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LIMIT THEOREMS RELATED TO A CLASS OF OPERATOR–SELF–SIMILAR PROCESSES

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1. Introduction and results

An $\mathbf{R}^d$-valued $(d \geq 1)$ stochastic process $X = \{X(t)\}_{t \geq 0}$ is said to be operator-self-similar if there exists a linear operator D on $\mathbf{R}^d$ such that for each $c > 0$

$$\{X(ct)\} \overset{f.d.}{=} \{c^D X(t)\},$$

where $\overset{f.d.}{=}$ means the equality for all finite-dimensional distributions and

$$c^D = \exp\{(\ln c)D\} = \sum_{k=0}^{\infty} \frac{1}{k!} (\ln c)^k D^k.$$

We refer the reader to [HM1], [Sa] and [MM] for more information about operator-self-similar processes. In the present paper, we show limit theorems related to a class of operator-self-similar processes, as a direct extension of [KS].

A probability distribution μ on $\mathbf{R}^d$ is said to be full if μ is not concentrated on a proper hyperplane and a full distribution μ on $\mathbf{R}^d$ is called operator-stable if it is infinitely divisible and there exist an invertible linear operator B on $\mathbf{R}^d$ and a function $b : (0, \infty) \to \mathbf{R}^d$ such that for all $t > 0$,

$$\varphi(\theta)^t = \varphi(t^{B^*}\theta) e^{ib(t)}, \quad \theta \in \mathbf{R}^d,$$

where φ is the characteristic function of μ, B^* is the adjoint operator of B. μ is called strictly operator-stable if we can choose $b(t) \equiv 0$. In this paper, we always assume μ is a full strictly operator-stable on $\mathbf{R}^d$. However, Sharpe ([Sh]) showed that if 1 is not an eigenvalue of B, then the operator-stable law can be centered so as to become strictly operator-stable. Thus the assumption for the strict operator-stability is not so restrictive. So, in the present paper, we always assume

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$$\varphi(\theta)^t = \varphi(t^{B^*}\theta), \quad \theta \in \mathbf{R}^d. \tag{1}$$

The exponent B is not necessarily unique. Let $\Lambda_B = \max\{\mathrm{Re}\,\sigma : \sigma \in \sigma(B)\}$ and $\lambda_B = \min\{\mathrm{Re}\,\sigma : \sigma \in \sigma(B)\}$, where $\sigma(B)$ is the set of all eigenvalues of B. Then it is known ([Sh]) that $\lambda_B \geq \dfrac{1}{2}$ and a full operator-stable measure μ can be classified as follows:

(i) μ is Gaussian. In this case, $B = \dfrac{1}{2} I$ can always be taken as an exponent of μ.

(ii) μ is purely non-Gaussian. In this case, $\lambda_B > \dfrac{1}{2}$. When μ is d-dimensional α-stable measure, we can take $B = \dfrac{1}{\alpha} I$.

(iii) μ is general. Theorem 1 in [HM2] allows us to consider the Gaussian component and the purely non-Gaussian component separately.

In this paper, we focus on purely non-Gaussian operator-stable laws, since Gaussian case $\left(B = \dfrac{1}{2} I\right)$ can be handled similarly to [KS]. The representation for the characteristic function of purely non-Gaussian operator-stable law with exponent B is known as follows:

$$\varphi(\theta) = \exp\Big\{i\langle\theta, c\rangle + \int_S \gamma(dx) \int_0^\infty [e^{i\langle\theta, s^B x\rangle} - 1 - i\langle\theta, s^B x\rangle I_Q(s^B x)] \frac{1}{s^2} ds\Big\}, \tag{2}$$

where

$\theta \in \mathbf{R}^d, \quad c \in \mathbf{R}^d,$
$S = \{x \in \mathbf{R}^d : \|x\| = 1 \text{ and } \|t^B x\| > 1 \text{ for all } t > 1\},$
$Q = \{x \in \mathbf{R}^d : \|x\| \leq 1\},$
γ is a probability measure on S,
$\langle\ ,\ \rangle$ is the inner product in $\mathbf{R}^d$.

Let Z_B be a purely non-Gaussian operator-stable random vector with exponent B and let $\{\xi(k)\}_{k\in Z}$ be i.i.d. $\mathbf{R}^d$-valued random variables such that they belong to be domain of normal attraction of Z_B, namely

$$n^{-B} \sum_{k=1}^{n} \xi(k) \xrightarrow{w} Z_B. \tag{3}$$

Let $\{S_n\}_{n=0}^{\infty}$ be an integer-valued random walk independent of $\{\xi(k)\}$ such that

$$\frac{1}{n^{1/\alpha}} S_n \xrightarrow{w} Z_\alpha, \tag{4}$$

where Z_α is one-dimensional α-stable with $1 < \alpha \leq 2$. In this paper, we are concerned with a sequence of dependent stationary random vectors $\{\xi(S_k)\}_{k=0}^{\infty}$ and study the asymptotic behavior of its cumulative sum

$$W_n = \sum_{k=1}^{n} \xi(S_k).$$

Kesten and Spitzer ([KS]) called this a random walk in random scenery when $d = 1$, and proved that with a suitable normalization, $W_{[nt]}$ converges weakly to a self-similar process represented by a stable integral whose integrand is a local time.

To describe our theorem, we need some preliminaries. Let $\{Y(t)\}_{t\geq 0}$ be an α-stable Lévy process with right continuous sample paths such that the distribution of $Y(1)$ is the same as that of Z_α in (4). Since $1 < \alpha \leq 2$, $L_t(x)$, the local time of $Y(\cdot)$ at x, exists and we can take a version of $L_t(x)$ (denoted by $L_t(x)$ again) which is continuous in (t, x). Let $\{Z_B(t)\}_{t\in\mathbf{R}}$ be an $\mathbf{R}^d$-valued Lévy process independent of $\{Y(t)\}_{t\geq 0}$ such that the distribution of $Z_B(1)$ is the same as that of Z_B in (3). This $\{Z_B(t)\}$ is called an operator-stable Lévy process or operator-stable motion with exponent B. Each component $\{Z_B^{(i)}(t)\}$, $i = 1,2,\cdots, d$, of $\{Z_B(t)\}$ is also a real-valued (not necessary stable) Lévy process. Hence the stochastic integral

$$\Delta^{(i)}(t) = \int_{-\infty}^{\infty} L_t(x)\, dZ_B^{(i)}(x)$$

can be defined for each i as in [KS]. The $\mathbf{R}^d$-valued stochastic process whose i-th component is $\Delta^{(i)}(t)$ is denoted by

$$\Delta(t) = \int_{-\infty}^{\infty} L_t(x)\, dZ_B(x),$$

where $L_t(x)$ is a random scalar and Z_B is a random vector.

