G. Cao
Nagoya Math. J.
Vol. 144 (1996), 171–177

A NOTE ON CHARACTERISTIC EQUATION OF TOEPLITZ OPERATORS ON THE SPACES A_k

GUANGFU CAO

1. Preliminaries

Let k be any integer, $k \geq 0$. The k-th Bergman measure on unit ball B of C^n, μ_k, is given by

$$d\mu_k = \frac{\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(k+1)}(1 - |w|^2)^k dv(w).$$

Note that μ_0 is simply normalized Lebesque measure on B. The k-th Bergman space, A_k, is defined as the space of analytic functions on B which are square integrable with respect to the measure μ_k. Note that $A_k = H^2(\mu_k)$, where $H^2(\mu_k)$ be the $L^2(\mu_k)$-closure of the ball algebra A, and that $A_j \subset A_k$ for $j \leq k$. The standard orthonormal base for A_k is given by

$$e^k_\alpha = c(a, k, n)z^\alpha = c(a, k, n)r_1^{\alpha_1}e^{i\alpha_1\theta_1} \cdots r_n^{\alpha_n}e^{i\alpha_n\theta_n}$$

where $c(\alpha, k, n)$ is a constant number such that $c(\alpha, k, n)\|z^\alpha\| = 1$. Let P_k denote the projection of $L^2(\mu_k)$ onto A_k. Note that $L^\infty(\mu_k) = L^\infty(B) = \{f : f \text{ is essentially bounded on } B \text{ with respect to Lebesque measure on } B\}$. Also $H^\infty(\mu_k)$, the *weak**-closure of the polynomials in z in $L^\infty(B)$, is the set $\{f : f \in L^\infty(B)$ and $fA_k \subseteq A_k\} = H^\infty$, the set of bounded analytic functions on B. For $f \in L^\infty(B)$, $\|f\|_\infty$ denotes the essential supremum of f on B. For any $\varphi \in L^\infty(B)$ and for any $k \geq 0$, we define a Toeplitz operator $T^{(k)}_\varphi : A_k \to A_k$ as follows:

$$T^{(k)}_\varphi f = P_k(\varphi f) \quad (f \in A_k).$$

It can be seen easily that

$$T^{(k)}_\varphi f(z) = \int_B \frac{\varphi(\zeta)f(\zeta)}{(1 - \langle z, w\rangle)^{k+n+1}} d\mu_k(\zeta)$$

Received November 8, 1994.

(consult Rudin [1]). The set of all bounded linear operator on A_k is written as $L(A_k)$, clearly, $T_\varphi^{(k)} \in L(A_k)$. It is well-known that equation $T_{\bar{z}}TT_z = T$ characterize the Toeplitz operators on Hardy space in one complex variable. A. M. Davie and N. P. Jewell [2] proved that $\sum_{i=1}^n T_{\bar{z}_i}TT_{z_i} = T$ characterizes Toeplitz operators on Hardy space of several complex variables. D. H. Yu and Sh. H. Sun [3] proved that $T \in L(H^2)$ is a Toeplitz operator iff equation $T_\eta^* TT_\eta = T$ is hold for each inner function η. In [4], for $n = 1$, N. P. Jewell raised the following.

PROBLEM. *Is there a set of operator equations which characterize Toeplitz operators on the weighted Bergman spaces of one complex variable?*

In next section, we answer negatively the problem.

2. Theorems

THEOREM 1. *Let B be a set of operator equations and A be the set of bounded linear operators on the k-th weighted Bergman space A_k which satisfy B. If A contains all Toeplitz operators on A_k, then A is weak*-dense in $L(A_k)$.*

Proof. If A is not weak*-dense in $L(A_k)$, there exists a nonzero trace class operator S such that $\mathrm{tr}(ST) = 0$ for any $T \in A$. Then there exist $\{f_t\}$ in A_k such that $S = \sum_{t=1}^\infty f_t \otimes e_t$, where $\{e_t\}$ is the orthonormal basis of A_k and $f_t \otimes e_t$ is a 1-rank operator on A_k. Without loss of generality, one can assume that $\{e_t\}_{t=1}^\infty = \{e_\alpha^k\}_{\alpha \in Z^{+n}}$, where $e_\alpha^k = c(n, k, \alpha)z^\alpha$. For convenience, we replace f_i by f_α. Note $S^* = \sum_\alpha e_\alpha^k \otimes f_\alpha$, so

$$S^*S = (\sum_\alpha e_\alpha^k \otimes f_\alpha)(\sum_\alpha f_\alpha \otimes e_\alpha^k) = \sum_\alpha \| f_\alpha \|_2^2 \, e_\alpha \otimes e_\alpha^k.$$

