

$$\int_{\Delta(0, r^{\varepsilon-1})} |\omega_+| = \int_{\Delta(0, r^{\varepsilon})} |\theta_r^* \omega_+|. \tag{1.4}$$

From the assumption $\Delta\left(\frac{1}{4}, 1\right) \subset \Delta(r^2, 1)$. Hence

$$\int_{\Delta(1/3,1/2)} |\theta_r^* \omega|_\Delta^2 du_\Delta \leq C_1 \|\omega\|_1^2 \tag{1.5}$$

for some constant C_1, since the L^p-norms are all locally equivalent for holomorphic functions.

Clearly

$$\int_{\Delta(1/3,1/2)} |\theta_r^* \omega_+|^2 du_\Delta \leq \int_{\Delta(1/3,1/2)} |\theta_r^* \omega|_\Delta^2 du_\Delta. \tag{1.6}$$

Moreover, by Cauchy's inequality and the maximum principle, there exists a constant C_2 such that, for any holomorphic function f on $\Delta(0,1)$ with a pole of order at most one at 0, the inequality

$$|f(z)|^2 \leq C_2 |z|^{-2} \int_{\Delta(1/3,1/2)} |f(\zeta)|^2 d\mu_\Delta(\zeta) \tag{1.7}$$

holds for any $z \in \Delta\left(0,\frac{1}{3}\right)$.

Combining (1.4) through (1.7) one immediately gets (1.1). The inequality (1.2) is reduced to (1.1) by the inversion $z \to z^{-1}$.

In view of (1.1) and (1.2), (1.3) is a consequence of the equality

$$\frac{\int_r^{r^{1-\varepsilon}} \frac{dt}{t} + \int_{r^{\varepsilon-1}}^{r^{-1}} \frac{dt}{t}}{\int_{r^{1-\varepsilon}}^{r^{\varepsilon-1}} \frac{dt}{t}} = \frac{\varepsilon}{1-\varepsilon}.$$

□

For simplicity we set

$$JS(r) = \mathbf{C} \cdot z^{-2} dz^{\otimes 2} \subset A_2^1(\Delta(r, r^{-1})) \text{ for any } r > 0$$

and define a bijective linear map T_r from $A_2^1(\Delta(r, r^{-1}))$ to $A_2^1(\Delta(0, r^{-1})) \oplus A_2^1(\Delta(r, \infty)) \oplus JS(r)$ by $T_r(\omega) = (\omega_+, \omega_-, \omega_0)$. Then as an immediate consequence of Lemma 1.1, we obtain the following decomposition theorem.

THEOREM 1.2. *For any $\varepsilon > 0$ there exists an $r > 0$ such that*

$$(1-\varepsilon)\|\omega\|_1 \leqq \|T_r(\omega)\|_1 \leqq (1+\varepsilon)\|\omega\|_1 \tag{1.8}$$

for all $\omega \in A_2^1(\Delta(r, r^{-1}))$. Here the L^1-norm $\|\ \|_1$ of $T_r(\omega)$ is defined as the sum of those of the components.

We shall generalize Theorem 1.2 from the annuli to those Riemann surfaces that are pinchable to two hyperbolic surfaces. For that we first recall the following (cf. [O-2] or [D]).

THEOREM 1.3. *Let M be a complex manifold of dimension n that admits a complete Kähler metric and let (E, h) be a positive Hermitian line bundle over M. Then, for any $q > 0$ and any $\bar{\partial}$-closed (n, q)-form v on M which is square integrable with respect to $\theta = \sqrt{-1}\,\bar{\partial}\partial \log h$ and h, there exists an $(n, q-1)$-form u on M such that $\bar{\partial} u = v$ and $\| u \|_{h,\theta} \leqq \| v \|_{h,\theta}$. Here $\| \ \|_{h,\theta}$ denotes the L^2-norm with respect to h and θ.*

Applying this to Riemann surfaces we shall prove a lemma which generalizes the property (1.1) of the correspondence $\omega \to \omega_+$.