Define W_t for $t > 0$ by

$$W_t = W_{[t]} + (t - [t])(W_{[t]+1} - W_{[t]}),$$

where $[t]$ is the integer part of t. Our theorems are the following.

THEOREM 1. *Let* $D=\left(1-\frac{1}{\alpha}\right)I+\frac{1}{\alpha}B$. *Then any finite dimensional distribution of* $\{n^{-D}W_{nt}\}_{t\geq 0}$ *converges to that of* $\{\Delta(t)\}_{t\geq 0}$. $\{\Delta(t)\}_{t\geq 0}$ *is operator-self-similar with exponent* D *and has stationary increments.*

The latter half of Theorem 1 is easily seen by the definition of $\Delta(t)$.

THEOREM 2. $\{n^{-D}W_{nt}\}_{t\geq 0}$ *converges weakly to* $\{\Delta(t)\}_{t\geq 0}$ *in the space* $C([0,\infty):\mathbf{R}^d)$, *provided that* $\xi(0)$ *is symmetric in the sense that* $\xi(0)\stackrel{d}{=}-\xi(0)$ *when* $\lambda_B\leq 1\leq \Lambda_B$.

The idea of the proofs of these theorems is found in [KS]. The only technical difference in the proof of Theorem 1 comes from the fact that the characteristic function of operator-stable random vector (eq. (2)) does not have a simple form like that of one-dimensional stable random variable. This technical point can be dealt with the basic relation (1) and observations given in Lemmas 4 and 7 below. (Lemma 4 is trivial for the one-dimensional case.) The rest of the argument is exactly the same as in [KS].

For the proof of Theorem 2, we need some estimates for the "tail" behavior of the random vector belonging to the domain of normal attraction of operator-stable law. It will be recognized as in [W] that in the multidimensional case $P\{\|n^{-B}\xi\|\in A\}$ should be estimated instead of $P\{\|\xi\|\in A\}$. (See Lemmas 9, 11 and 12 below.) The estimates presented here can also be applied to a functional version of operator-stable limit theorem and other weak convergence theorem (see [M]).

We give here a brief remark on the extra condition of the symmetricity of $\xi(0)$ for the case $\lambda_B\leq 1\leq\Lambda_B$. When $d=1$, this case ($\lambda_B=\Lambda_B=1$) corresponds to the so-called Cauchy case where the index of stability is 1, and we often assume some conditions related to the symmetricity of $\xi(0)$. Such conditions are needed for the estimates for the tail behavior of random variables. However, the condition here is rather technical. The essential point would be whether 1 is an eigenvalue of B or not. From this point of view, the extra condition in Theorem 2 might be weakened, although we do not try it in this paper.

We end this section with a remark about the i-th component $\Delta^{(i)}(t)$ of the $\mathbf{R}^d$-valued stochastic process $\Delta(t)$. If B is diagonalizable over $\mathbf{R}$, then $Z_B^{(i)}(t)$ is one-dimensional stable ([H]). Thus $\Delta^{(i)}(t)$ is nothing but the process appearing in [KS]. Therefore it is self-similar. However if B is not semi-simple, then $Z_B^{(i)}(t)$ is *not* stable ([H]). Thus this process is not covered by [KS]. If B is not semi-simple, nor is D. Then it follows from Theorem 5.1 in [M] that $\Delta^{(i)}(t)$ is *not* self-similar. Therefore the $\mathbf{R}$-valued process $\Delta^{(i)}(t)$ is different from that in [KS].

2. Proof of Theorem 1

In the following, $\|\cdot\|$ stands for the ordinary Euclidean norm.

The first step of the proof is to represent W_n as

$$W_n = \sum_{k=0}^{n} \xi(S_k) = \sum_{u \in \mathbf{Z}} N_n(u)\xi(u), \tag{5}$$

where $N_n(u)$ is the number of visits of the random walk $\{S_n\}$ to the point u in the time interval $[0, n]$. All that are necessary about the occupation time $N_n(u)$ of $\{S_n\}$ and the local time $L_t(x)$ are found in [KS]. We collect some of them which we need later as lemmas. Consider the linear interpolation of $N_n(u)$ as W_t as follows:

$$N_t(u) = N_{[t]}(u) + (t - [t])(N_{[t]+1}(u) - N_{[t]}(u)).$$

For $-\infty < a < b < \infty$, define

$$T_t^n(a, b) = \frac{1}{n} \sum_{n^{\frac{1}{\alpha}} a \leq u < n^{\frac{1}{\alpha}} b} N_{nt}(u)$$

and

$$\Gamma_t(a, b) = \int_a^b L_t(u)\, du.$$

LEMMA 1 ([KS]). *For any* $t_1, t_2, \cdots, t_k \geq 0$,

$$\{T_{t_j}^n(a_j, b_j), 1 \leq j \leq k\} \xrightarrow{w} \{\Gamma_{t_j}(a_j, b_j), 1 \leq j \leq k\}.$$

LEMMA 2 ([KS]). *For any* $p \geq 1$,

$$\sup_{u \in Z} E[N_n(u)^p] = O(n^{p(1-\frac{1}{\alpha})}) \tag{6}$$

and

$$P\{N_n(u) > 0 \text{ for some } u \text{ with } |u| > An^{\frac{1}{\alpha}}\} \leq \varepsilon(A) \text{ for } n \geq 1, \tag{7}$$

where $\varepsilon(A) \to 0$ *as* $A \to \infty$ *and* $\varepsilon(A)$ *is independent of* n.

In what follows, C denotes an absolute constant which may differ from one inequality to another. Let $f = \log \varphi$, where φ is the characteristic function of Z_B defined in (2). We are going to show three lemmas.

