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ON p-ADIC DEDEKIND SUMS

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§1. Introduction

For positive integers h, k and m, the higher-order Dedekind sums are defined by

$$S_{m+1}^{(r)}(h, k) = \sum_{a=0}^{k-1} \bar{B}_{m+1-r}\left(\frac{a}{k}\right)\bar{B}_r\left(\frac{ha}{k}\right), \quad 0 \leq r \leq m+1,$$

where $\bar{B}_n(x)$, $n \geq 0$, are the Bernoulli functions (§2). If m is odd and $(h, k) = 1$, the sum $S_{m+1}^{(m)}(h, k)$ is identical with the higher-order Dedekind sum of Apostol [1],

$$s_m(h, k) = \sum_{a=1}^{k-1} \frac{a}{k} \bar{B}_m\left(\frac{ha}{k}\right).$$

Recently, Rosen and Snyder [6] constructed a p-adic continuous function $S_p(s\,; h, k)$ for an odd prime p, which takes the values

$$S_p(m\,; h, k) = \begin{cases} k^m s_m(h, k) - p^{m-1}k^m s_m((p^{-1}h)_k, k), & \text{if } (k, p) = 1, \\ k^m s_m(h, k), & \text{if } k = p, \end{cases}$$

at positive integers m such that $m + 1 \equiv 0 \pmod{p-1}$; here $(p^{-1}h)_k$ denotes the integer x such that $0 \leq x < k$ and $px \equiv h \pmod{k}$.

The purpose of this paper is to extend this result of them to $k^m S_{m+1}^{(r)}(h, k)$ for every h, k and $r \geq 1$. To this end, we use an expression of $k^m S_{m+1}^{(r)}(h, k)$ in terms of the Euler numbers ([2], [3]) and a p-adic continuous function which interpolates these numbers ([7], [8]).

Let p be a prime number and Z_p the ring of rational p-adic integers. Let $e = p - 1$ or $e = 2$ according as $p > 2$ or $p = 2$. In §§2-3, we shall prove the following

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THEOREM 1. *Let h, k and r be fixed integers ≥ 1. Then, there exists a continuous function $S_p(s\,;r,\,h,\,k)$ on Z_p, which satisfies*

$$S_p(m\,;r,\,h,\,k) = k^m S_{m+1}^{(r)}(h,\,k) - p^{m-r}k^m S_{m+1}^{(r)}(ph,\,k)$$

for all integers m such that $m \geq r$ and $m+1 \equiv 0 \pmod e$.

In §4, we shall discuss about a special value and a continuity property of our function $S_p(s\,;r,\,h,\,k)$, assuming that $(h,\,k)=1$.

§2. Preliminaries

Let C_p be the completion of an algebraic closure of the rational p-adic number field Q_p, $|\ \ |$ the valuation on C_p normalized so that $|p| = p^{-1}$, $\mathcal{O}$ the ring of integers in C_p and Z the ring of rational integers. Throughout, we fix p and consider algebraic numbers to be contained in C_p.

For each root of unity $\rho \neq 1$, we define the numbers $E_n(\rho)$, $n \geq 0$, by

$$\frac{\rho}{e^t - \rho} = \sum_{n=0}^{\infty} E_n(\rho)\,\frac{t^n}{n!}.$$

Here, $\dfrac{1-\rho}{\rho}E_n(\rho) = H_n(\rho)$, $n \geq 0$, are the Euler numbers with the parameter ρ.

If ρ satisfies the condition that $\rho^{p^n} \neq 1$, for all $n \geq 0$, we can define a finitely additive $\mathcal{O}$-valued measure μ_ρ on Z_p by

$$\mu_\rho(a + p^N Z_p) = \frac{\rho^{p^N - a}}{1 - \rho^{p^N}}, \quad 0 \leq a < p^N, \quad N \geq 0.$$

Let Z_p^* denote the group of units in Z_p. We know by [7], [8] that

$$\int_{Z_p} x^n d\mu_\rho(x) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} a^n \frac{\rho^{p^N-a}}{1-\rho^{p^N}} = E_n(\rho), \quad n \geq 0 \tag{1}$$

and

$$\int_{Z_p^*} x^n d\mu_\rho(x) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1}{}^{*}\, a^n \frac{\rho^{p^N-a}}{1-\rho^{p^N}} = E_n(\rho) - p^n E_n(\rho^p), \quad n \geq 0, \tag{2}$$

where $\sum^*$ means to take sum over all integers prime to p in the given range.

Let c be an integer >1 and $E_n(1) = \dfrac{B_{n+1}}{n+1}$, $n \geq 0$, where B_n, $n \geq 0$, are the Bernoulli numbers defined by $\dfrac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \dfrac{t^n}{n!}$. Then, it follows at once from the identity

$$\sum_{\eta^c=1} \frac{\rho\eta}{e^t - \rho\eta} = \frac{c\rho^c}{e^{ct} - \rho^c}$$

that

$$\sum_{\eta^c=1} E_n(\rho\eta) = c^{n+1} E_n(\rho^c), \quad n \geq 0 \tag{3}$$

for every root of unity ρ. If $\rho^c = 1$, the formula (3) is equivalent to that

$$\sum_{\eta^c=1, \eta\neq 1} E_n(\eta) = (c^{n+1} - 1) \frac{B_{n+1}}{n+1}, \quad n \geq 0.$$

Let $B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$, $n \geq 0$, be the Bernoulli polynomials and let $\{x\}$ denote the smallest real number $t \geq 0$ such that $x - t \in Z$, for a real number x. Then we have $\bar{B}_n(x) = B_n(\{x\})$ except for the case $n = 1$ and $x \in Z$ ($\bar{B}_1(x) = 0$ for $x \in Z$). Therefore we get without difficulty that

$$S^{(r)}_{m+1}(h, k) = \sum_{a=0}^{k-1} B_{m+1-r}\left(\frac{a}{k}\right) B_r\left(\left\{\frac{ha}{k}\right\}\right), \quad 1 \leq r \leq m \tag{4}$$

for all odd integers m (unless $r = m = 1$). If $r = m = 1$, the right hand side of (4) is equal to $S^{(1)}_2(h, k) + \frac{1}{4}$.