Let R be any hyperbolic Riemann surface of the form $\hat{R} \setminus \{p\}$ for some Riemann surface $\hat{R}$ and $p \in \hat{R}$, and let z be a local coordinate around p which maps a neighbourhood D of p biholomorphically onto the unit disc. We put $D_r := \{q \in D \mid |z(q)| < r\}$ for $r \in (0,1)$. By ρ_r we shall denote the restriction map from $A_2^1(R)$ to $A_2^1(\hat{R} \setminus \bar{D}_r)$.

LEMMA 1.4. *For any $r \in \left(0, \frac{1}{9}\right)$ one can find a continuous linear map ϕ_r from $A_2^1(\hat{R} \setminus \bar{D}_r)$ to $A_2^1(R)$ so that $\{\phi_r\}_{r \in (0,1/9)}$ satisfies the following.*

(1.9)
$$\phi_r \circ \rho_r = \mathrm{id} \quad \textit{for all } r.$$

(1.10) *There exists a constant $C_3 = C_3(R, D, p)$ such that for all $\varepsilon \in \left(0, \frac{1}{2}\right)$ with $\varepsilon \log r < -\log 3$,*

$$\int_{\hat{R} \setminus \bar{D}_{r^\varepsilon}} |\phi_r(\omega) - \omega| + \int_{D_{r^\varepsilon} \setminus \{p\}} |\phi_r(\omega)| \leqq C_3 \varepsilon \|\omega\|_1$$

holds for all $\omega \in A_2^1(\hat{R} \setminus \bar{D}_r)$.

Proof. We fix any point p' on $\partial D_{1/2}$ and a covering map π from Δ onto R satisfying $\pi(0) = p'$. Then we define a real-valued continuous function ϕ on R by

$$\phi(q) = \sup_{z \in \pi^{-1}(q)} \log(1 - |z|^2).$$

Clearly ϕ is continuous and enjoys the following two properties.

(1.11) There exists a constant C_4 such that
$C_4^{-1}\phi(q) \leq \mathrm{dist}(p', q) \leqq C_4\phi(q)$ for any $q \in R$, where $\mathrm{dist}(p', q)$ denotes the distance between p' and q with respect to $d\sigma_R^2$.

$$\sqrt{-1}\left(\partial\bar{\partial}\pi^*\phi + \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}\right) \geqq 0. \tag{1.12}$$

As a consequence of (1.11) we have

$$\int_R e^{(1+\eta)\phi} d\mu_R < \infty \quad \text{for any } \eta > 0. \tag{1.13}$$

Let $\chi : \mathbf{R} \to \mathbf{R}$ be any C^∞ function satisfying $\chi \mid (-\infty, 1/3) \equiv 0$ and $\chi \mid (1/2, \infty) \equiv 1$. Then we define ϕ_r as follows.

Given any $\omega \in A_2^1(\hat{R} \setminus \bar{D}_r)$ we define a $K_R^{\otimes 2}$-valued (0,1)-form v on R by

$$v = \begin{cases} (\omega - (\omega \mid D \setminus \bar{D}_r)_+)\bar{\partial}\chi(|z|) & \text{on } D \setminus D_{1/3} \\ 0 & \text{otherwise.} \end{cases}$$

Since the pointwise norm of $\bar{\partial}\chi(|z|)$ is bounded by $\mathrm{const}\cdot\mathrm{dist}(\partial D_1, \partial D_{1/3})$, applying Lemma 1.1 to estimate the norm of $\omega - (\omega \mid D \setminus \bar{D}_r)_+$ from above, we obtain by a straightforward application of the Cauchy-Schwartz inequality,

$$\sup_{\omega \neq 0} \frac{\| v \|_{2,3\phi/2}}{\| \omega \|_1} \leqq C_5 |\log r|^{-1} \tag{1.14}$$

for some constant C_5 which depends only on $\mathrm{dist}(\partial D_1, \partial D_{1/3})$.