Furthermore, $\| S \|_{C_1} = \mathrm{tr}((S^*S)^{\frac{1}{2}}) = \sum_\alpha \| f_\alpha \|_2$. Hence, $\sum_\alpha \| f_\alpha \|_2 \leq \infty$, consequently, $\sum_\alpha f_\alpha e_\alpha^k \in L^1$. If A contains all Toeplitz operators on A_k, then for any $\varphi \in L^\infty(B)$, we have $T_\varphi^{(k)} \in A$. Thus

$$\begin{aligned}
\mathrm{tr}(T_\varphi^{(k)} S) &= \sum_{\alpha \in Z^{+n}} \langle T_\varphi^{(k)} S e_\alpha^k, e_\alpha^k \rangle \\
&= \sum_{\alpha \in Z^{+n}} \langle \varphi(\sum_{\beta \in Z^{+n}} f_\beta \otimes e_\beta^k) e_\alpha^k, e_\alpha^k \rangle \\
&= \sum_{\alpha \in Z^{+n}} \langle \varphi f_\alpha, e_\alpha^k \rangle \\
&= \sum_{\alpha \in Z^{+n}} \int_B \varphi f_\alpha \bar{e}_\alpha^k d\mu_k
\end{aligned}$$

$$= \int_B \varphi(\sum_{\alpha \in Z^{+n}} f_\alpha \overline{e_a^k}) d\mu_k = 0.$$

Since φ is arbitrary, we easily see that $\sum f_\alpha(z) \overline{e_a^k}(z) = 0$ for any $z \in B$. Suppose f_α has series expansion $f_\alpha = \sum_{\beta \in Z^{+n}} a_{\alpha\beta} e_\beta^k$, then

$$\begin{aligned}
\sum_\alpha f_\alpha(z) \overline{e_a^k}(z) &= \sum_\alpha \sum_\beta a_{\alpha\beta} \overline{e_\alpha^k} e_\beta^k(z) \\
&= \sum_\alpha, \sum_\beta a_{\alpha\beta} c(n, \alpha, k) c(n, \beta, k) \bar{z}^\alpha z^\beta \\
&= \sum_{\alpha\beta} a_{\alpha\beta} c(n, \alpha, k) c(n, \beta, k) r^{\alpha+\beta} e^{i(\beta-\alpha)\theta} \\
&= \sum_{t \in Z^{+n}} [\sum_{\alpha+\beta=t} a_{\alpha\beta} c(n, \alpha, k) c(n, \beta, k) e^{i(\beta-\alpha)\theta}] r^t = 0,
\end{aligned}$$

where

$$\theta = (\theta_1, \ldots, \theta_n),\ 0 \le \theta_i \le 2\pi,\ (\beta - \alpha)\theta = \sum (\beta_i - \alpha_i)\theta_i,$$
$$r = (r_1, \cdots, r_n),\ 0 \le \| r \| < 1.$$

So for each $t \in Z^{+n}$,

$$\sum_{\alpha+\beta=t} a_{\alpha\beta} c(n, \alpha, k) c(n, \beta, k) e^{i(\beta-\alpha)\theta} = 0$$

i.e.

$$\sum_{\alpha+\beta=t} a_{\alpha\beta} c(n, \alpha, k) c(n, \beta, k) e^{i(t-2\alpha)\theta} = 0.$$

Clearly, $\{e^{i(t-2\alpha)\theta}\}$ is linear independent, so $a_{\alpha\beta} = 0$ for $\alpha + \beta = t$. Hence, for any $\alpha \in Z^{+n}$, $\beta \in Z^{+n}$, we have $a_{\alpha\beta} = 0$ and so $S = 0$. It contradicts that $S \neq 0$. This completes the proof.

Frankfurt [5] proved that no bounded operator T on A_0 satisfies the operator equation $B_0^* T B_0 = T$, where B_0 is the Bergman shift on $A_0(D)$ and D is the unit disc. We can extend this result to the case $A_k(B)$. In fact, we have the following.

THEOREM 2. *There isn't nonzero bounded operator* T *on* $A_k(B)$ *such that* $\sum_{i=1}^n T_{\bar{z}_i}^{(k)} T T_{z_i}^{(k)} = T$.

To prove Theorem 2, we need some lemmas. The proof of Lemma 1 is related to that of Proposition 2.4 in [4].