Now, by the equality

$$\frac{te^{\{\frac{a}{k}\}t}}{e^t - 1} = \frac{1}{k} \sum_{\zeta^k=1} \left(\sum_{b=0}^{k-1} \frac{te^{\frac{b}{k}t}}{e^t - 1} \zeta^{-b}\right) \zeta^a,$$

we have

$$k^n B_n\left(\left\{\frac{a}{k}\right\}\right) = n \sum_{\zeta^k=1} E_{n-1}(\zeta) \zeta^a, \quad n \geq 1. \tag{5}$$

Therefore we obtain the formula of [2], [3],

$$k^m S^{(r)}_{m+1}(h, k) = (m + 1 - r) r \sum_{\zeta^k=1} E_{m-r}(\zeta^h) E_{r-1}(\zeta^{-1}), \quad 1 \leq r \leq m, \tag{6}$$

for all odd m (unless $r = m = 1$). If $r = m = 1$, the formula (6) holds for

$k\left(S_2^{(1)}(h, k) + \frac{1}{4}\right)$.

§3. Definition of $S_p(s\,; r, h, k)$

In this section, we give a proof of Theorem 1 mentioned in introduction. Let h, k and r denote positive integers and ζ a root of unity. Let $q = p$ or $q = 4$ according as $p > 2$ or $p = 2$.

Suppose first that $\zeta^{p^n} \neq 1$ for all $n \geq 0$. Let

$$G(s\,; r, \zeta) = \int_{Z_p^*} \omega(x)^{-1}\langle x\rangle^s \frac{1}{x^r}\, d\mu_\zeta(x), \quad s \in Z_p, \tag{7}$$

where ω is the Teichmüller character with conductor q and $\langle x\rangle = \omega(x)^{-1}x$ for $x \in Z_p^*$.

Let exp and log denote the p-adic exponential and logarithm functions, respectively. Then, since $\langle x\rangle \equiv 1 \pmod q$ for $x \in Z_p^*$, $\log\langle x\rangle \equiv 0 \pmod q$ and $\langle x\rangle^s = \exp(s \log\langle x\rangle)$. Therefore $G(s\,; r, \zeta)$ is an analytic function of s in Z_p with an expansion

$$\begin{gathered} G(s\,; r, \zeta) = \sum_{n=0}^{\infty} c_{n,r}(\zeta)(s + 1 - r)^n, \\ c_{n,r}(\zeta) = \int_{Z_p^*} \omega^{-r}(x)\, \frac{(\log\langle x\rangle)^n}{n!} \frac{1}{x}\, d\mu_\zeta(x), \\ |\, c_{n,r}(\zeta)\,| \leq \left|\frac{q^n}{n!}\right| \leq (q^{-1}p^{\frac{1}{p-1}})^n. \end{gathered} \tag{8}$$

Now, as e is the order of ω, we have, by (2),

$$G(m\,; r, \zeta) = \int_{Z_p^*} x^{m-r} d\mu_\zeta(x) = E_{m-r}(\zeta) - p^{m-r}E_{m-r}(\zeta^p) \tag{9}$$

for all integers m such that $m \geq r$ and $m + 1 \equiv 0 \pmod e$.

Next, suppose that $\zeta^{p^n} = 1$ for some $n \geq 0$. Choose an integer $c > 1$ so that $|\, c - 1\,| \leq |\, q\,|$ and $\zeta^c = \zeta$. Let

$$F_c(s\,; r, \zeta) = \sum_{\eta^c = 1, \eta \neq 1} G(s\,; r, \zeta\eta).$$

Then, it follows from (9) and (3) that

$$F_c(m\,; r, \zeta) = (c^{m+1-r} - 1)(E_{m-1}(\zeta) - p^{m-r}E_{m-r}(\zeta^p))$$

for all $m \geq r$, $m + 1 \equiv 0 \pmod e$.

Now, we consider the power series

$$U_{c,r}(s) = \sum_{n=0}^{\infty} B_n \frac{(\log c)^{n-1}}{n!} (s+1-r)^n.$$

Since $|B_n| \leq |\frac{1}{p}|$ for all n (by the von Staudt-Clausen Theorem) and $|\frac{(\log c)^{n-1}}{n!}| \leq |\frac{q^{n-1}}{n!}|$, this power series defines an analytic function of $s \in Z_p$ and is equal to $\frac{s+1-r}{c^{s+1-r}-1}$ for $s \neq r-1$. Let

$$G(s\,; r, \zeta) = \frac{1}{s+1-r} U_{c,r}(s) F_c(s\,; r, \zeta), \quad \text{for } s \neq r-1,$$
$$= \frac{1}{c^{s+1-r}-1} F_c(s\,; r, \zeta).$$

Then the value of this function G is independent of the choice of c, and

$$G(m\,; r, \zeta) = E_{m-r}(\zeta) - p^{m-r} E_{m-r}(\zeta^p) \tag{10}$$

for all $m \geq r$, $m+1 \equiv 0 \pmod e$. We define the function $S_p(s\,; r, h, k)$ by

$$S_p(s\,; r, h, k) = (s+1-r) r \sum_{\zeta^k=1} G(s\,; r, \zeta^h) E_{r-1}(\zeta^{-1}),$$

and show that this function $S_p(s\,; r, h, k)$ satisfies the properties described in Theorem 1.