LEMMA 3 (The joint distribution of $\Delta(t)$). *For any $t_1, t_2, \dots, t_k \geq 0$ and $\theta_1, \theta_2, \dots, \theta_k \in \mathbf{R}^d$,*

$$E\left[\exp\left\{i \sum_{j=1}^{k} \langle \theta_j, \Delta(t_j) \rangle\right\}\right] = E\left[\exp\left\{\int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u)\theta_j\right) du\right\}\right].$$

Proof. The assertion easily follows from the facts that

$$\int_0^{\infty} L_t(u)\, dZ_B(u) = \lim_{n\to\infty} \sum_{l=0}^{\infty} L_t(u_l^n)[Z_B(u_{l+1}^n) - Z_B(u_l^n)] \quad \text{w.p.1},$$

where $0 = u_0^n < u_1^n < \cdots$ is a suitable sequance satisfying

$$\lim_{l\to\infty} u_l^n = \infty, \quad \lim_{n\to\infty} \max_l (u_{l+1}^n - u_l^n) = 0,$$

and that

$$E[e^{i\langle \theta, Z_B(u_{l+1}^n) - Z_B(u_l^n)\rangle}] = \varphi(\theta)^{u_{l+1}^n - u_l^n},$$

as in Lemma 5 in [KS]. □

LEMMA 4. *Let $\beta = 1$ when $\Lambda_B < 1$ and let $0 < \beta < \dfrac{1}{\Lambda_B}$ when $\Lambda_B \geq 1$. Then for any θ_1 and $\theta_2 \in \mathbf{R}^d$, we have*

$$| f(\theta_1) - f(\theta_2) | \leq C\{\| \theta_1 - \theta_2 \|(1 + \| \theta_1 \| + \| \theta_2 \|) + \| \theta_1 - \theta_2 \|^{\beta}\}.$$

Proof. By (2),

$$\begin{aligned} f(\theta_1) - f(\theta_2) &= i\langle \theta_1 - \theta_2, c\rangle \\ &\quad + \int_S \gamma(dx) \int_{\{\|s^B x\| \leq 1\}} [e^{i\langle \theta_1, s^B x\rangle} - e^{i\langle \theta_2, s^B x\rangle} \\ &\qquad - i\langle \theta_1 - \theta_2, s^B x\rangle] \frac{1}{s^2} ds \\ &\quad + \int_S \gamma(dx) \int_{\{\|s^B x\| > 1\}} [e^{i\langle \theta_1, s^B x\rangle} - e^{i\langle \theta_2, s^B x\rangle}] \frac{1}{s^2} ds. \end{aligned}$$

Observe that if $0 < \beta \leq 1$,

$$| e^{i\xi_1} - e^{i\xi_2} | \leq 2^{(1-\beta)/\beta} | \xi_1 - \xi_2 |^{\beta}.$$

For, if $| \xi_1 - \xi_2 | \geq 2^{1/\beta}$, then $| e^{i\xi_1} - e^{i\xi_2} | \leq 2 \leq | \xi_1 - \xi_2 |^{\beta}$. If $| \xi_1 - \xi_2 | < 2^{1/\beta}$, then

$$| e^{i\xi_1} - e^{i\xi_2} | \leq | \xi_1 - \xi_2 | = | \xi_1 - \xi_2 |^{1-\beta} | \xi_1 - \xi_2 |^{\beta} \leq 2^{(1-\beta)/\beta} | \xi_1 - \xi_2 |^{\beta}.$$

Thus we have

$$\begin{aligned} | f(\theta_1) - f(\theta_2) | &\leq C \| \theta_1 - \theta_2 \| \\ &\quad + 2 \| \theta_1 - \theta_2 \| (\| \theta_1 \| + \| \theta_2 \|) \int_S \gamma(dx) \int_{\{\|s^B x\| \leq 1\}} \frac{\| s^B x \|^2}{s^2} ds \\ &\quad + 2^{(1-\beta)/\beta} \| \theta_1 - \theta_2 \|^{\beta} \int_S \gamma(dx) \int_{\{\|s^B x\| > 1\}} \frac{\| s^B x \|^{\beta}}{s^2} ds. \end{aligned}$$

Recall that $\lambda_B > \frac{1}{2}$ since μ is purely non-Gaussian operator-stable with exponent B. Hence

$$\int_S \gamma(dx) \int_{\{\|s^B x\| \leq 1\}} \frac{\| s^B x \|^2}{s^2} ds < \infty.$$

On the other hand, since $\beta < \frac{1}{\Lambda_B}$,

$$\int_S \gamma(dx) \int_{\{\|s^B x\| > 1\}} \frac{\| s^B x \|^{\beta}}{s^2} ds < \infty.$$

Altogether we conclude the lemma.

LEMMA 5. *For any $t_1, t_2, \ldots, t_k \geq 0$ and $\theta_1, \theta_2, \ldots, \theta_k \in \mathbf{R}^d$,*

$$\sum_{u \in \mathbf{Z}} f\Big(n^{-D^*} \sum_{j=1}^{k} N_{nt_j}(u) \theta_j\Big) \xrightarrow{w} \int_{-\infty}^{\infty} f\Big(\sum_{j=1}^{k} L_{t_j}(u) \theta_j\Big) du.$$

Proof. Since $n^{-D^*} = n^{-(1-\frac{1}{\alpha})} n^{-\frac{1}{\alpha} B^*}$, we have, by the use of the relation (1),

$$\begin{aligned} &\sum_{u \in \mathbf{Z}} f\Big(n^{-D^*} \sum_{j=1}^{k} N_{nt_j}(u) \theta_j\Big) \\ &\quad = \sum_{u \in \mathbf{Z}} \log \varphi\Big(n^{-(1-\frac{1}{\alpha})} \sum_{j=1}^{k} N_{nt_j}(u) n^{-\frac{1}{\alpha} B^*} \theta_j\Big) \\ &\quad = \sum_{u \in \mathbf{Z}} n^{-\frac{1}{\alpha}} \log \varphi\Big(n^{-(1-\frac{1}{\alpha})} \sum_{j=1}^{k} N_{nt_j}(u) \theta_j\Big). \end{aligned}$$

Thus it is enough to show that

$$\sum_{u\in\mathbf{Z}} n^{-\frac{1}{\alpha}} f\Big(n^{-(1-\frac{1}{\alpha})}\sum_{j=1}^{k} N_{nt_j}(u)\theta_j\Big) \xrightarrow{w} \int_{-\infty}^{\infty} f\Big(\sum_{j=1}^{k} L_{t_j}(u)\theta_j\Big)\,du. \tag{8}$$