Regarding v as a K_R-valued (1,1)-form on R and applying Theorem 1.3 for $M = R$ by utilizing the curvature inequality (1.12) with respect to the regularizations of ϕ that approximate ϕ from above (cf. [Ri]) we obtain a section u of $K_R^{\otimes 2}$ over R satisfying

$$\begin{cases} \bar{\partial}u = v \\ u \perp \mathrm{Ker}\,\bar{\partial} \\ \| u \|_{2,3\phi/2}^2 \leqq 2 \| v \|_{2,3\phi/2}^2. \end{cases} \tag{1.15}$$

Then we put

$$\phi_r(\omega) = \begin{cases} (\omega \mid D \setminus \bar{D}_r)_+ + \chi(|z|)(\omega - (\omega \mid D \setminus \bar{D}_r)_+) - u & \text{on } D \setminus \{p\} \\ \omega + (\chi(|z|) - 1)(\omega - (\omega \mid D \setminus \bar{D}_r)_+) - u & \text{on } \hat{R} \setminus D_{1/3}. \end{cases}$$

The desired estimate for $\Phi_r(\omega) - \omega$ and $\Phi_r(\omega)$ follows directly from (1.2), (1.3), (1.14), (1.15) and

$$\int_R |u| \leqq \|u\|_{2,3\phi/2}\left(\int_R e^{3\phi/2} d\mu_R\right)^{1/2}. \qquad \square$$

To provide a situation under which we can generalize Theorem 1.2, let R_i $(i = 1, 2)$ be two hyperbolic Riemann surfaces of the form $\hat{R}_i \setminus \{p_i\}$, and let $z_i : D_i \to \Delta$ be local coordinates around p_i. Then for any $r \in (0,1)$ we define a Riemann surface $R[r]$ by patching $\hat{R}_i \setminus \{|z_i| \leqq r\}$ along $D_i \setminus \{|z_i| \leqq r\}$ via the correspondence

$$\begin{array}{ccc} D_1 \setminus \{|z_1| \leqq r\} & \longrightarrow & D_2 \setminus \{|z_2| \leqq r\} \\ \cup & & \cup \\ q & \longrightarrow & z_2^{-1}(r/z_1(q)). \end{array}$$

We put $D_i(r) = \{q \in \hat{R}_i \mid |z_i(q)| < r\}$. Then the following generalizes Theorem 1.2.

THEOREM 1.5. *For any* $\varepsilon > 0$ *there exist an* $r_0 > 0$*, depending on* ε *and* $\mathrm{dist}(\partial D_i(1), \partial D_i(1/3))$ $(i = 1,2)$*, and a bijective linear map* $\mathcal{T}_r$ *from* $A_2^1(R[r])$ *to* $A_2^1(R_1) \oplus A_2^1(R_2) \oplus JS(r)$ *satisfying*

$$(1.16) \qquad (1-\varepsilon)\|\omega\|_1 \leqq \|\mathcal{T}_r(\omega)\|_1 \leqq (1+\varepsilon)\|\omega\|_1$$

for all $\omega \in A_2^1(R[r])$.

Proof. Let $D_i(r) = \{q \in \hat{R}_i \mid |z_i(q)| < r\}$. By Lemma 1.4 there exist a constant C_6 and continuous linear maps Φ_{ir} from $A_2^1(\hat{R}_i \setminus \overline{D_i(r)})$ onto $A_2^1(R_i)$ for $r \in \left(0, \frac{1}{9}\right)$ such that

$$(1.17) \qquad \int_{\hat{R}_i \setminus \overline{D_i(r^\varepsilon)}} |\Phi_{ir}(\omega_i) - \omega_i| + \int_{D_i(r^\varepsilon)\setminus\{p_i\}} |\Phi_{ir}(\omega_i)| \leqq C_6 \varepsilon \|\omega_i\|_1$$

for all $\omega_i \in A_2^1(\hat{R}_1 \setminus \overline{D_i(r)})$ provided that $r \in (0, \exp(-(\log 3)/\varepsilon))$.