The function S_p is analytic in Z_p and in particular is continuous. Further by (9), (10) and (6) we have

$$\begin{aligned} S_p(m\,; r, h, k) &= (m+1-r) r \sum_{\zeta^k=1} (E_{m-r}(\zeta^h) - p^{m-r} E_{m-r}(\zeta^{ph})) E_{r-1}(\zeta^{-1}) \\ &= k^m S_{m+1}^{(r)}(h, k) - p^{m-r} k^m S_{m+1}^{(r)}(ph, k) \end{aligned}$$

for all $m \geq r$, $m+1 \equiv 0 \pmod e$. This completes the proof of Theorem 1.

Let d be a positive integer. Since $S_{m+1}^{(r)}(dh, dk) = d^{r-m} S_{m+1}^{(r)}(h,k)$ ([2]), we have

$$\begin{aligned} S_p(m\,; r, dh, dk) &= (dk)^m S_{m+1}^{(r)}(dh, dk) - p^{m-r} (dk)^m S_{m+1}^{(r)}(pdh, dk) \\ &= d^r k^m S_{m+1}^{(r)}(h, k) - p^{m-r} d^r k^m S_{m+1}^{(r)}(ph, k) \\ &= d^r S_p(m\,; r, h, k) \end{aligned}$$

for all $m \geq r$, $m+1 \equiv 0 \pmod e$. Hence by analyticity we obtain

$$S_p(s\,;r,\,dh,\,dk) = d^r S_p(s\,;r,\,h,\,k), \quad s \in Z_p.$$

Therefore, when we discuss the property of $S_p(s\,;r,\,h,\,k)$, it is sufficient to consider in the case where $(h,\,k) = 1$. Similarly, if $(k,\,p) > 1$, we can write the formula of Theorem 1 as

$$S_p(m\,;r,\,h,\,k) = k^m S_{m+1}^{(r)}(h,\,k) - k^m S_{m+1}^{(r)}(h,\,kp^{-1}),$$

for m such that $m \geq r,\; m + 1 \equiv 0 \pmod{e}$.

Remark 1. Let $(h,\,k) = 1$ and $p > 2$. Take an integer $h^* > 0$ such that $hh^* \equiv 1 \pmod{k}$. Then by the property $S_{m+1}^{(1)}(h^*,\,k) = S_{m+1}^{(m)}(h,\,k)$ of Dedekind sums, it follows that

$$S_p(m,\,1,\,h^*,\,k) = \begin{cases} k^m s_m(h,\,k) - p^{m-1}k^m s_m((p^{-1}h)_k,\,k), & \text{if } (k,\,p) = 1, \\ k^m s_m(h,\,k), & \text{if } k = p, \end{cases}$$

for all $m \geq 1,\, m + 1 \equiv 0 \pmod{p-1}$. Therefore the function $S_p(s\,;1,\,h^*,\,k)$ gives the Rosen-Snyder's $S_p(s\,;h,\,k)$.

Remark 2. If $p = 2$ or 3, then Theorem 1 holds for $r = 1$ and $m = 1$, so

$$S_p(1\,;1,\,h,\,k) = \begin{cases} k\,s(h,\,k) - k\,s(ph,\,k), & \text{if } (k,\,p) = 1, \\ k\,s(h,\,k) - k\,s(h,\,kp^{-1}), & \text{if } (k,\,p) = p, \end{cases}$$

where $s(h,\,k) = S_2^{(1)}(h,\,k),\; (h,\,k) = 1$, denote the ordinary Dedekind sums.

For any integer $\nu \geq 0$, let $p^{\bar{\nu}}$ be the least common multiple of q and p^ν. Let $c = 1 + p^{\bar{\nu}}$. Then the function $S_p(s\,;r,\,h,\,p^\nu)$ is defined by

$$S_p(s\,;r,\,h,\,p^\nu) = U_{c,r}(s)\; r \sum_{\zeta^{p^\nu}=0} F_c(s\,;r,\,\zeta^h)E_{r-1}(\zeta^{-1}). \tag{11}$$

Let $(h,\,k) = 1,\; k > 1$ and let

$$\bar{S}_p(s\,;r,\,h,\,k) = (s + 1 - r)r \sum_{\zeta^k=1,\,\zeta^{p^\nu}\neq 1} G(s\,;r,\,\zeta^h)E_{r-1}(\zeta^{-1}), \tag{12}$$

where $k = k_0 p^\nu,\; (k_0,\,p) = 1$, and G on the right is the analytic one defined by (7). Then the function $S_p(s\,;r,\,h,\,k)$ is separated as

$$S_p(s\,;r,\,h,\,k) = \bar{S}_p(s\,;r,\,h,\,k) + S_p(s\,;r,\,h,\,p^\nu).$$

Finally, if r is odd, then we see from the definition of Dedekind sums that $S_{m+1}^{(r)}(h,\,1) = S_{m+1}^{(r)}(h,\,2) = 0$ for odd $m \geq r$. Hence it follows from Theorem 1 and the analyticity of S_p that

$$S_p(s\,;r,\,h,\,1)=S_p(s\,;r,\,h,\,2)=0,\quad s\in Z_p,$$

if r is odd.

§4. Properties of $S_p(s\,;r,\,h,\,k)$

It is the purpose of this section to estimate the p-adic absolute values $|a_n|$, $n \geq 0$, of the coefficients of

$$S_p(s\,;r,\,h,\,k)=\sum_{n=0}^{\infty}a_n(s+1-r)^n,\quad a_n\in Q_p,$$

in the case where $(h,\,k)=1$. We write $k=k_0p^{\nu}$, $(k_0,\,p)=1$, $\nu\geq 0$, and consider separately about $S_p(s\,;r,\,h,\,p^{\nu})$ and $\bar{S}_p(s\,;r,\,h,\,k)$. Let $p^{\bar{\nu}}$ denote the least common multiple of q and p^{ν} as before.