The following argument is very similar to that in [KS]. For some small $\tau > 0$ and large M, define

$$A_{n,l} = \{u \in \mathbf{Z} : l\tau n^{\frac{1}{\alpha}} \le u < (l+1)\tau n^{\frac{1}{\alpha}}\}, \quad l \in \mathbf{Z},$$

$$U(\tau, M, n) = \sum_{|u|>M\tau n^{\frac{1}{\alpha}}} n^{-\frac{1}{\alpha}} f\Big(n^{-(1-\frac{1}{\alpha})}\sum_{j=1}^{k} N_{nt_j}(u)\theta_j\Big)$$

and

$$V(\tau, M, n) = \sum_{|l|\le M} |A_{n,l}|\, n^{-\frac{1}{\alpha}} f\Big(n^{-(1-\frac{1}{\alpha})}\frac{1}{\tau n^{\frac{1}{\alpha}}}\sum_{y\in A_{n,l}}\sum_{j=1}^{k} N_{nt_j}(y)\theta_j\Big),$$

where $|A_{n,l}|$ is the number of integers in $A_{n,l}$. Then

$$\begin{aligned} I &:= \sum_{u\in\mathbf{Z}} n^{-\frac{1}{\alpha}} f\Big(n^{-(1-\frac{1}{\alpha})}\sum_{j=1}^{k} N_{nt_j}(u)\theta_j\Big) - U(\tau, M, n) - V(\tau, M, n) \\ &= \sum_{|l|\le M}\sum_{u\in A_{n,l}} n^{-\frac{1}{\alpha}} \Big\{ f\Big(n^{-(1-\frac{1}{\alpha})}\sum_{j=1}^{k} N_{nt_j}(u)\theta_j\Big) \\ &\qquad - f\Big(n^{-(1-\frac{1}{\alpha})}\frac{1}{\tau n^{\frac{1}{\alpha}}}\sum_{y\in A_{n,l}}\sum_{j=1}^{k} N_{nt_j}(y)\theta_j\Big)\Big\}. \end{aligned}$$

Set, for a moment,

$$g_j = N_{nt_j}(u) \quad \text{and} \quad h_j = \frac{1}{\tau n^{\frac{1}{\alpha}}}\sum_{y\in A_{n,l}} N_{nt_j}(y).$$

By Lemma 4,

$$\begin{aligned} E[|I|] &\le C(2M+1)\,|A_{n,l}|\, n^{-\frac{1}{\alpha}} \sup_{u\in A_{n,l}} \Big\{ E\Big[n^{-(1-\frac{1}{\alpha})}\Big\|\sum_{j=1}^{k}(g_j - h_j)\theta_j\Big\| \\ &\qquad \Big(1 + n^{-(1-\frac{1}{\alpha})}\Big\|\sum_{j=1}^{k} g_j\theta_j\Big\| + n^{-(1-\frac{1}{\alpha})}\Big\|\sum_{j=1}^{k} h_j\theta_j\Big\|\Big)\Big] \\ &\qquad + E\Big[n^{-\beta(1-\frac{1}{\alpha})}\Big\|\sum_{j=1}^{k}(g_j - h_j)\theta_j\Big\|^{\beta}\Big]\Big\} \\ &\le CM\tau \sup_{u\in A_{n,l}} \Big\{ n^{-(1-\frac{1}{\alpha})}\Big(E\Big[\Big\|\sum_{j=1}^{k}(g_j - h_j)\theta_j\Big\|^2\Big]\Big)^{1/2} \\ &\qquad \Big(1 + n^{-2(1-\frac{1}{\alpha})} E\Big[\Big\|\sum_{j=1}^{k} g_j\theta_j\Big\|^2\Big] + n^{-2(1-\frac{1}{\alpha})} E\Big[\Big\|\sum_{j=1}^{k} h_j\theta_j\Big\|^2\Big]\Big)^{1/2} \end{aligned}$$

$$+ n^{-\beta(1-\frac{1}{\alpha})}\left(E\left[\left\|\sum_{j=1}^{k}(g_j-h_j)\theta_j\right\|^2\right]\right)^{\beta/2}\Bigg\}$$

$$\le CM\tau \sup_{u\in A_{n,l}}\Bigg\{n^{-(1-\frac{1}{\alpha})}\left(E\left[\left\|\sum_{j=1}^{k}(g_j-h_j)^2\right]\sum_{j=1}^{k}\|\theta_j\|^2\right)^{1/2}$$

$$\left(1+n^{-2(1-\frac{1}{\alpha})}E\left[\sum_{j=1}^{k}g_j^2\right]\sum_{j=1}^{k}\|\theta_j\|^2\right.$$

$$\left.+ n^{-2(1-\frac{1}{\alpha})}E\left[\sum_{j=1}^{k}h_j^2\right]\sum_{j=1}^{k}\|\theta_j\|^2\right)^{1/2}$$

$$+ n^{-\beta(1-\frac{1}{\alpha})}\left(E\left[\sum_{j=1}^{k}(g_j-h_j)^2\right]\sum_{j=1}^{k}\|\theta_j\|^2\right)^{\beta/2}\Bigg\}.$$

In [KS], it is proved that

$$\sup_{u\in A_{n,l}} E[|g_j-h_j|^2] \le C\tau^{\alpha-1}n^{2-\frac{2}{\alpha}}.$$

Also by (6) in Lemma 2,

$$\sup_{u\in\mathbf{Z}} E[N_n(u)^2] = O(n^{2-\frac{2}{\alpha}}).$$

Hence we have

$$E[|I|] \le CM(\tau^{\frac{\alpha}{2}+\frac{1}{2}} + \tau^{1+\frac{\beta}{2}(\alpha-1)}) = CM\tau(\tau^{\frac{1}{2}(\alpha-1)} + \tau^{\frac{\beta}{2}(\alpha-1)}).$$

As to $U(\tau, M, n)$, as in [KS], we see that for large n and for each $\eta > 0$, we can take $M\tau$ so large that

$$P\{U(\tau, M, n) \ne 0\} \le \eta.$$

Recall $\alpha > 1$. Then take τ so small that

$$CM\tau(\tau^{\frac{1}{2}(\alpha-1)} + \tau^{\frac{\beta}{2}(\alpha-1)}) \le \eta^2.$$

Then we can conclude that for such τ, M and large n,

$$P\left\{\left|\sum_{u\in\mathbf{Z}} f\left(n^{-D^*}\sum_{j=1}^{k}N_{nt_j}(u)\theta_j\right) - V(\tau, M, n)\right| > \eta\right\} \le 2\eta. \tag{9}$$

By the above consideration, it is enough to show the convergence of $V(\tau, M, n)$ in order to prove the lemma. By the use of the notation and the statement of Lemma 1, we have

$$V(\tau, M, n) = \sum_{|l|\le M}\frac{|A_{n,l}|}{n^{\frac{1}{\alpha}}} f\left(\frac{1}{\tau}\sum_{j=1}^{k}T_{t_j}^n(l\tau, (l+1)\tau)\theta_j\right)$$

which, as $n \longrightarrow \infty$, weakly converges to

$$\tau \sum_{|l| \leq M} f\Big(\sum_{j=1}^{k} \frac{1}{\tau} \int_{l\tau}^{(l+1)\tau} L_{t_j}(y) dy \theta_j\Big), \tag{10}$$

where we have used $\dfrac{|A_{n,l}|}{n^{\frac{1}{\alpha}}} \longrightarrow \tau$.

