K. Cho
Nagoya Math. J.
Vol. 147 (1997), 63–69

A GENERALIZATION OF KITA AND NOUMI'S VANISHING THEOREMS OF COHOMOLOGY GROUPS OF LOCAL SYSTEM

KOJI CHO

Abstract. We prove vanishing theorems of cohomology groups of local system, which generalize Kita and Noumi's result and partially Aomoto's result. Main ingredients of our proof are the Hodge to de Rham spectral sequence and Serre's vanishing theorem in algebraic geometry.

§1. Introduction

Vanishing theorems of cohomology groups of local system are important to study the theory of hypergeometric functions (cf. [AK]). In fact, several vanishing theorems of cohomology groups of local system are known. For example, Kita and Noumi [KN] obtained a vanishing theorem by using the technique of a filtration attached to logarithmic forms. Our aim is to generalize their result and give a simple algebro-geometric proof. Of course there is a tradeoff: While their proof is concrete and elementary, ours is abstract and depends on big theorems.

In order to state our theorem, we begin with the definition on the sheaf of logarithmic 1-forms along a divisor adapted in this paper, which is different from the one given in [S]. Let M be a complex projective manifold of dimension n, D an effective reduced divisor on M. Let $D = \sum_{j=1}^{m} D_j$ be the irreducible decomposition of D and f_j the defining equation of D_j. For $x \in M$, Assume $x \in D_{j_i}, i = 1, \dots, k$ and $x \notin D_j$ for $j \neq \{1, \dots, m\} \setminus \{j_1, \dots, j_k\}$. Then the sheaf $\Omega^1(\log D)$ of logarithmic 1-forms along D is defined as follows: $\psi \in \Omega^1(\log D)_x$ if and only if $\psi = \sum_{i=1}^{k} h_k \frac{df_{j_i}}{f_{j_i}} + \varphi$, where h_i is a holomorphic function at x and φ is a holomorphic 1-form at x. We also define $\Omega^p(\log D)$ as $\wedge^p \Omega^1(\log D)$. Let $\mathcal{E}$ be a locally free sheaf on M. A meromorphic connection ∇ on $\mathcal{E}$ is said to be an integrable holomorphic

Received December 18, 1995.
Revised February 2, 1996.

connection with logarithmic poles along D, if it satisfies $\nabla^2 = 0$ and

$$\nabla\colon \mathcal{E} \to \Omega^1(\log D) \otimes \mathcal{E}.$$

Now our theorem is:

THEOREM 1. *Let M be a complex projective manifold of dimension n, D an effective reduced divisor on M, $D = \sum_{j=1}^m D_j$ the irreducible decomposition of D. We consider a locally free sheaf $\mathcal{E}$ with an integrable holomorphic connection ∇ with logarithmic poles along D*

$$\nabla\colon \mathcal{E} \to \Omega^1(\log D) \otimes \mathcal{E}.$$

Let $\mathcal{L}$ be a local system defined by ∇ on $U := M \setminus D$. Namely, $\mathcal{L} := \mathrm{Ker}(\nabla|_U\colon \mathcal{E}|_U \to \Omega^1(\log D) \otimes \mathcal{E}|_U)$. We assume that there exists a subset $J \subset \{1, \ldots, m\}$ satisfying

(1) *$D' := \sum_{j \in J} D_j$ is a support of an effective ample divisor H*

(2) *D is simple normal crossing in a neighborhood of D'*

(3) *the monodromies of $\mathcal{L}$ around D_j (for any $j \in J$) do not have an eigenvalue 1.*

Then

$$H^k(U, \mathcal{L}) = 0, \qquad k \neq n.$$

Hence we have

$$\dim H^n(U, \mathcal{L}) = (-1)^n \operatorname{rank}(\mathcal{L}) \chi_{\mathrm{top}}(U).$$

One can also prove the following theorem which partially generalizes Aomoto's vanishing theorem [A].