Then we put

$$\mathcal{T}_r(\omega) = (\Phi_{1r}(\omega \mid \hat{R}_1 \setminus \overline{D_1(r)}), \Phi_{2r}(\omega \mid \hat{R}_2 \setminus \overline{D_2(r)}), ((\theta_{\sqrt{r}}^{-1} \circ z_1^{-1})^*(\omega \mid D_1 \setminus \{|z_1| \leqq r\}))_0).$$

By (1.17) $\mathcal{T}_r$ satisfies (1.16) for sufficiently small r. In particular, $\mathcal{T}_r$ is then injective and the image of $\mathcal{T}_r$ is closed. Therefore the surjectivity of $\mathcal{T}_r$ will follow from the assertion that, for any $(\omega_1, \omega_2, \omega_3)$ in $A_2^1(R_1) \oplus A_2^1(R_2) \oplus \Lambda(\sqrt{r})$ there exists an $\omega \in A_2^1(R[r])$ such that

$$\| \mathcal{T}_r(\omega) - (\omega_1, \omega_2, \omega_3) \|_1 \leq \frac{1}{2} \| (\omega_1, \omega_2, \omega_3) \|_1.$$

For that one has only to repeat the argument in the proof of Lemma 1.4. The detail is thus left to the reader. □

Note that the above $R[r]$ is uniquely determined by the triples $\{(R_i, D_i, z_i)\}_{i=1,2}$ and r. To make this dependence more explicit, we shall denote $R[r]$ by $(R_1, D_1, z_1) \#_r (R_2, D_2, z_2)$. We note that one may take $r_0 = \text{const} \cdot \min(\text{dist}(\partial D_1(1), \partial D_1(1/3)), \text{dist}(\partial D_2(1), \partial D_2(1/3)) \cdot \exp(-(\log 3)/\varepsilon))$.

§2. Quasi-irreducible decomposition of normed vector spaces

Let $(L_i, \| \|_{(i)})$ $(i = 1, 2)$ be normed vector spaces. For any $(v_1, v_2) \in L_1 \oplus L_2$ we put

$$\| (v_1, v_2) \|_1 = \| v_1 \|_{(1)} + \| v_2 \|_{(2)}$$

and

$$\| (v_1, v_2) \|_\infty = \max\{\| v_1 \|_{(1)}, \| v_2 \|_{(2)}\}.$$

We shall call the normed vector space $(L_1 \oplus L_2, \| \|_1)$ (resp. $(L_1 \oplus L_2, \| \|_\infty)$) the 1-direct sum (resp. the ∞-direct sum) of $(L_1, \| \|_{(1)})$ and $(L_2, \| \|_{(2)})$. For any $K \geqq 1$, a bijective linear map $\Phi : L_1 \to L_2$ is said to be a K-quasi-isometry if $K^{-1} \| v \|_{(1)} \leqq \| \Phi(v) \|_{(2)} \leqq K \| v \|_{(1)}$ for all $v \in L_1$. By an abuse of language we shall call the quantity

$$\begin{aligned} &\text{dist}((L_1, \| \|_{(1)}), (L_2, \| \|_{(2)})) \quad (\in [0,\infty]) \\ &:= \inf\{\log K \mid \text{there exists a } K\text{-quasi-isometry between } (L_i, \| \|_{(i)})\} \end{aligned}$$

the distance between $(L_1, \| \|_{(1)})$ and $(L_2, \| \|_{(2)})$. A normed vector space $(L, \| \|)$ will be said to be 1-K-irreducible (resp. ∞-K-irreducible) if $L \neq \{0\}$ and L is not K-quasi-isometric to the 1-direct sum (resp. the ∞-direct sum) of two nontrivial subspaces of L equipped with the induced norm. Moreover we say that $(L, \| \|)$ is purely 1-K-irreducible (resp. purely ∞-K-irreducible) if every (nontrivial) subspace of L is 1-K-irreducible (resp. ∞-K-irreducible) in the above sense. The following is obvious from the definition.