Finally, the continuity of $\sum_{j=1}^{k} L_{t_j}(u)\theta_j$ as a function of u and the fact that $L_{t_j}(\cdot)$ has a.s. compact support imply that as $\tau \rightarrow 0$ and $M \rightarrow \infty$, (10) converges to

$$\int_{-\infty}^{\infty} f\Big(\sum_{j=1}^{k} L_{t_j}(u)\theta_j\Big)\, du.$$

This together with (9) shows (8), completing the proof of the lemma. □

We now return to the proof of the theorem. Denote the characteristic function of $\xi(u)$ by

$$\lambda(\theta) = E[e^{i\langle\theta,\xi(u)\rangle}], \quad \theta \in \mathbf{R}^d.$$

Then by (5)

$$\begin{aligned} I_n := E\left[\exp\Big\{i \sum_{j=1}^{k} \langle\theta_j, n^{-D} W_{nt_j}\rangle\Big\}\right] \\ = E\left[\exp\Big\{i \sum_{j=1}^{k} \langle\theta_j, n^{-D} \sum_{u\in\mathbf{Z}} N_{nt_j}(u)\xi(u)\rangle\Big\}\right] \\ = E\left[\prod_{u\in\mathbf{Z}} \lambda\Big(n^{-D^*} \sum_{j=1}^{k} N_{nt_j}(u)\theta_j\Big)\right]. \end{aligned} \tag{11}$$

We need more lemmas.

LEMMA 6.

$$\lim_{n\to\infty} \sup_{u\in\mathbf{Z}} N_n(u) n^{-D^*}\theta = 0 \textit{ in probability.}$$

Proof. By (6) and (7) in Lemma 2, we have for some $p \geq 1$,

$$\begin{aligned} & P\Big\{\sup_{u\in\mathbf{Z}} N_n(u)\, \| n^{-D^*}\theta \| > \eta\Big\} \\ & \leq P\{N_n(u) > 0 \text{ for some } u \text{ with } |u| > An^{\frac{1}{\alpha}}\} \\ & \qquad + P\Big\{\sup_{|u|\leq An^{\frac{1}{\alpha}}} N_n(u)\, \| n^{-D^*}\theta \| > \eta\Big\} \end{aligned}$$

$$\leq \varepsilon(A) + \sum_{|u| \leq An^{\frac{1}{\alpha}}} \frac{1}{\eta^p} E[N_n(u)^p] \| n^{-D^*} \theta \|^p$$

$$= \varepsilon(A) + \sum_{|u| \leq An^{\frac{1}{\alpha}}} \frac{1}{\eta^p} O(n^{p(1-\frac{1}{\alpha})}) n^{-p(1-\frac{1}{\alpha})} \| n^{-\frac{1}{\alpha}B^*} \theta \|^p$$

$$= \varepsilon(A) + O(n^{\frac{1}{\alpha}} \| n^{-\frac{1}{\alpha}B^*} \theta \|^p). \tag{12}$$

Since for any $\varepsilon > 0$,

$$\| n^{-\frac{1}{\alpha}B^*} \| \leq C n^{-\frac{1}{\alpha}(\lambda_B - \varepsilon)},$$

if we take p such that $\frac{1}{\alpha} - \frac{1}{\alpha}(\lambda_B - \varepsilon)p < 0$, the last term in (12) converges to 0 for fixed η and A. If we next let A tend to infinity, then the desired conclusion follows. □

LEMMA 7 (Lemma 6.1 of [MM]). Under (3), $\log \lambda(\theta) \sim \log \varphi(\theta)$ as $\theta \to 0$.

We now return to (11). By Lemmas 6 and 7,

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} E\left[\prod_{u\in\mathbf{Z}} \varphi\left(n^{-D^*} \sum_{j=1}^{k} N_{nt_j}(u)\theta_j\right)\right]$$

$$= \lim_{n\to\infty} E\left[\exp\left\{\sum_{u\in\mathbf{Z}} f\left(n^{-D^*} \sum_{j=1}^{k} N_{nt_j}(u)\theta_j\right)\right\}\right]$$

$$= E\left[\exp\left\{\int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u)\theta_j\right) du\right\}\right] \text{ (by Lemma 5)}$$

$$= E\left[\exp\left\{i \sum_{j=1}^{k} \langle \theta_j, \Delta(t_j) \rangle\right\}\right] \text{ (by Lemma 3).}$$

The proof of Theorem 1 is thus completed.

3. Proof of Theorem 2

We prove the tightness of $\{n^{-D} W_{nt}\}$ by showing that for each $T > 0$ and any $\eta > 0$

$$\lim_{n\to\infty} \limsup_{\delta \downarrow 0} P\{ \sup_{\substack{0 \leq t,s \leq T \\ |t-s| \leq \delta}} \| \Delta_t^n - \Delta_s^n \| \geq \eta \} = 0, \tag{13}$$

where $\Delta_t^n = n^{-D} W_{nt}$. To this end, as in [KS], we first approximate Δ_t^n by $\bar{\Delta}_t^n$ plus a linear function $E_n t$ such that $\bar{\Delta}_t^n$ has the second moments, E_n are bounded and

$$\limsup_{n\to\infty} P\Big\{\sup_{t\le T} \| \Delta_t^n - \bar{\Delta}_t^n - E_n t\| \ge \frac{1}{2}\eta\Big\} \le \frac{\varepsilon}{2},$$

and then use Kolmogorov's moment criteria for $\bar{\Delta}_t^n$.