LEMMA. *Suppose $\zeta^{p^n}\neq 1$ for all $n\geq 0$. Then,*

$$\int_{Z_p^*}\omega^{-r}(x)\,\frac{1}{x}\,d\mu_{\zeta}(x)=\begin{cases}\log(1-\zeta)-\dfrac{1}{p}\log\,(1-\zeta^{p}), & \text{if } r\equiv 0 \pmod{e},\\[2ex] \dfrac{\tau(\omega^{-r})}{q}\displaystyle\sum_{a=0}^{q-1}\omega^{r}(a)\log(1-\zeta\zeta_q^{a}), & \text{if } r\not\equiv 0 \pmod{e},\end{cases}$$

where ζ_q is a primitive q-th root of unity, and $\tau(\omega^{-r})=\sum_{i=0}^{q-1}\omega^{-r}(i)\,\zeta_q^{i}$.

Proof. Let $f(X)$ be the unique power series in $\mathcal{O}[[X]]$ such that

$$f(X)\equiv\sum_{a=0}^{p^n-1}\mu_{\zeta}(a+p^nZ_p)\,(1+X)^a\pmod{P_n(X)}$$

for all $n\geq 0$, where $P_n(X)=(1+X)^{p^n}-1$. Then it follows immediately from the above congruences that $f(X)=\dfrac{\zeta}{1+X-\zeta}$. Therefore, we can calculate the value of this integral following the theory of Γ-transform, namely, e.g. along the argument of [5] (pp. 45-48). This completes the proof. The assertion for the case where $r\equiv 0 \pmod{e}$ is obtained also in [9].

Let $c=1+p^{\bar{\nu}}$, and let $F_c(s\,;r,\,\zeta)$ and $U_{c,r}(s)$ be the functions defined in §3. In the sequel we write $F^{(\nu)}(s\,;r,\,\zeta)$ and $U_r^{(\nu)}(s)$ for the functions F_c and U_c, respectively.

PROPOSITION 1. *For each root of unity ζ such that $\zeta^{p^{\nu}}=1$, let*

$$F^{(\nu)}(s\,;r,\zeta)=\sum_{n=0}^{\infty}b_{n,r}^{(\nu)}(\zeta)(s+1-r)^{n},\quad b_{n,r}^{(\nu)}(\zeta)\in C_{p}.$$

(a) *When $r\equiv 0 \pmod e$,*

$$b_{0,r}^{(\nu)}(\zeta)=\begin{cases}\left(1-\dfrac{1}{p}\right)\log c, & \text{if } \zeta=1,\\ -\dfrac{1}{p}\log c, & \text{if } \zeta^{p}=1,\ \zeta\neq 1,\\ 0, & \textit{otherwise};\end{cases}$$

(b) *when $r\not\equiv 0 \pmod e$,*

$$b_{0,r}^{(\nu)}(\zeta)=\begin{cases}\dfrac{\tau(w^{-r})}{q}\,\omega^{r}(i)\log c, & \text{if } \zeta=\zeta_{q}^{-i},\ (i,p)=1,\\ 0, & \textit{otherwise};\end{cases}$$

and

$$\text{(c)}\qquad b_{n,r}^{(\nu)}(\zeta)=\sum_{a=0}^{p^{\bar{\nu}}-1}{}^{*}\,\omega^{-r}(a)\zeta^{-a}\left(\frac{(\log a)^{n}}{n!}+\frac{q^{n}}{n!}q^{-1}p^{\bar{\nu}}\xi_{a}^{(n)}\right),\quad n\geq 1,$$

where $\xi_{a}^{(n)}$ are rational p-adic integers independent of ζ.

Proof. Since

$$b_{n,r}^{(\nu)}(\zeta)=\sum_{\eta^{c}=1,\eta\neq 1}\int_{Z_{p}^{*}}\omega^{-r}(x)\frac{(\log\langle x\rangle)^{n}}{n!}\frac{1}{x}\,d\mu_{\zeta\eta}(x),\quad n\geq 0, \tag{13}$$

the assertions (a), (b) for $n=0$ immediately follow from Lemma and the fact that

$$\sum_{\eta\neq 1}\log(1-\zeta\eta)=\begin{cases}\log c, & \text{if } \zeta=1,\\ 0, & \text{if } \zeta\neq 1\end{cases}$$

for any p^{ν}-th root of unity ζ. Let $n\geq 1$. In order to prove the assertion (c), we write

$$\begin{aligned}b_{n,r}^{(\nu)}(\zeta)&=\sum_{\eta\neq 1}\lim_{N\to\infty}\sum_{a=0}^{p^{\bar{\nu}+N}-1}{}^{*}\,\omega^{-r}(a)\frac{(\log a)^{n}}{n!}\frac{1}{a}\frac{(\zeta\eta)^{p^{\bar{\nu}+N}-a}}{1-(\zeta\eta)^{p^{\bar{\nu}+N}}}\\ &=\sum_{\eta\neq 1}\lim_{N\to\infty}\sum_{a=0}^{p^{\bar{\nu}}-1}{}^{*}\sum_{b=0}^{p^{N}-1}\omega^{-r}(a)\frac{(\log(a+p^{\bar{\nu}}b))^{n}}{n!(a+p^{\bar{\nu}}b)}\frac{\zeta^{-a}\eta^{-a}(\eta^{-1})^{p^{N}-b}}{1-(\eta^{-1})^{p^{N}}}\end{aligned}$$

so that

$$b_{n,r}^{(\nu)}(\zeta) = \sum_{a=0}^{p^{\bar\nu}-1}{}^{*} \omega^{-r}(a)\zeta^{-a} \sum_{\eta\neq 1} \eta^{a} \int_{Z_p} \frac{(\log (a+p^{\bar\nu}x))^n}{n!\,(a+p^{\bar\nu}x)} d\mu_\eta(x), \quad n \geq 1.$$

