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HYPONORMAL TOEPLITZ OPERATORS ON $H^2(T)$ WITH POLYNOMIAL SYMBOLS

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Let T be the unit circle on the complex plane, $H^2(T)$ be the usual Hardy space on T, T_ϕ be the Toeplitz operator with symbol $\phi \in L^\infty(T)$, C. Cowen showed that if f_1 and f_2 are functions in H^2 such that $f = f_1 + \bar{f}_2$ is in L^∞, then T_f is hyponormal if and only if $f_2 = c + T_{\bar{g}} f_1$ for some constant c and some function g in H^∞ with $\|g\|_\infty \leq 1$ [1]. Using it, T. Nakazi and K. Takahashi showed that the symbol of hyponormal Toeplitz operator T_ϕ satisfies $\phi - g = k\bar{\phi}$, $g \in H^\infty$ and $k \in H^\infty$ with $\|k\| \leq 1$ [2], and they described the ϕ solving the functional equation above. Both of their conditions are hard to check, T. Nakazi and K. Takahashi remarked that even "the question about polynomials is still open" [2]. Kehe Zhu gave a computing process by way of Schur's functions so that we can determine any given polynomial ϕ such that T_ϕ is hyponormal [3]. Since no closed-form for the general Schur's function is known, it is still valuable to find an explicit expression for the condition of a polynomial ϕ such that T_ϕ is hyponormal and depends only on the coefficients of ϕ, here we have one, it is elementary and relatively easy to check. We begin with the most general case and the following Lemma is essential.

LEMMA 1. *If $f, g \in H^2(T)$ and $\phi = f + \bar{g} \in L^\infty(T)$, then T_ϕ is hyponormal if and only if the (bounded) operator A on l^2*

$$(1) \qquad A = (A_{ij}) \equiv (A_f(i, j) - A_g(i, j)) \equiv (\langle S^{*^j} f, S^{*^i} f\rangle - \langle S^{*^j} g, S^{*^i} g\rangle) \quad i, j \geqq 1$$

is non-negative where S refers to the unilateral shift on $H^2(T)$.

Proof. By definition T_ϕ is hyponormal when $T_\phi^* T_\phi - T_\phi T_\phi^* \geqq 0$, i.e. $(T_{f+\bar{g}})^* T_{f+\bar{g}} - T_{f+\bar{g}}(T_{f+\bar{g}})^* = (T_f^* T_f - T_f T_f^*) - (T_g^* T_g - T_g T_g^*) \geqq 0$, the Lemma

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is no other than to find out the matrix form of $T_\phi{}^*T_\phi - T_\phi T_\phi{}^*$.

Put $f = \sum_{k=0}^{\infty} f_k z^k$, $g = \sum_{l=0}^{\infty} g_l z^l$, let $\{z^n\}_{n=1}^{\infty}$ be the natural base for $H^2(T)$ since

$$T_f{}^*T_f - T_f T_f{}^* = H_{\bar f}{}^* H_{\bar f} \tag{2}$$

where $H_{\bar f}$ refers to the Hankel operator with symbol $\bar f$ (consult [4] for the definition and related properties of a Hankel operator), for any pair of non-negative integers i, j, $i \geq j$, we have

$$\begin{aligned} &\langle (T_f{}^*T_f - T_f T_f{}^*) z^j, z^i \rangle = \langle H_{\bar f}{}^* H_{\bar f} z^j, z^i \rangle \\ &= \langle H_{\bar f} z^j, H_{\bar f} z^i \rangle_{L^2(T)} = \langle \sum_{l=j+1}^{\infty} \bar f_l z^{j-l}, \sum_{k=i+1}^{\infty} \bar f_k z^{i-k} \rangle_{L^2(T)} \\ &= \sum_{k=j+1}^{\infty} \bar f_k f_{i-j+k} \end{aligned} \tag{3}$$

since $T_f{}^*T_f - T_f T_f{}^*$ is self-adjoint (We temporarily disregard the boundedness of T_f, since $\{z^n\}_{n=0}^{\infty}$ are obviously in H^∞, the above computation has no problem). The element of the upper half of the matrix A_f is $\sum_{l=j+1}^{\infty} \bar f_{l+i-j} f_l$ respectively, thus we have