PROPOSITION 2.1. *If* $1 \leqq \dim L < \infty$ *and the subset* $\{v \in L \mid \|v\| = 1\}$ *is a* C^1*-smooth hypersurface of* L, *then there exists a* $K > 1$ *such that* L *is both purely* 1-K*-irreducible and purely* ∞-K*-irreducible.*

For any finite Riemann surface R, as is well known $A_2^1(R)$ is finite dimensional and the unit sphere of $A_2^1(R)$ is C^1-smooth (cf. [F-K], [R], [E-K]). Accordingly we have the following.

PROPOSITION 2.2. *For any finite Riemann surface* R *with* $A_2^1(R) \neq \{0\}$, *there exists a* $K > 1$ *such that* $A_2^1(R)$ *is purely* 1-K*-irreducible.*

COROLLARY 2.3. *Under the above situation the dual space of* $A_2^1(R)$ *is purely* ∞-K*-irreducible for the same* K.

For any countably many normed vector spaces $\{(L_i, \| \ \|_{(i)})\}_{i \in \mathbf{N}}$ we put

$$\bigoplus_{i \in \mathbf{N}}^1 L_i = \{f \in \prod_{i \in \mathbf{N}} L_i \mid \|f\|_1 := \sum_{i \in \mathbf{N}} \|f_i\|_{(i)} < \infty\}$$

and

$$\bigoplus_{i \in \mathbf{N}}^\infty L_i = \{f \in \prod_{i \in \mathbf{N}} L_i \mid \|f\|_\infty := \sup_{i \in \mathbf{N}} \|f_i\|_{(i)} < \infty\}.$$

Here f_i denotes the i-th component of f.

THEOREM 2.4. *Let* $K \in (1, \sqrt[4]{2})$ *and let* $\{(L_i, \| \ \|_{(i)})\}_{i \in \mathbf{N}}$, $\{(L'_i, \| \ \|'_{(i)})\}_{i \in \mathbf{N}}$ *be two sequences of finite dimensional normed vector spaces such that* $\bigoplus_{i \in \mathbf{N}}^\infty L_i$ *and* $\bigoplus_{i \in \mathbf{N}}^\infty L'_i$ *are mutually* K*-quasi-isometric. Suppose that* $(L_i, \| \ \|_{(i)})$ *and* $(L'_i, \| \ \|'_{(i)})$ *are all purely* ∞-K^3*-irreducible. Then there exists a bijective map* $\tau : \mathbf{N} \to \mathbf{N}$ *such that* $(L_i, \| \ \|_{(i)})$ *and* $(L'_{\tau(i)}, \| \ \|'_{\tau(i)})$ *are mutually* K^2*-quasi-isometric for all* $i \in \mathbf{N}$.

Proof. For simplicity we shall suppress the subscripts for the norm. Let Ψ : $\bigoplus_{i \in \mathbf{N}}^\infty L_i \to \bigoplus_{i \in \mathbf{N}}^\infty L'_i$ be any K-quasi-isometry. For any $i \in \mathbf{N}$ and for any $u \in L_i$ satisfying $\|u\| = 1$ we put

$$\Psi(u) = \sum_{j \in \mathbf{N}} v_j, \quad v_j \in L'_j.$$

Since Ψ is a K-quasi-isometry we have $K^{-1} \leqq \sup_j \|v_j\| \leqq K$.