Since the sum on the right over $\eta \neq 1$ $(\eta^c = 1)$ is a rational p-adic integer independent of ζ, it is sufficient to show that this sum is congruent to $\frac{(\log a)^n}{n!}$ modulo $\frac{q^{n-1}}{n!} p^{\bar\nu}$, for each a. Now since $\log (a+p^{\bar\nu}x) \equiv \log a \pmod{p^{\bar\nu}}$, $\frac{1}{a+p^{\bar\nu}x} \equiv \frac{1}{a} \pmod{p^{\bar\nu}}$ and $\log a \equiv 0 \pmod q$, we have

$$\frac{(\log (a+p^{\bar\nu}x))^n}{a+p^{\bar\nu}x} \equiv \frac{(\log a)^n}{a} \pmod{q^{n-1}p^{\bar\nu}}, \quad n \geq 1.$$

On the other hand by making use of (1) and (5), we obtain

$$\sum_{\eta\neq 1} \eta^a \int_{Z_p} d\mu_\eta(x) = \sum_{\eta\neq 1} \eta^a E_0(\eta) = c\,B_1\left(\frac{a}{c}\right) - B_1$$

$$\text{(because } 0 \leq a \leq p^{\bar\nu} - 1 < c\text{)}$$

$$= a - \frac{p^{\bar\nu}}{2} \equiv a \pmod{p^{\bar\nu}-1}.$$

Hence

$$\sum_{\eta\neq 1} \eta^a \int_{Z_p} \frac{(\log (a+p^{\bar\nu}x))^n}{n!\,(a+p^{\bar\nu}x)} d\mu_\eta(x) \equiv \frac{(\log a)^n}{n!} \quad \left(\bmod \frac{q^{n-1}}{n!} p^{\bar\nu}\right), \quad n \geq 1,$$

as desired. This completes the proof of Proposition 1.

Now, for $\nu \geq 1$, let

$$T_r^{(\nu)}(s) = r \sum_{\zeta^{p^\nu}=1} F^{(\nu)}(s\,;r,\zeta^h) E_{r-1}(\zeta^{-1}),$$

where $(h, p) = 1$. Then, by (11), we have $S_p(s\,;r, h, p^\nu) = U_r^{(\nu)}(s)\,T_r^{(\nu)}(s)$.

Let $B_{n,\omega^{-r}}$, $n \geq 0$, denote the generalized Bernoulli numbers for the character ω^{-r}, defined by

$$\sum_{a=0}^{q-1} \frac{\omega^{-r}(a)te^{at}}{e^{qt}-1} = \sum_{n=0}^{\infty} B_{n,\omega^{-r}} \frac{t^n}{n!}.$$

PROPOSITION 2. *Let* $\nu \geq 1$ $(\nu \geq 2$ *if* $p = 2,\ r \not\equiv 0 \pmod{e})$ *and*

$$T_r^{(\nu)}(s) = \sum_{n=0}^{\infty} t_{n,r}^{(\nu)}(s+1-r)^n, \quad t_{n,r}^{(\nu)} \in Q_p.$$

Then,

$$\text{(a)} \qquad t_{0,r}^{(\nu)} = \begin{cases} (1-p^{r-1})B_r \log c, & \text{if } r \equiv 0 \pmod{e}, \\ \omega^r(h) B_{r,\omega^{-r}} \log c, & \text{if } r \not\equiv 0 \pmod{e} \end{cases}$$

and

$$\text{(b)} \quad t_{n,r}^{(\nu)} \equiv \frac{(\log(1+q))^n}{n!} h^r \sum_{a=0}^{p^{\bar{\nu}}-1}{}^{*} v(a)^n (1+q)^{rv(a)} \left(\operatorname{mod} \frac{q^n}{n!} q^{-1} p^{\bar{\nu}}\right), \quad n \geq 1,$$

where $v(a)$ *belongs to* Z_p *and determined uniquely by* $\langle a \rangle = (1+q)^{v(a)}$, *for each integer* a *prime to* p.

Proof. By the definition of $T_r^{(\nu)}$, we have

$$t_{n,r}^{(\nu)} = r \sum_{\zeta^{p^\nu}=1} b_{n,r}^{(\nu)}(\zeta^h) E_{r-1}(\zeta^{-1}), \quad n \geq 0.$$

(a) Let $r \equiv 0 \pmod{e}$. Then, by Proposition 1(a),

$$t_{0,r}^{(\nu)} = r \sum_{\zeta^p=1, \zeta \neq 1} \left(-\frac{1}{p} \log c\right) E_{r-1}(\zeta^{-1}) + r\left(1-\frac{1}{p}\right) \log c\, E_{r-1}(1).$$

The right hand side reduces to $(1-p^{r-1})B_r \log c$ by making use of the formula (3). Next, let $r \not\equiv 0 \pmod{e}$. Then by Proposition 1(b),

$$\begin{aligned} t_{0,r}^{(\nu)} &= r \sum_{i=0}^{q-1}{}^{*} b_{0,r}^{(\nu)}(\zeta_q^{-ih}) E_{r-1}(\zeta_q^i) \\ &= r \frac{\tau(\omega^{-r})}{q} \omega^r(h) \log c \sum_{i=0}^{q-1} \omega^r(i) E_{r-1}(\zeta_q^i). \end{aligned}$$

Now, from the equality

$$\frac{\tau(\omega^{-r})}{q} \sum_{i=0}^{q-1} \omega^r(i) \frac{\zeta_q^i}{e^t - \zeta_q^i} = \sum_{a=0}^{q-1} \frac{\omega^{-r}(a) e^{at}}{e^{qt}-1}$$

we have

$$\frac{\tau(\omega^{-r})}{q} \sum_{i=0}^{q-1} \omega^r(i)\, E_{r-1}(\zeta_q^i) = \frac{1}{r} B_{r,\omega^{-r}}.$$