THEOREM 2. *Let M be a complex projective manifold of dimension n, D an effective reduced divisor on M, $D = \sum_{j=1}^m D_j$ the irreducible decomposition of D. We consider a locally free sheaf $\mathcal{E}$ with an integrable holomorphic connection ∇ with logarithmic poles along D*

$$\nabla\colon \mathcal{E} \to \Omega^1(\log D) \otimes \mathcal{E}.$$

Let $\mathcal{L}$ be a local system defined by ∇ on $U := M \setminus D$. We assume that there exists a subset $J \subset \{1, \ldots, m\}$ satisfying

(1) *$D' := \sum_{j \in J} D_j$ is a support of an effective ample divisor*

(2) *Let $B := \{x \in D' \mid D$ is not simple normal crossing at $x\}$ and $J' := \{j \mid D_j \cap B \neq \emptyset\}$. Then the eigenvalues of monodromies of $\mathcal{L}$ around D_j (for any $j \in J \cup J'$) are generic.*

Then

$$H^k(U, \mathcal{L}) = 0, \qquad k \neq n.$$

Hence we have

$$\dim H^n(U, \mathcal{L}) = (-1)^n \operatorname{rank}(\mathcal{L}) \chi_{\mathrm{top}}(U).$$

Remark. In the condition (2) of the above theorem, we mean *generic* if the following condition is satisfied: There exists an embedded resolution $\pi: \tilde{M} \to M$ such that $\pi^{-1}(D_b)$ is simple normal crossing, where $D_b := \sum_{j \in J \cup J'} D_j$, and that none of eigenvalues of the residues of $\pi^* \nabla$ around the irreducible components of $\pi^{-1}(D_b)$ is not integer.

Since the proofs of Theorems 1 and 2 are almost identical, we only give a proof of Theorem 1.

§2. Proof of Theorem 1

Since U is affine $H^k(U, \mathcal{L}) = 0, k > n$. We show that $H^k(U, \mathcal{L}) = 0, k < n$. By the condition (2) and Hironaka's theorem on resolution of singularities, we can find a blowing up $\pi\ \tilde{M} \to M$ along nonsingular centers such that $\tilde{D} := \pi^{-1}(D)$ is a simple normal crossing divisor and the exceptional locus is contained in $\tilde{D}$ and does not meet $\pi^{-1}(D')\,(\simeq D')$, which will be also denoted D'.

Let E be an effective divisor supported in the exceptional divisor such that $-E$ is π-ample. It follows from the condition (1) that for sufficiently large $N > 0$, $\tilde{H} := \pi^*(NH) - E$ is very ample.

Let $(\tilde{\mathcal{E}}, \tilde{\nabla})$ be the canonical extension of $\pi|_U{}^* \mathcal{L}\,(\simeq \mathcal{L})$ defined by Deligne [D1]. In particular, the eigenvalues of residues of $\tilde{\nabla}$ are not positive integers. Moreover from the condition (3) and the construction of $\tilde{M}$, the eigenvalues of residues of $\tilde{\nabla}^{(-\nu)}$ are not positive integers for $\nu > 0$, where $\tilde{\nabla}^{(-\nu)}$ is a connection with logarithmic poles induced by $\tilde{\nabla}$ on $\tilde{\mathcal{E}}(-\nu \tilde{H})$. Thus by the Deligne theorem [D1, II, Théorème 6.2]

$$H^k(U, \mathcal{L}) \simeq \mathbf{H}^k(\tilde{M}, (\Omega^\bullet(\log \tilde{D}) \otimes \tilde{\mathcal{E}}(-\nu \tilde{H}), \tilde{\nabla}^{(-\nu)})) \qquad \text{for } \nu > 0.$$

By the Serre duality and vanishing theorem, for $q < n$

$$H^q(\tilde{M}, \Omega^p(\log \tilde{D}) \otimes \tilde{\mathcal{E}}(-\nu \tilde{H})) = 0 \qquad \text{for sufficiently large } \nu > 0.$$