$$A_f = \begin{pmatrix} \sum_{l=1}^{\infty} |f_l|^2, & \sum_{l=2}^{\infty} \bar f_{l-1} f_l, & \sum_{l=3}^{\infty} \bar f_{l-2} f_l, & \sum_{l=4}^{\infty} \bar f_{l-3} f_l, \dots & \\ \sum_{l=2}^{\infty} f_{l-1} \bar f_1, & \sum_{l=2}^{\infty} |f_l|^2, & \sum_{l=3}^{\infty} \bar f_{l-1} f_l, & \sum_{l=4}^{\infty} \bar f_{l-2} f_l, \dots & \\ \sum_{l=3}^{\infty} f_{l-2} \bar f_l, & \sum_{l=3}^{\infty} f_{l-1} \bar f_l, & \sum_{l=3}^{\infty} |f_l|^2, & \sum_{l=4}^{\infty} f_{l-1} \bar f_l, \dots & \\ \sum_{l=4}^{\infty} f_{l-3} \bar f_l, & \sum_{l=4}^{\infty} f_{l-2} \bar f_l, & \sum_{l=4}^{\infty} f_{l-1} \bar f_l, & \sum_{l=4}^{\infty} |f_l|^2, \dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} \| S^* f \|^2, & \langle S^{*2} f, S^* f \rangle, & \langle S^{*3} f, S^* f \rangle, & \langle S^{*4} f, S^* f \rangle, & \cdots \\ \langle S^* f, S^{*2} \rangle, & \| S^{*2} f \|^2, & \langle S^{*3} f, S^{*2} f \rangle, & \langle S^{*4} f, S^{*2} f \rangle, & \cdots \\ \langle S^* f, S^{*3} f \rangle, & \langle S^{*2} f, S^{*3} f \rangle, & \| S^{*3} f \|^2, & \langle S^{*4} f, S^{*3} f \rangle, & \cdots \\ \langle S^* f, S^{*4} f \rangle, & \langle S^{*2} f, S^{*4} f \rangle, & \langle S^{*3} f, S^{*4} f \rangle, & \| S^{*4} f \|^2, \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

the Lemma is proved.

From the matrix form of $T_\phi{}^*T_\phi - T_\phi T_\phi{}^*$, we have an explanation for the fact that T_ϕ is hyponormal, the analytic part of ϕ must be in some sense "larger" than it's co-analytic part, namely we have

COROLLARY 1. *Suppose* $\phi \in L^\infty(T)$, $\phi = f + \bar g$, $f, g \in H^2(T)$ *and* T_ϕ *is hyponormal, then the following inequalities hold*

(5) $$\| S^{*^i} f \|^2 = \sum_{l=i}^{\infty} | f_l |^2 \geqq \| S^{*^i} g \|^2 = \sum_{l=i}^{\infty} | g_l |^2 \quad \forall i = 1,2,\cdots,$$

where S^* is the backward shift on $H^2(T)$.

Proof. It is enough to take $h \in H^2(T)$ such that the coefficient of z^n is zero for all n except $n = i$ where it equals 1 and compute $\langle (A_f - A_g)h, h \rangle$.

In particular, when f is a polynomial, we have the following

THEOREM 1. *If $T_{f+\bar{g}}$ is a hyponormal Toeplitz operator where $f = \sum_{k=0}^{n} f_k z^k$, $f_n \neq 0$, $g \in H^\infty$, then g must be a polynomial with order less or equal to n, $g = \sum_{l=0}^{n} g_l z^l$, and the finite matrix.*

(6) $$\begin{pmatrix} \sum_{l=1}^{n} (| f_l |^2 - | g_l |^2), & \sum_{l=1}^{n} (\bar{f}_{l-1} f_l - \bar{g}_{l-1} g_l), & \cdots, & \bar{f}_1 f_n - \bar{g}_1 g_n \\ \sum_{l=2}^{n} (f_{l-1} \bar{f}_l - g_{l-1} \bar{g}_l), & \sum_{l=2}^{n} (| f_l |^2 - | g_l |^2), & \cdots, & \bar{f}_2 f_n - \bar{g}_2 g_n \\ \sum_{l=3}^{n} (f_{l-2} \bar{f}_l - g_{l-2} \bar{g}_l), & \sum_{l=3}^{n} (f_{l-1} \bar{f}_l - g_{l-1} \bar{g}_l), & \cdots, & \bar{f}_3 f_n - \bar{g}_3 g_n \\ \sum_{l=4}^{n} (f_{l-3} \bar{f}_l - g_{l-3} \bar{g}_l), & \sum_{l=4}^{n} (f_{l-2} \bar{f}_l - g_{l-2} \bar{g}_l), & \cdots, & \bar{f}_4 f_n - \bar{g}_4 g_n \\ \cdots, & \cdots, & \cdots, & \cdots, \\ f_1 \bar{f}_n - g_1 \bar{g}_n, & f_2 \bar{f}_n - g_2 \bar{g}_n, & \cdots, & | f_n |^2 - | g_n |^2 \end{pmatrix}$$