Hence $t_{0,r}^{(\nu)} = \omega^r(h) B_{r,\omega^{-r}} \log c$, as claimed.
(b) Let $n \geq 1$, then it follows from Proposition 1(c) that

$$t_{n,r}^{(\nu)} = \sum_{a=0}^{p^{\bar{\nu}}-1}{}^{*} \omega^{-r}(a) \left(\frac{(\log a)^n}{n!} + \frac{q^n}{n!} q^{-1} p^{\bar{\nu}} \xi_a^{(n)} \right) r \sum_{\zeta^{p^\nu}=1} \zeta^{ha} E_{r-1}(\zeta).$$

By (5) and the von Staudt-Clausen Theorem, we have

$$r \sum_{\zeta} \zeta^{ha} E_{r-1}(\zeta) = p^{\nu r} B_r\left(\left\{\frac{ha}{p^\nu}\right\}\right) \equiv h^r a^r \pmod{p^{\nu-1}},$$

and hence

$$\begin{aligned} t_{n,r}^{(\nu)} &\equiv h^r \sum_{a=0}^{p^{\bar{\nu}}-1}{}^{*} \langle a \rangle^r \frac{(\log a)^n}{n!} \quad \left(\bmod \frac{q^n}{n!} q^{-1} p^{\bar{\nu}}\right) \\ &= \frac{(\log(1+q))^n}{n!} h^r \sum_{a=0}^{p^{\bar{\nu}}-1}{}^{*} v(a)^n (1+q)^{rv(a)}. \end{aligned}$$

This completes the proof of Proposition 2.

Now, let $p^\nu > q$, so we write ν for $\bar{\nu}$. Let $A_\mu^{(n)} = \sum_{i=0}^{p^\mu-1} i^n (1+q)^{ri}$, $\mu \geq 1$, $n \geq 1$. Then,

$$\sum_{a=0}^{p^\nu-1}{}^{*} v(a)^n (1+q)^{rv(a)} \equiv e A_\mu^{(n)} \pmod{p^\mu},$$

where $q^{-1} p^\nu = p^\mu$, $\mu \geq 1$. By induction on μ it follows that

$$A_\mu^{(n)} \equiv \begin{cases} p^\mu B_n \pmod{p^\mu}, & \text{if } p > 2, \\ 0 \pmod{p^{\mu-1}}, & \text{if } p = 2, \end{cases}$$

for all $\mu \geq 1$ and $n \geq 1$. Hence we have

$$\sum_{a=0}^{p^\nu-1}{}^{*} v(a)^n (1+q)^{rv(a)} \equiv \begin{cases} -q^{-1} p^\nu B_n \pmod{q^{-1}p^\nu}, & \text{if } p > 2, \\ 0 \pmod{q^{-1}p^\nu}, & \text{if } p = 2. \end{cases}$$

By Proposition 2(b) and the von Staudt-Clausen Theorem, we therefore obtain

(14) $t_{1,r}^{(\nu)} \equiv 0 \pmod{p^\nu}$, $t_{n,r}^{(\nu)} \equiv 0 \left(\bmod \frac{p^{n-2+\nu}}{n!}\right)$, $n \geq 2$, if $p > 2$, $\nu \geq 2$,

(15) $t_{n,r}^{(\nu)} \equiv 0 \left(\bmod \frac{p^n}{n!}\right)$, $n \geq 1$, if $p > 2$, $\nu = 1$,

(16) $t_{n,r}^{(\nu)} \equiv 0 \left(\bmod \frac{q^{n-1}}{n!} p^\nu\right)$, $n \geq 1$, if $p = 2$, $\nu > 2$.

For $p = 2$, $0 \leq \nu \leq 2$, we see, more exactly,

$$(17) \qquad b_{n,r}^{(\nu)}(\zeta) = \sum_{a=0}^{q-1}{}^{*} \omega^{-r}(a)\zeta^{-a}\frac{q^n}{n!}\xi^{(n)}, \quad (\zeta^{2^\nu} = 1, \nu \leq 2),$$

where $\xi^{(n)}$ is a 2-adic integer independent of both ζ and a. Indeed, we can see, by a little calculation, that

$$\eta^3 \int_{Z_2} \frac{(\log(3+4x))^n}{3+4x} d\mu_\eta(x) = \eta^{-1} \int_{Z_2} \frac{(\log(1+4x))^n}{1+4x} d\mu_{\eta^{-1}}(x),$$

for all $\eta \neq 1$, $\eta^5 = 1$, and hence

$$\xi^{(n)} = \sum_{\eta^5=1, \eta\neq 1} \eta \int_{Z_2} \frac{(\log(1+qx))^n}{q^n(1+qx)} d\mu_\eta(x).$$

From this expression of $b_{n,r}^{(\nu)}(\zeta)$ we obtain, in the same manner as in the proof of Proposition 2(b),

$$(18) \qquad t_{n,r}^{(\nu)} \equiv 0 \left(\mathrm{mod}\, \frac{2q^n}{n!}\right), \quad n \geq 1, \quad \text{if } p = 2, \nu = 1,2.$$

By these results obtained above, we can now prove the following

PROPOSITION 3. *Let*

$$S_p(s; r, h, p^\nu) = \sum_{n=0}^{\infty} a_n (s+1-r)^n, \quad a_n \in Q_p,$$

where $\nu \geq 1$ ($\nu \geq 2$ *if* $p = 2$, $r \not\equiv 0 \pmod e$) *and* $(h, p) = 1$. *Then,*

$$\text{(a)} \qquad a_0 = \begin{cases} (1 - p^{r-1})B_r, & \text{if } r \equiv 0 \pmod e, \\ \omega^r(h) B_{r,\omega^{-r}}, & \text{if } r \not\equiv 0 \pmod e, \end{cases}$$

$$\text{(b)} \qquad |a_1| \leq 1, \ |a_n| \leq \left|\frac{p^{n-2}}{n!}\right|, \ n \geq 2, \quad \text{if } p > 2,$$

$$|a_n| \leq \left|\frac{q^{n-1}}{n!}\right|, \ n \geq 1, \qquad \text{if } p = 2.$$