Considering the Hodge to de Rham spectral sequence [D2, 1.4]

$$H^q(\tilde{M}, \Omega^p(\log \tilde{D}) \otimes \tilde{\mathcal{E}}(-\nu \tilde{H})) \Longrightarrow \mathbf{H}^k(\tilde{M}, (\Omega^\bullet(\log \tilde{D}) \otimes \tilde{\mathcal{E}}(-\nu \tilde{H}), \tilde{\nabla}^{(-\nu)})),$$

we get

$$H^k(U, \mathcal{L}) = 0, \qquad k < n.$$

Since $\chi(\mathcal{L}) = \operatorname{rank}(\mathcal{L})\chi_{\mathrm{top}}(U)$, we get the last assertion. □

§3. Comparison with Kita and Noumi's results and some examples

In order to compare our theorem with Kita and Noumi's results, for the convenience of the reader, we will explain their result in the case of local system of rank one. Let $P_0(x) := x_0, P_1(x), \ldots, P_m(x)$ be distinct homogeneous polynomials with $n+1$ variables $x = (x_0, x_1, \ldots, x_n)$, D_j an effective divisor on $\mathbf{P}^n$ defined by $P_j(x) = 0$, $D := \sum_{j=0}^m D_j$, and U its complement. We consider an integrable holomorphic connection ∇ with logarithmic poles along D

$$\nabla \colon \mathcal{O} \to \Omega^1(\log D).$$

Let us denote $\mathcal{L}$ a local system of rank one on U defined by ∇, namely

$$\mathcal{L} := \operatorname{Ker} \nabla|_U.$$

Set $\bar{P}_j(x_1, \ldots, x_n) := P_j(0, x_1, \ldots, x_n) \quad (j = 1, \ldots, m)$.

THEOREM. ([KN]) *Assume the following conditions:*

(1) *For any* $1 \le r \le \min(m, n-1)$ *and* $1 \le j_1 < \ldots < j_r \le m$

$$\operatorname{height}(d\bar{P}_{j_1} \wedge \ldots \wedge d\bar{P}_{j_r}, \bar{P}_{j_1}, \ldots, \bar{P}_{j_r}) \ge n,$$

where $(d\bar{P}_{j_1} \wedge \ldots \wedge d\bar{P}_{j_r}, \bar{P}_{j_1}, \ldots, \bar{P}_{j_r})$ *denotes the ideal of* $\mathbf{C}[x_1, \ldots, x_n]$ *generated by all* r*-minors of the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial \bar{P}_{j_1}}{\partial x_1} & \cdots & \frac{\partial \bar{P}_{j_1}}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \bar{P}_{j_r}}{\partial x_1} & \cdots & \frac{\partial \bar{P}_{j_r}}{\partial x_n} \end{pmatrix}$$

and $\bar{P}_{j_1}, \ldots, \bar{P}_{j_r}$.

(2) *For any* $1 \leq s \leq \min(m,n)$ *and* $1 \leq j_1 < \ldots < j_s \leq m$, $\bar{P}_{j_1}, \ldots, \bar{P}_{j_r}$ *is a regular sequence.*

If moreover the residue of ∇ *along a hyperplane* D_0 *is not an integer, then*

$$H^k(U, \mathcal{L}) = 0, \qquad k \neq n.$$

Remark. One can easily check that the above conditions (1), (2) are equivalent to saying that the divisor D is simple normal crossing in a neighborhood of a hyperplane D_0.

As an application of our Theorems, we will give two interesting examples in the theory of hypergeometric functions.

EXAMPLE 1. Let us consider the arrangement in $\mathbf{P}^2$ with the homogeneous coordinates $(u_0 : u_1 : u_2)$ defined by $P_0 \cdot \ldots \cdot P_4 = 0$, where $P_0 := u_0, P_1 := u_1, P_2 = u_2, P_3 = u_0 - u_1 - u_2, P_4 := u_1 u_2 - x_1 u_0 u_1 - x_2 u_0 u_2$ and $x_1 \neq 0, x_2 \neq 0$. We also let

$$\nabla = d + \omega \wedge = d + \sum_{i=1}^{4} \alpha_i \frac{dp_i}{p_i} \wedge,$$

where $p_i = P_i(1, u_1, u_2), \alpha_i \in \mathbf{C}, \quad i = 1, \ldots 4$. Then by Theorem 1, if α_3 is not an integer, $H^k(\mathcal{L}) = 0, k \neq 2$ and hence $\dim H^k(\mathcal{L}) = 4$.