is non-negative.

Proof. Since $S^{*^i} f \equiv 0 \ \forall \ i > n$ by Lemma 1, all the components in A_f are zeros except the first n rows and rays, so by Corollary 1, $g_k = 0 \ \forall \ k > n$, the rest of the proof is trivial. we are done.

We give some examples, they are Example 6 and a special case of Example 7 respectively in [3].

EXAMPLE 1. Put $\phi = a_0 + a_1 z + a_2 z^2 + \overline{b_0 + b_1 z + b_2 z^2}$ and

(7) $$A_2 = \begin{pmatrix} | a_1 |^2 + | a_2 |^2 - | b_1 |^2 - | b_2 |^2, & \bar{a}_1 a_2 - \bar{b}_1 b_2 \\ a_1 \bar{a}_2 - b_1 \bar{b}_2, & | a_2 |^2 - | b_2 |^2 \end{pmatrix}.$$

The non-negativity conditions of this matrix A_2 are

(8) $$\begin{aligned} &\text{(i)} \quad | a_1 |^2 + | a_2 |^2 \geqq | b_1 |^2 + | b_2 |^2 \text{ and } | a_2 |^2 \geqq | b_2 |^2, \\ &\text{(ii)} \quad (| a_1 |^2 + | a_2 |^2 - | b_1 |^2 - | b_2 |^2)(| a_2 |^2 - | b_2 |^2) - \\ &\qquad - (a_1 \bar{a}_2 - b_1 \bar{b}_2)(\bar{a}_1 a_2 - \bar{b}_1 b_2) \\ &\qquad = (| a_2 |^2 - | b_2 |^2)^2 - | a_1 b_2 - b_1 a_2 |^2 \geqq 0, \\ &\text{(iii)} \quad | a_2 |^2 \geqq | b_2 |^2 + | a_1 b_2 - b_1 a_2 |. \end{aligned}$$

It is easy to check (iii) implies (i) and (ii), so (iii) is the necessary and sufficient condition for that T_ϕ is hyponormal.

EXAMPLE 2. Put $\phi = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \overline{b_0 + b_1 z + b_2 z^2}$,

$$(9)\quad A_3 = \begin{pmatrix} |a_1|^2 + |a_2|^2 + |a_3|^2 - |b_1|^2 - |b_2|^2, & \bar{a}_1 a_2 - \bar{b}_1 b_2 + \bar{a}_2 a_3, & \bar{a}_1 a_3 \\ a_1 \bar{a}_2 - b_1 \bar{b}_2 + a_2 \bar{a}_3, & |a_2|^2 + |a_3|^2 - |b_2|^2, & \bar{a}_2 a_3 \\ a_1 \bar{a}_3, & a_2 \bar{a}_3, & |a_3|^2 \end{pmatrix}$$

and

$$(10)\quad \det A_3 = |a_3|^2 \begin{vmatrix} |a_1|^2 + |a_2|^2 + |a_3|^2 - |b_1|^2 - |b_2|^2, & \bar{a}_1 a_2 - \bar{b}_1 b_2 + \bar{a}_2 a_3, & \bar{a}_1 \\ a_1 \bar{a}_2 - b_1 \bar{b}_2 + a_2 \bar{a}_3, & |a_2|^2 + |a_3|^2 - |b_2|^2, & \bar{a}_2 \\ a_1, & a_2, & 1 \end{vmatrix}.$$

A computation shows that T_ϕ is hyponormal if and only if the following (11) is true.

$$(11)\quad |a_3|^2 \geqq |b_2|^2 + |a_3 b_1 - a_2 b_2|$$

Of course, we can give more examples (through routine computation), but I feel it probably looks more natural to give the condition in matrix form.

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