In particular,

$$\text{(c)} \qquad |S_p(s; r, h, p^\nu) - S_p(s'; r, h, p^\nu)| \leq |s - s'|, \quad s, s' \in Z_p.$$

Proof. Let $U_r^{(\nu)}(s) = \sum_{n=0}^{\infty} u_n (s+1-r)^n$. Then,

$$(19) \qquad u_0 = \frac{1}{\log c}\ (c = 1 + p^{\bar\nu}) \text{ and } |u_n| = \left|B_n \frac{p^{\bar\nu(n-1)}}{n!}\right|, \quad n \geq 0,$$

so the assertion (a) is obvious from Proposition 2(a). We further know by Proposition 2(a) and the von Staudt-Clausen Theorem for the Bernoulli (resp. generalized Bernoulli) numbers, that $|t_{0,r}^{(\nu)}| = |p^{\bar{\nu}-1}|$. Thus, the assertion (b) follows from (14)-(16), (18) and (19), by taking the power series product of $U_r^{(\nu)}$ and $T_r^{(\nu)}$. The last assertion (c) is an immediate consequence of the fact that $|a_n| \leq 1$ for all $n \geq 1$. This completes the proof of Proposition 3.

PROPOSITION 4. *Let $(h, k) = 1$ and $k > 1$. Then, for $\bar{S}_p(s; r, h, k)$, we have*

$$\bar{S}_p(s; r, h, k) = \sum_{n=1}^{\infty} \bar{a}_n (s+1-r)^n, \quad |\bar{a}_n| \leq |r\frac{q^{n-1}}{(n-1)!}|, \quad n \geq 1,$$

and hence

$$|\bar{S}_p(s; r, h, k) - \bar{S}_p(s'; r, h, k)| \leq |r||s-s'|, \quad s, s' \in Z_p.$$

Moreover, if $p = 2$ and $r > 1$, we see $|\bar{a}_n| \leq |2r\frac{q^{n-1}}{(n-1)!}|, \quad n \geq 1$, *and*

$$|\bar{S}_2(s; r, h, k) - \bar{S}_2(s'; r, h, k)| \leq |2r||s-s'|, \quad s, s' \in Z_2.$$

Proof. Recalling that $(1-\zeta)^{n+1}E_n(\zeta) \in Z[\zeta]$, $n \geq 0$, we have $|E_n(\zeta)| \leq 1$, if $|\zeta - 1| = 1$. Let $k = k_0 p^{\nu}$, $(k_0, p) = 1$. Then by the definition (12) of $\bar{S}_p$,

$$\bar{a}_n = r \sum_{\zeta^k = 1, \zeta^{p^\nu} \neq 1} c_{n-1,r}(\zeta^h) E_{r-1}(\zeta^{-1}), \quad n \geq 1.$$

Hence, by (8), the first half of this proposition is obvious.

Now, in general, it follows from the definition of $E_n(\zeta)$ that

$$(20) \qquad E_0(\zeta^{-1}) = -E_0(\zeta) - 1; \quad E_{r-1}(\zeta^{-1}) = (-1)^r E_{r-1}(\zeta), \; r > 1,$$

for every root of unity ζ. On the other hand, we can see by a little calculation that

$$(21) \qquad c_{n,r}(\zeta^{-1}) = (-1)^r c_{n,r}(\zeta), \quad n \geq 0, \quad r \geq 1,$$

for all ζ, $|\zeta - 1| = 1$. Let $p = 2$ and $r > 1$. Then, by cupling the terms for ζ and ζ^{-1} in the above expression of $\bar{a}_n$ (note that $\zeta \neq \zeta^{-1}$), we get the second half. This completes the proof of Proposition 4.

Since $S_p(s; r, h, 1) = 0$ for r odd (§3), $\bar{S}_p(s; r, h, k) = S_p(s; r, h, k)$ if $(h, k) = (k, p) = 1$ and $r \not\equiv 0 \pmod 2$. In this case, Proposition 4 describes the property of $S_p(s; r, h, k)$. For r even, we obtain the following

PROPOSITION 5. *For even positive integer r, let*

$$S_p(s\,;r,h,1)=\sum_{n=0}^{\infty}a'_n(s+1-r)^n,\quad a'_n\in Q_p.$$

Then,

$$a'_0=\begin{cases}\left(1-\frac{1}{p}\right)B_r, & \text{if } r\equiv 0 \pmod e,\\ 0, & \text{if } r\not\equiv 0 \pmod e,\end{cases}$$

$$|a'_1|\le\left|\frac{1}{p}\right|,\quad |a'_n|\le\left|\frac{p^{n-3}}{n!}\right|,\quad n\ge 2,\qquad \text{if } p>2,\ r\equiv 0 \pmod e,$$

$$|a'_1|\le|r|,\quad |a'_n|\le\left|\frac{rp^{n-2}}{n!}\right|,\quad n\ge 2,\qquad \text{if } p>2,\ r\not\equiv 0 \pmod e,$$

$$|a'_1|\le\left|\frac{1}{p}\right|,\quad |a'_n|\le\left|\frac{2q^{n-2}}{n!}\right|,\quad n\ge 2,\qquad \text{if } p=2.$$