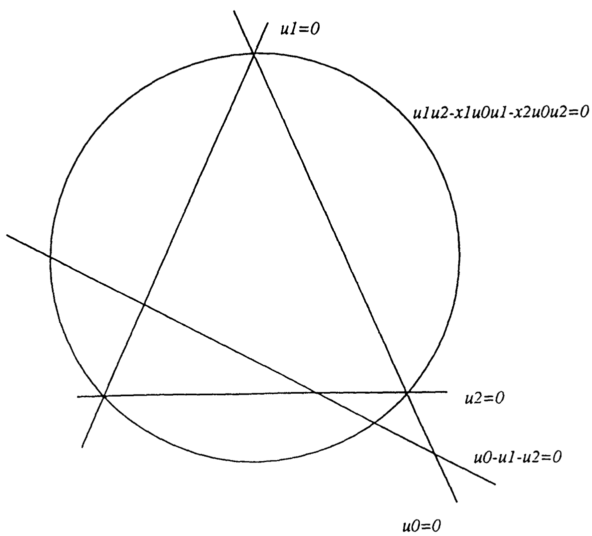

FIG. 1

EXAMPLE 2. Let us consider the arrangement in $\mathbf{P}^2$ with the homogeneous coordinates $(u_0 : u_1 : u_2)$ defined by $l_0 \cdot \ldots \cdot l_5 = 0$, where $l_0 := u_0, l_1 := u_1, l_2 = u_2, l_3 = u_0 - u_1, l_4 := u_0 - u_2, l_5 := u_1 - u_2$ and $\nabla = d + \omega\wedge = d + \sum_{i=0}^{5} \alpha_i \frac{dl_i}{l_i}\wedge$, where $\alpha_0, \ldots, \alpha_5 \in \mathbf{C}$ and $\sum_{i=0}^{5} \alpha_i = 0$. If, for example, none of $\alpha_0, \alpha_0 + \alpha_1 + \alpha_3$ and $\alpha_0 + \alpha_2 + \alpha_4$ is an integer, it follows from Theorem 2 that $H^k(\mathcal{L}) = 0, k \neq 2$ and $\dim H^k(\mathcal{L}) = 2$.

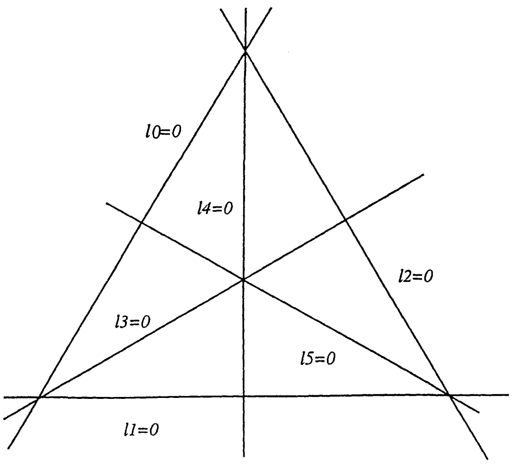

FIG. 2

REFERENCES

[A] K. Aomoto, *Un théorèm du type de Matsushima-Murakami concernant l'intégrale des fonctions multiformes*, J. Math. Pures Appl., **52** (1973), 1–11.

[AK] K. Aomoto and M. Kita, The Theory of Hypergeometric Functions (in Japanese), Springer, 1994.

[D1] P. Deligne, Equations différentielles à points singuliers réguliers, Lect. Notes in Math. 163, Springer, 1970.

[D2] ———, *Théorie de Hodge* II, Publ. Math. I.H.E.S., **40** (1971), 5–58.

[H] R. Hartshorne, Algebraic Geometry, GTM 52, Springer, 1977.

[KN] M. Kita and M. Noumi, *On the structure of cohomology groups attached to integrals of certain many valued analytic functions*, Japan. J.Math., **9** (1983), 113–157.

[S] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo, **Sec. IA 27** (1980), 265–291.

Graduate School of Mathematics
Kyushu University
Fukuoka 810
Japan
cho@math.kyushu-u.ac.jp