Let us put

$$\Psi^{-1}(v_j) = \sum_{k\in\mathbf{N}} u_k^j, \quad u_k^j \in L_k.$$

Clearly $\| u_k^j \| \leqq \frac{1}{2} K^2 \leqq K^{-2}$ if $k \neq i$, so that

$$K^{-2} \leqq \sup_j \| u_i^j \| \leqq K^2.$$

In particular there exists a j such that $K^{-2} \leqq \| v_j \| \leqq K^2$ and $K^{-3} \leqq \| u_i^j \| \leqq K^3$. Let us take another $u' \in L_i$ with $\| u' \| = 1$ and $l \in \mathbf{N}$ such that $K^{-2} \leqq \| v'_l \| \leqq K^2$ and $K^{-3} \leqq \| u_i'^l \| \leqq K^3$. Here v'_l and $u_i'^l$ are defined similarly as v_j and u_k^j. Then j and l must be equal because L_i is purely ∞-K^3-irreducible.

Hence the map

$$\begin{array}{cccc} \Psi_i : & L_i & \rightarrow & L'_j \\ & \cup & & \cup \\ & u & \rightarrow & v_j \end{array}$$

is well-defind and gives a K^2-quasi-isometry onto a subspace of L'_j.

Interchanging and role of L_i and L'_j we have a K^2-quasi-isometry from L'_j onto a subspace of L_i for the same i and j. Since $\dim L_i$ and $\dim L'_j$ are finite, it follows that Ψ_i is surjective. Thus we may put $j = \tau(i)$. □

§3. Non-equivalence of Teichmüller spaces

Let $\{R_i\}_{i\in\mathbf{N}}$ be a sequence of finite and hyperbolic Riemann surfaces of the form $R_1 = \hat{R}_1 \setminus \{p_1\}$ and $R_i = \hat{R}_i \setminus \{p_i, q_{i-1}\}$ for $i \geqq 2$. Here p_i and q_{i-1} are distinct points of $\hat{R}_i$. We choose any local coordinates $z_i : D_i \rightarrow \Delta$ (resp. $w_i : E_i \rightarrow \Delta$) around p_i (resp. q_i) so that $D_i \cap E_{i-1} = \emptyset$ for all $i \geqq 2$.

Then we put $\mathcal{R}_i = (R_i, (D_i, z_i), (E_{i-1}, w_{i-1}))$ where $(E_0, w_0) := \emptyset$, $|\mathcal{R}_i| = R_i$ and

$$\mathcal{R} = \{\mathcal{R}_i\}_{i\in\mathbf{N}}.$$

Given any sequence $r = \{r_i\}_{i\in\mathbf{N}} \subset (0,1)$ we define Riemann surfaces $\mathcal{R}_{\leqq i}[r]$, inductively on i, by defining $\mathcal{R}_{\leqq 1}[r] = R_1$ and

$$\mathcal{R}_{\leqq i+1}[r] = (\mathcal{R}_{\leqq i}[r], D_i, z_i) \#_{r_i} (R_{i+1}, E_i, w_i).$$

We then define $\mathcal{R}[r]$ as the inductive limit of $\{\mathcal{R}_{\leqq i}[r]\}_{i\in\mathbf{N}}$.

Letting $\mathcal{R}_{i,1} = (\mathcal{R}_{i+1}, (D_{i+1}, z_{i+1}), \emptyset)$, $\mathcal{R}_{i,j} = (\mathcal{R}_{i+j}, (D_{i+j}, z_{i+j}), (E_{i+j}, w_{i+j}))$ for $j \geqq 2$,

$$\mathscr{R}_{>i} = \{\mathscr{R}_{1,j}\}_{j\in\mathbf{N}}$$

and $r_{>i} = \{r_{i+j}\}_{j\in\mathbf{N}}$ we put $\mathscr{R}_{>i}[r] = \mathscr{R}_{>i}[r_{>i}]$.