For any $\varepsilon > 0$, choose large A such that $\varepsilon(AT^{-\frac{1}{\alpha}}) \le \frac{\varepsilon}{4}$, where $\varepsilon(\cdot)$ is the one defined in (7) in Lemma 2. Then we have

$$\begin{aligned}(14)\qquad & P\{N_{nt}(u) > 0 \text{ for some } |u| > An^{\frac{1}{\alpha}} \text{ and } t \le T\}\\ &\le P\{N_{nt}(u) > 0 \text{ for some } |u| > An^{\frac{1}{\alpha}}\}\\ &\le \varepsilon(AT^{-\frac{1}{\alpha}}) \le \frac{\varepsilon}{4}.\end{aligned}$$

We need several lemmas, where we always assume (3). For notational simplicity, we write ξ for $\xi(0)$ in the following. Let

$$c_n(G) = nP\{\| n^{-B}\xi \| \in G\}, \quad G \in \mathfrak{B}((0, \infty)),$$

$$M(F) = \int_S \gamma(dx) \int_0^\infty I_F(s^B x) \frac{1}{s^2} ds, \quad F \in \mathfrak{B}(\mathbf{R}^d \setminus \{0\})$$

and

$$c(G) = M(\{x : \| x \| \in G\}), \quad G \in \mathfrak{B}((0, \infty)).$$

Note that under (3), by the general central limit theorem for infinitely divisible laws in $\mathbf{R}^d$ (cf. Proposition 1.8.17 in [JM]),

$$nP\{n^{-B}\xi \in F\} \to M(F)$$

for every Borel set F which is bounded away from the origin and $M(\partial F) = 0$, and

$$(15)\qquad \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} n\int_{\|x\|<\varepsilon} \langle\theta, x\rangle^2 P\{n^{-B}\xi \in dx\} = 0, \quad \theta \in \mathbf{R}^d.$$

(Recall that we are dealing with purely non-Gaussian case.) Assume for a moment that $\| \cdot \|$ is the "invariant norm" of [HJV]. In their norm, $c(\{y\}) = 0$ for each $y > 0$. Then by eq. (7) in [W], we have

LEMMA 8. *For every $y > 0$,*

$$c_n([y, \infty)) \to c([y, \infty)).$$

LEMMA 9. (i) *Let $\rho > 0$. Then*

$$\sup_n \int_0^\rho y^2 c_n(dy) < \infty.$$

(ii)

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \int_0^\varepsilon y^2 c_n(dy) < \infty.$$

Proof. Suppose $\{\theta_1, \ldots, \theta_d\}$ is an orthonormal basis for $\mathbf{R}^d$. Then $\| x \|^2 = \sum_{i=1}^d \langle \theta_i, x \rangle^2$. Since

$$\int_0^\varepsilon y^2 c_n(dy) = n \int_{\|x\|<\varepsilon} \| x \|^2 P\{n^{-B}\xi \in dx\},$$

we conclude the lemma by (15) with $\theta = \theta_1, \ldots, \theta_d$. □

LEMMA 10. *Let* $\rho > 0$.
(i) *If* $\lambda_B > 1$, *then*

$$\int_0^\rho yc(dy) < \infty.$$

(ii) *If* $\Lambda_B < 1$, *then*

$$\int_\rho^\infty yc(dy) < \infty.$$

Proof. We have

$$c((y, \infty)) = M(\{x : \| x \| > y\}) = \int_S \gamma(dx) \int_0^\infty I[\| s^B x \| > y] \frac{1}{s^2} ds.$$

Note that for any $\delta > 0$ there exists $C_1 > 0$ such that

$$\| s^B \| \le \begin{cases} C_1 s^{\lambda_B - \delta} & \text{if } s \le 1, \\ C_1 s^{\Lambda_B + \delta} & \text{if } s > 1. \end{cases} \tag{16}$$

By the use of (16), we have

$$\begin{aligned} c((y, \infty)) &\le \int_0^1 I[s > C_2 y^{1/(\lambda_B - \delta)}] \frac{1}{s^2} ds \\ &\quad + \int_1^\infty I[s > C_2 y^{1/(\Lambda_B + \delta)}] \frac{1}{s^2} ds \\ &=: I_1(y) + I_2(y), \end{aligned}$$

for some $C_2 > 0$.

(i) As $y \to 0$, $I_2(y) = O(1)$ and $I_1(y) = O(y^{-1/(\lambda_B - \delta)})$. If $\lambda_B > 1$, we can find $\delta > 0$ such that $1/(\lambda_B - \delta) < 1$. Thus $\int_0^\rho c((y, \infty))dy < \infty$, which concludes (i).

(ii) As $y \to \infty$, $I_1(y) = o(1)$ and $I_2(y) = O(y^{-1/(\Lambda_B + \delta)})$. Thus, if $\Lambda_B < 1$, we have $\int_\rho^\infty c((y, \infty))dy < \infty$, concluding (ii). □

LEMMA 11. *Let $\rho > 0$. If $\lambda_B > 1$, then*

$$\sup_n \int_0^\rho y c_n(dy) < \infty.$$

Proof. It is obvious that for every $n \geq 1$

$$\int_0^\rho y c_n(dy) < \infty,$$

and also

$$\int_0^\rho y c(dy) < \infty$$

by Lemma 10 (i). Note that $c_n(\cdot)$ and $c(\cdot)$ are Lévy measures on $(0, \rho)$, namely $\int_0^\rho (y^2 \wedge 1) c_n(dy) < \infty$ and $\int_0^\rho (y^2 \wedge 1) c(dy) < \infty$. Hence, by Lemmas 8 and 9 (ii), a convergence theorem of infinitely divisible laws (cf. Corollary 1.8.16 in [JM]) implies that the characteristic function

$$f_n(\theta) := \exp\left\{\int_0^\rho (e^{i\theta y} - 1) c_n(dy)\right\}, \quad \theta \in \mathbf{R}$$

converges to

$$f(\theta) := \exp\left\{\int_0^\rho (e^{i\theta y} - 1) c(dy)\right\}, \quad \theta \in \mathbf{R}.$$

Thus

$$\lim_{n\to\infty} \int_0^\rho (e^{i\theta y} - 1) c(dy)$$

exists. This together with Lemma 9 (i) concludes the lemma. □

LEMMA 12. *Let $\rho > 0$. If $\Lambda_B < 1$, then*

$$\sup_n \int_\rho^\infty y c_n(dy) < \infty.$$

Proof. We first show the statement when ξ is symmetric. Let $\varepsilon > 0$, and choose a so large that

$$2P\left\{\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\| > a\right\} < \varepsilon \text{ for all } n,$$

which is possible by tightness, (see eq. (3)). Thus

$$2P\left\{\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\| > y\right\} < \varepsilon \text{ for all } y \geq a \text{ and for all } n.$$