Proof. By (11), we obtain

$$S_p(s\,;r,h,1)=U_r^{(0)}(s)\,F^{(0)}(s\,;r,1)B_r.$$

If we let $F^{(0)}(s\,;r,1)=\sum_{n=0}^{\infty}b_{n,r}^{(0)}(s+1-r)^n$, then Proposition 1(a)(b), (13) and (17) lead, respectively, to

$$b_{0,r}^{(0)}=\begin{cases}\left(1-\frac{1}{p}\right)\log(1+q) & \text{if } r\equiv 0 \pmod e,\\ 0, & \text{if } r\not\equiv 0 \pmod e,\end{cases}$$

$$b_{n,r}^{(0)}\equiv 0\left(\bmod\ \frac{p^n}{n!}\right),\ n\ge 1,\qquad \text{if } p>2,$$

$$b_{n,r}^{(0)}=\frac{2q^n}{n!}\,\xi^{(n)}\equiv 0\left(\bmod\ \frac{2q^n}{n!}\right),\ n\ge 1,\quad \text{if } p=2.$$

On the other hand if we let $U_r^{(0)}(s)=\sum_{n=0}^{\infty}u_n(s+1-r)^n$, then

$$u_0=\frac{1}{\log(1+q)},\quad |u_n|=\left|B_n\frac{q^{n-1}}{n!}\right|,\quad n\ge 1.$$

Since, moreover, $\left|\frac{B_n}{n}\right|\le 1$ if $1<n\not\equiv 0 \pmod e$ and $|B_n|=\left|\frac{1}{p}\right|$ if $0<n\equiv 0 \pmod e$, in the same manner as in the proof of Proposition 3, the result follows.

THEOREM 2. *Suppose that $(h,k)=1$ and $(k,p)>1$.*

(a) *If $p=2$, $k=2k_0$, $(k_0,2)=1$ and $r\not\equiv 0 \pmod e$, then*

$$S_2(r-1\,;r,\,h,\,k)=0,$$
$$|\,S_2(s\,;r,\,h,\,k)-S_2(s'\,;r,\,h,\,k)\,|\le|\,q\,|\,|\,s-s'\,|,\quad s,\,s'\in Z_2.$$

(b) *Otherwise,*

$$S_p(r-1\,;r,\,h,\,k)=\begin{cases}(1-p^{r-1})B_r, & \text{if } r\equiv 0 \pmod e,\\ \omega^r(h)B_{r,\omega^{-r}}, & \text{if } r\not\equiv 0 \pmod e,\end{cases}$$
$$|\,S_p(s\,;r,\,h,\,k)-S_p(s'\,;r,\,h,\,k)\,|\le|\,s-s'\,|,\quad s,\,s'\in Z_p.$$

Proof. Let $p=2$ and $r\not\equiv 0 \pmod 2$. Since $S_2(s\,;r,\,h,\,2)=0$, the function $S_2(s\,;r,\,h,\,2k_0)=\bar{S}_2(s\,;r,\,h,\,2k_0)$ has the expansion

$$S_2(s\,;r,\,h,\,2k_0)=\sum_{n=1}^{\infty}a_n(s+1-r)^n,\quad a_n=r\sum_{\zeta^k=1,\zeta^2\ne 1}c_{n-1,r}(\zeta^h)E_{r-1}(\zeta^{-1}).$$

Now, since

$$\mu_{-\zeta}(a+2^NZ_2)=\frac{(-\zeta)^{2^N-a}}{1-(-\zeta)^{2^N}}=-\mu_\zeta(a+2^NZ_2),\ 0\le a<2^N,\ (a,\,2)=1,$$

we have $d\mu_{-\zeta}(x)=-d\mu_\zeta(x)$, $x\in Z_2^*$, so that

$$c_{n,r}(-\zeta)=-c_{n,r}(\zeta),\quad n\ge 0,\quad r\ge 1.$$

Hence

$$a_n=r\sum_{\zeta^{k_0}=1,\zeta\ne 1}c_{n-1,r}(\zeta^h)\,(E_{r-1}(\zeta^{-1})-E_{r-1}(-\zeta^{-1})),\quad n\ge 1.$$

Write $d_n(\zeta)$, $\zeta\ne 1$, for the summand on the right. Then, since

$$E_{r-1}(\zeta^{-1})-E_{r-1}(-\zeta^{-1})=2^rE_{r-1}(\zeta^{-2})-2\,E_{r-1}(-\zeta^{-1})\equiv 0 \pmod 2,$$

we have $|\,d_n(\zeta)\,|\le\left|\dfrac{2q^{n-1}}{(n-1)!}\right|$. On the other hand, it follows from (20) and (21) that $d_n(\zeta)=d_n(\zeta^{-1})$. Now the order of ζ is odd ($\ne 1$), so clearly $\zeta\ne\zeta^{-1}$. Hence we have

$$|\,a_n\,|\le\left|\frac{q^n}{(n-1)!}\right|\le|\,q\,|,\quad n\ge 1.$$

Therefore the assertion (a) is proved. The assertion (b) is obvious from Propositions 3 and 4. This completes the proof of Theorem 2.

Since $S_p(s; r, h, k) = \bar{S}_p(s; r, h, k) + S_p(s; r, h, 1)$ if $(k, p) = 1$, we similarly obtain from Propositions 4 and 5 the following

THEOREM 3. *Suppose that* $(h, k) = 1$ *and* $(k, p) = 1$.

(a) *If* $r \equiv 0 \pmod{e}$, *then*

$$S_p(r-1; r, h, k) = \left(1 - \frac{1}{p}\right) B_r,$$

$$| S_p(s; r, h, k) - S_p(s'; r, h, k) | \leq \left| \frac{1}{p} \right| | s - s' |, \quad s, s' \in Z_p.$$

(b) *If* $r \not\equiv 0 \pmod{e}$, *then*

$$S_p(r-1; r, h, k) = 0,$$

$$| S_p(s; r, h, k) - S_p(s'; r, h, k) | \leq | r | | s - s' |, \quad s, s' \in Z_p.$$

$$(\leq | 2r | | s - s' | \text{ if } p = 2,\ r > 1).$$

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