Note that

$$\mathscr{R}[r] = (\mathscr{R}_{\leqq i}[r], D_i, z_i) \#_{r_i} (R_{>i}[r], E_i, w_i).$$

We shall call $\mathscr{R}$ a decorated sequence of Riemann surfaces and $\mathscr{R}[r]$ the r-connected sum of $\mathscr{R}$. $\mathscr{R}_i$ shall be referred to as the i-th component of $\mathscr{R}$.

Once for all we shall fix $\mathscr{R}$ and r. Then, for any Riemann surface R' which is K-quasiconformally equivalent to $\mathscr{R}[r]$, one can find a decorated sequence $\mathscr{R}' = \{R_i'\}_{i\in\mathbf{N}}$ and a sequence $r' = \{r_i'\}_{i\in\mathbf{N}} \subset (0,1)$ satisfying the following properties.

(3.1) $\mathscr{R}'[r']$ is conformally equivalent to R'.

(3.2) $|\mathscr{R}_i'|$ is K-quasiconformally equivalent to $|\mathscr{R}_i|$ for every i.

(3.3) $$r_i^K \leqq r_i' \leqq r_i^{K^{-1}}.$$

In fact, this is a consequence of two obvious facts that the quantity $m_r := -\log r$ attached to $\Delta(r, 1)$ satisfies

$$K^{-1} m_{r_2} \leqq m_{r_1} \leqq K m_{r_2}$$

if $\Delta(r_1, 1)$ and $\Delta(r_2, 1)$ are K-quasiconformally equivalent to each other, and that any K-quasiconformal homeomorphism from $\Delta(r_1, 1)$ to $\Delta(r_2, 1)$ is extendable to a K-quasiconformal automorphism of $\mathbf{C}\setminus\{0\}$ by iteration of the reflections along the circles centered at 0.

THEOREM 3.1. *Let $\mathscr{R}$ be any decorated sequence of finite and hyperbolic Riemann surfaces. Then there exist two sequences $r = \{r_i\}$, $\varepsilon = \{\varepsilon_i\} \subset (0,1)$ with $\lim_{i\to\infty}(r_i + \varepsilon_i) = 0$ such that, for any quasiconformal deformation $(\mathscr{R}', r')$ of $(\mathscr{R}, r)$ one can find a $j \in \mathbf{N}$ such that for all $k \in \mathbf{N}$, $A_2^1(\mathscr{R}'_{\leqq j+k}[r'])$ are purely $1-(1+\varepsilon_{j+k})^3$-irreducible and $A_2^1(\mathscr{R}'[r'])$ (resp. $A_2^1(\mathscr{R}'_{>j+k}[r'])$) is $(1+\varepsilon_{j+k})$-quasi-isometric to $A_2^1(\mathscr{R}'_{\leqq j+k}[r']) \oplus A_2^1(\mathscr{R}'_{>j+k}[r']) \oplus \mathbf{C}$ (resp. $(\bigoplus^1_{m\in\mathbf{N}} A_2^1(R'_{j+k+m})) \oplus l^1_{\mathbf{C}}$). Here $l^1_{\mathbf{C}}$ denotes the space of absolutely summable sequence with values in $\mathbf{C}$.*

Proof. The assertion is a routine consequence of Theorem 1.5 and Proposition 2.2. The detail is left to the reader. □

Let $\mathscr{R}$ be as above and let $\mathscr{R}_i$ be its i-th component. If the Riemann surfaces $|\mathscr{R}_i|$, $i \in \mathbf{N}$, consist of only finitely many conformal equivalence classes, then we can choose r and ε independently of the order of $\mathscr{R}_i$ ($i \geqq 2$), i.e., letting $\mathscr{R}_\tau := \{\mathscr{R}_{\tau(i)}\}_{i\in\mathbf{N}}$ for any bijective map $\tau:\mathbf{N}\to\mathbf{N}$ satisfying $\tau(1)=1$, they can be required to satisfy the same condition as above for all $\mathscr{R}_\tau$. Moreover, in this situation, we can require that all the components $A_2^1(R'_{j+k+m})$ are purely $1-(1+\varepsilon_{j+k})^3$-irreducible. The following is then deduced from Theorem 2.1.