Since $\{\xi(k)\}$ are symmetric, we have

$$P\left\{\max_{1 \leq k \leq n} \| n^{-B}\xi(k) \| > y\right\} \leq 2P\left\{\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\| > y\right\}.$$

Thus

$$\begin{aligned}[P\{\| n^{-B}\xi \| \leq y\}]^n &= P\left\{\max_{1 \leq k \leq n} \| n^{-B}\xi(k) \| \leq y\right\} \\ &= 1 - P\left\{\max_{1 \leq k \leq n} \| n^{-B}\xi(k) \| > y\right\} \\ &\leq 1 - 2P\left\{\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\| > y\right\}\end{aligned}$$

so that, for any $y \geq a$

$$\begin{aligned}nP\{\| n^{-B}\xi \| > y\} &\leq n\left\{1 - \left[1 - 2P\left\{\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\| > y\right\}\right]^{1/n}\right\} \\ &\leq \frac{2}{1-\varepsilon} P\left\{\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\| > y\right\},\end{aligned}$$

since for a fixed $\varepsilon < 1$,

$$n\{1 - (1-x)^{1/n}\} \leq \frac{1}{1-\varepsilon} x, \quad \text{for any } 0 \leq x < \varepsilon.$$

Hence

$$\begin{aligned}\sup_n \int_a^\infty nP\{\| n^{-B}\xi \| > y\} dy \\ &\le \frac{2}{1-\varepsilon} \sup_n \int_a^\infty P\left\{\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\| > y\right\} dy \\ &\le \frac{2}{1-\varepsilon} \sup_n E\left[\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\|\right].\end{aligned}$$

By Theorem 3 in [HVW], if $\| \cdot \|$ is the ordinary Euclidean norm and $\Lambda_B < 1$,

$$E\left[\left\| n^{-B} \sum_{k=1}^n \xi(k) \right\|\right] \to E[\| Z_B \|]$$

and hence

$$\sup_n \int_a^\infty nP\{\| n^{-B}\xi \| > y\} dy < \infty$$

for the "invariant norm" as well as for the ordinary Euclidean norm. This implies

$$\sup_n \int_a^\infty y c_n(dy) < \infty.$$

On the other hand

$$\int_\rho^a y c_n(dy) \to \int_\rho^a y c(dy)$$

by Lemma 8, thus we conclude

$$\sup_n \int_a^\infty y c_n(dy) < \infty$$

when ξ is symmetric.

It remains to prove the lemma for the non-symmetric case and the following argument is a standard desymmetrization. For general ξ, let ξ' be an independent copy of ξ. Since $\xi - \xi'$ is symmetric, we have shown

$$\sup_n \int_\rho^\infty nP\left\{\| n^{-B}(\xi - \xi') \| > \frac{y}{2}\right\} dy =: K < \infty.$$

Let

$$g_n(z) = \int_\rho^\infty nP\left\{\| n^{-B}(\xi - z) \| > \frac{y}{2}\right\}dy.$$

Then

$$\sup_n E[g_n(\xi')] = K.$$

Let b be so large that $P\{\| \xi \| > b\} < \frac{1}{2}$. Also let

$$B_b = \{x \in \mathbf{R}^d : \| x \| \leq b\}$$

and

$$G_n = \{z \in \mathbf{R}^d : g_n(z) \leq 3K\}.$$

Then B_b is not contained in $\mathbf{R}^d \setminus G_n$, because if it were, we would have

$$K = \sup_n E[g_n(\xi')] \geq \sup_n E[g_n(\xi')I[\xi' \in B_b]]$$
$$> 3KE[I[\xi' \in B_b]] = 3KP\{\| \xi \| \leq b\} > \frac{3}{2}K,$$

which is impossible. Hence $B_b \cap G_n \neq \phi$. Let $z_n \in B_b \cap G_n$ for each $n \geq 1$. Since $\| z_n \| \leq b$, we have

$$\int_\rho^\infty nP\{\| n^{-B}\xi \| > y\}dy \leq g_n(z_n)$$

for large n. Since $g_n(z_n) \leq 3K$, the proof is complete. □

Remark. Lemmas 9-12 have been proved for the "invariant norm" of [HJV]. However, the compatibility of all norms on $\mathbf{R}^d$ implies the same conclusions for the ordinary Euclidean norm.

LEMMA 13. *If $\Lambda_B < 1$, then $E[\| \xi \|] < \infty$ and $E[\xi] = 0$.*

Proof. The first part follows from Theorem 3 in [HVW]. The second part can be shown by the same way as in the one-dimensional case. □

By Lemma 8, we can find a ρ such that for all large n

$$(2An^{\frac{1}{\alpha}} + 1)P\{\| n^{-\frac{1}{\alpha}B}\xi \| > \rho\} \leq \frac{\varepsilon}{4}, \tag{17}$$

for the "invariant norm". By the compatibility of all norms on $\mathbf{R}^d$ again, the same observation also follows for the ordinary Euclidean norm. In the following, once again the norm $\|\cdot\|$ stands for the ordinary Euclidean norm.

Set

$$\bar{\xi}(u) = \xi(u) I[\| n^{-\frac{1}{\alpha}B}\xi(u) \| \leq \rho],$$
$$E_n = n^{-D} E\left[\sum_{u\in\mathbf{Z}} N_n(u)\bar{\xi}(u)\right]$$

and

$$\bar{\Delta}_t^n = n^{-D} \sum_{u\in\mathbf{Z}} N_{nt}(u)\{\bar{\xi}(u) - E[\bar{\xi}(u)]\}.$$

Again, for notational simplicity, we write $\bar{\xi}$ for $\bar{\xi}(0)$ in the following.

LEMMA 14. *We have*

$$\| E[n^{-\frac{1}{\alpha}B}\bar{\xi}]\| = O(n^{-\frac{1}{\alpha}}), \tag{18}$$

provided that ξ is symmetric when $\lambda_B \leq 1 \leq \Lambda_B$.

Proof. When ξ is symmetric, the left hand side of (18) is 0. Hence it is enough to consider the case $\lambda_B > 1$ or $\Lambda_B < 1$.

When $\lambda_B > 1$,

$$\begin{aligned}\sup_n n^{\frac{1}{\alpha}} \| E[n^{-\frac{1}{\alpha}B}\bar{\xi}]\| &= \sup_n n^{\frac{1}{\alpha}} \| E[n^{-\frac{1}{\alpha}B}\xi I[\| n^{-\frac{1}{\alpha}B}\xi \| \leq \rho]]\| \\ &\leq \sup_n \int_0^\rho y c_{n^{1/\alpha}}(dy) < \infty\end{aligned}$$

by Lemma 11.