LEMMA 1. *Let $M_{z_1} \cdots M_{z_n}$ be multiplication by the coordinate functions on $L^2(B, d\mu_k)$. If there exists $T \in L(L^2)$ such that $\sum_{i=1}^n M_{z_i}^* T M_{z_i} = T$, then T commutes with $M_{z_i}, M_{z_i}^*$ $(i = 1, \ldots, n)$.*

Proof. For any positive integer m and $f, g \in L^2$, we have

$$\langle Tf, g\rangle = \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \langle TM_{z_1}^{k_1} \cdots M_{z_n}^{k_n} f, M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g\rangle$$

by $\sum_{i=1}^n M_{\bar{z}_i} T M_{z_i} = T$. Hence

$$\begin{aligned}
&\langle (TM_{z_1} - M_{z_1}T) f, g\rangle \\
&= \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \langle TM_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f, M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g\rangle \\
&\quad - \sum_{\sum_{i=1}^n k_i = m} \frac{(m+1)!}{(k_1+1)! k_2! \cdots k_n!} \langle TM_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f, M_{z_1}^* M_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} g\rangle \\
&\quad - \sum_{\sum_{i=2}^n k_i = m+1} \frac{(m+1)!}{k_2! \cdots k_n!} \langle TM_{z_2}^{k_2} \cdots M_{z_n}^{k_n} f, M_{z_1}^* M_{z_2}^{k_2} \cdots M_{z_n}^{k_n} g\rangle \\
&= \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \langle TM_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f, \left(1 - \frac{m+1}{k_1+1} M_{z_1}^* M_{z_1}\right) M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g\rangle \\
&\quad - \sum_{\sum_{i=2}^n k_i = m+1} \frac{(m+1)!}{k_2! \cdots k_n!} \langle TM_{z_2}^{k_2} \cdots M_{z_n}^{k_n} f, M_{z_1}^* M_{z_2}^{k_2} \cdots M_{z_n}^{k_n} g\rangle.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\|T\|^{-1} \langle (TM_{z_1} - M_{z_1}T) f, g\rangle \\
&\leq \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \| M_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f \| \left\| \left(1 - \frac{m+1}{k_1+1} M_{\bar{z}_1} M_{z_1}\right) M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \right\| \\
&\quad + \sum_{\sum_{i=2}^n k_i = m+1} \frac{(m+1)!}{k_2! \cdots k_n!} \| M_{z_2}^{k_2} \cdots M_{z_n}^{k_n} f, \| \| M_{\bar{z}_1} M_{z_2}^{k_2} \cdots M_{z_n}^{k_n} g \|.
\end{aligned}$$

Note for any $f \in L^2(B, d\mu_k)$

$$\begin{aligned}
&\| (M_{\bar{z}_1} M_{z_1} + \cdots + M_{\bar{z}_n} M_{z_n})^m f \| \\
&= \| (\Sigma_{\sum_{i=1}^n |z_i|^2})^m f \| \to 0 \ (m \to \infty)
\end{aligned}$$

and

$$\sum_{\sum_{i=1}^n p_i = m} \frac{m!}{p_1! \cdots p_n!} \| M_{z_1}^{p_1} \cdots M_{z_n}^{p_n} f \|^2$$

$$= \sum_{\sum_{i=1}^n p_i = m} \frac{m!}{p_1! \cdots p_n!} \langle (M_{\bar{z}_1} M_{z_1})^{p_1} \cdots (M_{\bar{z}_n} M_{z_n})^{p_n} f, f \rangle$$
$$= \langle (M_{\bar{z}_1} M_{z_1} + \cdots + M_{\bar{z}_n} M_{z_n})^m f, f \rangle$$
$$\leq \| (\textstyle\sum_{\sum_{i=1}^n |z_i|^2})^m f \| \| f \|.$$

By

$$\sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \| M_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f \| \| \left(1 - \frac{m+1}{k_1+1} M_{\bar{z}_1} M_{z_1}\right) M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|$$
$$\leq [\sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \frac{m+1}{k_1+1} \| M_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f \|^2]^{\frac{1}{2}}$$
$$[\sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \frac{k_1+1}{m+1} \| \left(1 - \frac{m+1}{k_1+1} M_{\bar{z}_1} M_{z_1}\right) M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|^2]^{\frac{1}{2}}$$