PROPOSITION 3.2. *Under the above situation,* $\mathscr{R}[r]$ *and* $\mathscr{R}_\tau[r]$ *are not quasiconformally equivalent if two sequences* $\{\dim A_2^1(|\mathscr{R}_i|)\}_{i\in\mathbf{N}}$ *and* $\{\dim A_2^1(|\mathscr{R}_{\tau(i)}|)\}_{i\in\mathbf{N}}$ *are cofinally non-equivalent.*

Let $\mathscr{T}(\mathscr{R}[r])$ denote the Teichmüller space of $\mathscr{R}[r]$. In virtue of Gardiner's theorem, the tangent space of $\mathscr{T}(\mathscr{R}[r])$ at the point $[(\mathscr{R}[r], \mathrm{id})] \in \mathscr{T}(\mathscr{R}[r])$ is isometric to $A_2^1(R[r])^*$ with respect to the Teichmüller infinitesimal form (cf. [G, Theorem 3.5]). It is also known from the work of Gardiner that any biholomorphism between Teichmüller spaces is an isometry with respect to the Teichmüller metric (cf. [G, Corollary 2.1]). Hence we have also the following consequence of Theorem 3.1.

PROPOSITION 3.3. *Under the above situation,* $\mathscr{T}(\mathscr{R}[r])$ *and* $\mathscr{T}(\mathscr{R}_\tau[r])$ *are not biholomorphically equivalent whenever* $\{\dim A_2^1(|\mathscr{R}_i|)\}$ *and* $\{\dim A_2^1(|\mathscr{R}_{\tau(i)}|)\}$ *are cofinally non-equivalent.*

As an immediate corollary to Proposition 3.3 we obtain the following.

THEOREM 3.4. *There exist uncountably many biholomorphically non-equivalent Teichmüller spaces.*

Furthermore, if r and ε are sufficiently rapidly decreasing, then the $(1+\varepsilon_j)^2$-quasi-isometry between $A_2^1(\mathscr{R}'_{\leqq j}[r])$ and $A_2^1(\mathscr{R}'_{\leqq j}[r'])$ will imply K_j-quasiconformal equivalence between $\mathscr{R}_{\leqq j}[r]$ and $\mathscr{R}'_{\leqq j}[r']$ with $\lim_{j\to\infty} K_j = 1$. Therefore we see that our r can be chosen so that the existence of a biholomorphic automorphism σ of $\mathscr{T}(\mathscr{R}[r])$ with $\sigma([(\mathscr{R}[r], f)]) = [(\mathscr{R}'[r'], g)]$ forces $\mathscr{R}[r]$ and $\mathscr{R}'[r']$ to be are conformally equivalent.

Under the above situation it is clear that, for any point $p \in \mathscr{T}(\mathscr{R}[r])$ one can find a locally closed submanifold $\mathscr{T}' \subset \mathscr{T}(\mathscr{R}[r])$ of arbitrarily large dimension passing through p, and a neighbourhood $U \ni p$, such that every biholomorphic automorphism σ of $\mathscr{T}(\mathscr{R}[r])$ satisfies $\sigma(p) \notin \mathscr{T}' \setminus U$.

According to the work of Earle and Gardiner [E-G] it is known that any biholomorphic automorphism of the Teichmüller space of a topologically finite Riemann surface with non-empty border is induced from a quasiconformal homeomorphism between conformally equivalent surfaces. Combining this fact with the above observation we obtain

PROPOSITION 3.5. *Under the above assumptions on* $\mathfrak{R}$ *and* r, $\mathcal{T}(\mathfrak{R}[r])$ *is not biholomorphically equivalent to the Teichmüller space of any topologically finite Riemann surface.*

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