When $\Lambda_B < 1$, by the use of Lemmas 12 and 13,

$$\begin{aligned}\sup_n n^{\frac{1}{\alpha}} \| E[n^{-\frac{1}{\alpha}B}\bar{\xi}]\| &= \sup_n n^{\frac{1}{\alpha}} \| E[n^{-\frac{1}{\alpha}B}\xi I[\| n^{-\frac{1}{\alpha}B}\xi \| \leq \rho]]\| \\ &= \sup_n n^{\frac{1}{\alpha}} \| E[n^{-\frac{1}{\alpha}B}\xi I[\| n^{-\frac{1}{\alpha}B}\xi \| > \rho]]\| \\ &\leq \sup_n \int_\rho^\infty y c_{n^{1/\alpha}}(dy) < \infty.\end{aligned}$$

This concludes the lemma. □

Let us return to the proof of Theorem 2. We have by Lemma 14,

$$\begin{aligned}\| E_n \| &= \left\| n^{-(1-\frac{1}{\alpha})} n^{-\frac{1}{\alpha}B} E\left[\sum_{u\in\mathbf{Z}} N_n(u)\bar{\xi}(u)\right]\right\| \\ &= \left\| n^{-(1-\frac{1}{\alpha})} E[n^{-\frac{1}{\alpha}B}\bar{\xi}] E\left[\sum_{u\in\mathbf{Z}} N_n(u)\right]\right\| \\ &= n^{-(1-\frac{1}{\alpha})} O(n^{-\frac{1}{\alpha}})(n+1) = O(1).\end{aligned}$$

We also have

$$\begin{aligned}&\Delta_t^n - \bar{\Delta}_t^n - E_n t \\ &= n^{-D}\sum_{u\in\mathbf{Z}} N_{nt}(u)[\xi(u) - (\bar{\xi}(u) - E[\bar{\xi}(u)])] - n^{-D}E\left[\sum_{u\in\mathbf{Z}} N_n(u)\bar{\xi}(u)\right] t \\ &= n^{-D}\sum_{u\in\mathbf{Z}} N_{nt}(u)[\xi(u) - \bar{\xi}(u)] + n^{-D}\sum_{u\in\mathbf{Z}} N_{nt}(u)E[\bar{\xi}(u)] \\ &\qquad - n^{-D}E\left[\sum_{u\in\mathbf{Z}} N_n(u)\bar{\xi}(u)\right] t\end{aligned}$$

$$=: n^{-D}\sum_{u\in\mathbf{Z}} N_{nt}(u)[\xi(u) - \bar{\xi}(u)] + Q_n(t), \tag{19}$$

where by Lemma 14 for $t \leq T$,

$$\begin{aligned}\| Q_n(t) \| &= \| n^{-D}E[\bar{\xi}](nt + 1 - (n+1)t) \| \\ &\leq Tn^{-(1-\frac{1}{\alpha})} \| E[n^{-\frac{1}{\alpha}B}\bar{\xi}] \| = O\left(\frac{1}{n}\right).\end{aligned}$$

It follows from (14) and (17) that

$$\begin{aligned}&P\{\sum_{u\in\mathbf{Z}} N_{nt}(u)[\xi(u) - \bar{\xi}(u)] \neq 0 \text{ for some } t \leq T\} \\ &\leq P\{\xi(u) \neq \bar{\xi}(u) \text{ for some } |u| \leq An^{\frac{1}{\alpha}}\} \\ &\qquad + P\{N_{nt}(u) > 0 \text{ for some } |u| > An^{\frac{1}{\alpha}} \text{ and } t \leq T\} \\ &\leq (2An^{\frac{1}{\alpha}} + 1)P\{\| n^{-\frac{1}{\alpha}B}\xi \| > \rho\} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}.\end{aligned}$$

Hence by (19) for any $\eta > 0$,

$$\limsup_{n\to\infty} P\left\{\sup_{t\leq T} \| \Delta_t^n - \bar{\Delta}_t^n - E_n t \| \geq \frac{1}{2}\eta\right\} \leq \frac{\varepsilon}{2}. \tag{20}$$

We finally show

$$E[\| \bar{\Delta}_t^n - \bar{\Delta}_s^n \|^2] \leq C(t-s)^{2-\frac{1}{\alpha}}. \tag{21}$$

If we could show (21), the relation (13), with the respective replacements of $\varDelta_t^n$ and η by $\bar{\varDelta}_t^n$ and $\frac{\eta}{2}$, would follow, and it together with (20) implies (13). We have

$$(22)\quad E[\|\bar{\varDelta}_t^n-\bar{\varDelta}_s^n\|^2]$$
$$=E\left[\left\| n^{-D}\sum_{u\in\mathbf{Z}}(N_{nt}(u)-N_{ns}(u))(\bar{\xi}(u)-E[\bar{\xi}(u)])\right\|^2\right]$$
$$=\sum_{u\in\mathbf{Z}}E[(N_{nt}(u)-N_{ns}(u))^2]\,n^{-2(1-\frac{1}{\alpha})}E[\|n^{-\frac{1}{\alpha}B}(\bar{\xi}(0)-E[\bar{\xi}(0)])\|^2]$$
$$\leq\sum_{u\in\mathbf{Z}}E[(N_{nt}(u)-N_{ns}(u))^2]\,n^{-2(1-\frac{1}{\alpha})}E[\|n^{-\frac{1}{\alpha}B}\bar{\xi}(0)\|^2],$$

where

$$\sup_n n^{\frac{1}{\alpha}}E[\|n^{-\frac{1}{\alpha}B}\bar{\xi}\|^2]=\sup_n n^{\frac{1}{\alpha}}E[\|n^{-\frac{1}{\alpha}B}\xi\|^2 I[\|n^{-\frac{1}{\alpha}B}\xi\|\leq\rho]]$$
$$(23)\qquad =\sup_n\int_0^\rho y^2 c_{n^{1/\alpha}}(dy)<\infty$$

by Lemma 9. On the other hand, Kesten and Spitzer ([KS]) showed

$$(24)\qquad \sum_{u\in\mathbf{Z}}E[(N_{nt}(u)-N_{ns}(u))^2]\leq C[(t-s)n]^{2-\frac{1}{\alpha}}.$$

Thus (21) is given from (22)-(24) and the proof is completed.

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