and

$$\sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \frac{k_1+1}{m+1} \| \left(1 - \frac{m+1}{k_1+1} M_{\bar{z}_1} M_{z_1}\right) M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|^2$$
$$= \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \frac{k_1+1}{m+1} [\| M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|^2$$
$$- 2 \frac{m+1}{k_1+1} Re \langle M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g, M_{\bar{z}_1} M_{z_1} M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \rangle$$
$$+ \left(\frac{m+1}{k_1+1}\right)^2 \| M_{\bar{z}_1} M_{z_1} M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|^2]$$
$$\leq \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} [\| M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|^2 + 2 \| M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|^2$$
$$+ \frac{m+1}{k_1+1} \| M_{z_1}^{k_1+1} M_{z_2}^{k_2} \cdots M_{z_n}^{k_n} g \|^2]$$
$$\leq 3 \langle \sum_{i=1}^n M_{\bar{z}_i} M_{z_i})^m g, g \rangle + \langle (\sum_{i=1}^n M_{\bar{z}_i} M_{z_i})^{m+1} g, g \rangle$$
$$\leq 3 \| (\sum_{i=1}^n M_{\bar{z}_i} M_{z_i})^m g \| . \| g \| + \| \sum_{i=1}^n M_{\bar{z}_i} M_{z_i})^{m+1} g \| . \| g \|,$$

we have

$$T M_{z_1} - M_{z_1} T = 0,$$

i.e.

$$T M_{z_1} = M_{z_1} T.$$

Similarly, $TM_{z_i} = M_{z_i}T$, for $i = 1,2,\cdots, n$. It shows the lemma.

LEMMA 2. *If* $T \in L(A_k)$ *satisfy* $\sum_{i=1}^n T_{\bar{z}_i}TT_{z_i} = T$. *Then there is* $S \in L(L^2)$ *with* $\| S \| = \| T \|$, $\sum_{i=1}^n M_{\bar{z}_i}SM_{z_i} = S$ *and such that* T *is the compression of* S *to* A_k.

Proof. It is similar to the proof of Lemma 2.5 in [2]. In fact, we can define ψ: $L(L^2) \to L(L^2)$ by

$$\psi(S) = \sum_{i=1}^{n} M_{\bar{z}_i}SM_{z_i}$$

then $\| \psi(S) \| \le \| S \|$. Let $T^{\sim}$ be any operator on L^2 whose compression is T, with $\| T^{\sim} \| = \| T \|$, let $S_m = \frac{1}{m} \sum_{i=1}^m \psi^i(T^{\sim})$, and let S be a weak operator topology limit point of $\{S_m\}$, then S has the required properties.

LEMMA 3. *If* $T \in L(A_k)$ *satisfies* $\sum_{i=1}^n T_{\bar{z}_i}^k TT_{z_k}^k = T$, *then* T *is a Toeplitz operator.*

Proof. If T satisfies the equation, and S is the operator given by Lemma 2, then Lemma 1 shows that S commutes with M_{z_k} and $M_{\bar{z}_k}$ $(k = 1, \ldots, n)$, so there is $\varphi \in L^\infty$ such that $S = M_\varphi$, consequently, $T = T_\varphi^{(k)}$.

Proof of Theorem 2. If there is $T \in L(A_k)$ such that $\sum_{i=1}^n T_{\bar{z}_i}^k TT_{z_i}^{(k)} = T$, then T is a Toeplitz operator on A_k, i.e., there is L^∞, such that $T = T_\varphi^{(k)}$. Note

$$\sum_{i=1}^{n} T_{\bar{z}_i}^{(k)} T_\varphi^{(k)} T_{\bar{z}_i}^{(k)} = T_{(\sum_{i=1}^n |z_i|^2)\varphi}^{(k)},$$

so $T_{(\sum_{i=1}^n |z_i|^2)\varphi}^{(k)} = T_\varphi^{(k)}$, and hence, $T_{(1-\sum_{i=1}^n |z_i|^2)\varphi}^{(k)} = 0$. Hence, $\varphi = 0$, consequently, $T = 0$. We complete the proof of Theorem 2.

The author is indebted to the referee for his many suggestions.

REFERENCES

[1] W. Rudin, Function Theory in the Unit Ball of C^n, Springer-verlag, New York Heidelberg Berlin, 1980.

[2] A. M. Davie and N. P. Jewell, Toeplitz operators in several complex variables., J. Funct. Anal., **26** (1977), 356–368.

[3] D. H. Yu and Sh. S. Sun, The w^*-closure of operator space, Acta Math. of China (to appear).
[4] N. P. Jewell, Toeplitz operators on the Bergman spaces and in several complex Variables, Proc. London Math. Soc., **41** (1980), 193–216.
[5] R. Frankfurt, Operator equations and weigted shifts, J. Funct. Anal., **62** (1978) 610–619.

Department of Mathematics
Sichuan University
Chengdu 610064, China