

FOURIER COEFFICIENTS AND HECKE EIGENVALUES

WINFRIED KOHNEN

Abstract. We will give certain asymptotic relations between p -eigenvalues and certain Fourier coefficients of Siegel cusp forms of genus g . In particular, it will turn out that potential strong bounds for the Fourier coefficients will imply potential strong bounds for the eigenvalues.

§1. Introduction

Let F be a holomorphic non-zero cuspidal Hecke eigenform of integral weight k on the Siegel modular group $\Gamma_g := Sp_g(\mathbf{Z})$ of genus g . We write $a(T)$ (T a positive definite half-integral matrix of size g) for the Fourier coefficients of F and λ_p (p a prime) for the eigenvalue of F under the usual Hecke operator T_p .

For arbitrary $g \geq 2$ Fourier coefficients and eigenvalues are no longer proportional, and estimates for the first in general cannot be deduced from those for the latter. In the present paper we are interested in the converse question. In sect.3 we shall prove that conjectural strong bounds for the $a(T)$ ($\det T \rightarrow \infty$) imply rather strong bounds for the λ_p ($p \rightarrow \infty$). In particular, if $g = 3$ a "generalized Ramanujan-Petersson conjecture" for the Fourier coefficients implies a "generalized Ramanujan-Petersson conjecture" for the eigenvalues (for a precise statement of these conjectures, cf. sect.2).

We therefore think that a closer study of the Fourier coefficients of Siegel cusp forms -so far somewhat neglected, as it seems- might be profitable.

Our result -in fact- is a refinement of an estimate for the difference $\lambda_p a(T) - a(pT)$ (T fixed $p \rightarrow \infty$) given in [3,sect.1]. As in [3] the proof is rather ad hoc, however in contrast to [3] we need a much more detailed description of the action of T_p on Fourier coefficients as given in [11].

Similar results as proved here can eventually be obtained also for the Hecke operators $T_{i,g-i}(p^2)$ ($0 \leq i < g$) which together with T_p generate the

local Hecke algebra at p . One can also hope that they can be extended to Maass wave forms and -much more general- to cuspidal automorphic forms on arbitrary reductive linear algebraic groups.

§2. Estimates for Fourier coefficients and eigenvalues

Both for the Fourier coefficients and the eigenvalues of F there are “generalized Ramanujan-Petersson conjectures” stating that

$$(1) \quad a(T) \ll_{F,\epsilon} (\det T)^{\frac{k}{2} - \frac{g+1}{4} + \epsilon} \quad (\epsilon > 0)$$

and

$$(2) \quad \lambda_p \ll_{g,\epsilon} p^{\frac{gk}{2} - \frac{g(g+1)}{4} + \epsilon} \quad (\epsilon > 0),$$

respectively. Conjecture (1) is due to Resnikoff-Saldaña [12] (the tribute to Ramanujan-Petersson here does not seem to be quite motivated) while (2) goes back to Satake [13], Kurokawa [9] and (in much more general context) to Langlands [10]. Actually (2) is an easy consequence of what is usually called the Ramanujan-Petersson conjecture for F , meaning that all the Satake p -parameters of F are of absolute value 1.

Using the analytic properties of the Rankin-Dirichlet series of F and the well-known asymptotic bounds for class numbers due to Siegel and Kitaoka, one can show that (1) is best possible. One should expect that (2) is also best possible, but we do not know any proof of this.

Both conjectures in general are known to be wrong. If $g = 2$, forms F that are Saito-Kurokawa lifts give counterexamples both to (1) and (2), cf. [1,4]. Also if $8|g$, there are certain theta series with spherical harmonics of weight $\frac{g}{2} + 1$ whose Fourier coefficients do not satisfy (1), cf. [1]. Nevertheless, there is some hope that (1) and (2) generically should be true.

If $g = 2$ and $k \geq 3$, recently Weissauer [15] proved the Ramanujan-Petersson conjecture in its stronger form for F not a Saito-Kurokawa lift using the trace formula. For arbitrary $g \geq 2$ one knows according to Faltings-Chai [5,chap.8,sect.5] that at least the product of the p -parameters of F (suitably normalized) is of absolute value 1.

Concerning the λ_p 's, rather good estimates -however still far away from (2)- have been obtained for arbitrary $g \geq 2$ using either local representation theory [3] or methods from arithmetic algebraic geometry [7]. According to our information, the best bound so far available (under the assumption $k \geq g + 1$) is

$$(3) \quad \lambda_p \ll_{g,\epsilon} p^{\frac{gk}{2} - \frac{g(g+2)}{8}}$$

([7]; if g is odd, a slightly better bound is proved).

Concerning estimates for the Fourier coefficients our knowledge at present is still extremely limited. After Hecke's bound

$$(4) \quad a(T) \ll_F (\det T)^{\frac{k}{2}}$$

the best result for arbitrary $g \geq 2$ (and $k > g + 1$) at present seems to be

$$a(T) \ll_{F,\epsilon} (\det T)^{\frac{k}{2} - \frac{1}{2g} - (1 - \frac{1}{g})\beta_g + \epsilon} \quad (\epsilon > 0)$$

where

$$\beta_g^{-1} := 4(g-1) + 4 \left[\frac{g-1}{2} \right] + \frac{2}{g+2}$$

[2].

Also for $4|g$ there are examples of forms F of arbitrary large weights with

$$a(T) \ll_{F,\epsilon} (\det T)^{\frac{k}{2} - \frac{1}{2} + \epsilon} \quad (\epsilon > 0)$$

[8].

§3. Statement of result

In [3, sect.1] the estimate

$$(5) \quad \lambda_p a(T) - a(pT) \ll_F (\det T)^{\frac{k}{2}} p^{\frac{gk}{2} - 1}$$

was shown. Let $\alpha \geq 0$ be fixed. Then as a consequence of (5) (choosing T with $a(T) \neq 0$) one obtains

$$(6) \quad \lambda_p \ll_F p^{\frac{gk}{2} - \min\{1, g\alpha\}} \quad (p \rightarrow \infty)$$

provided

$$(7) \quad a(T) \ll_F (\det T)^{\frac{k}{2} - \alpha} \quad (\det T \rightarrow \infty).$$

The proof of (5) only uses the standard explicit description of the action of T_p on Fourier coefficients together with the Hecke estimate (4) for the latter. Inspecting the proof and using the hypothesis (7) rather than (4), it is not difficult to improve somewhat upon the p -exponents in (5) and hence in (6). However, much better results can be obtained if one exploits a more detailed information of the action of T_p as given in [11]. We shall prove

THEOREM. Let $\alpha \geq 0$ be fixed and suppose that the bound

$$(8) \quad a(T) \ll_F (\det T)^{\frac{k}{2}-\alpha}$$

holds for all T . Then one has

$$(9) \quad \lambda_p a(T) - a(pT) \ll_F (\det T)^{\frac{k}{2}-\alpha} p^{\frac{gk}{2}-\kappa_\alpha} \quad (p > 2, p \nmid \det(2T))$$

where

$$\kappa_\alpha := \begin{cases} g\alpha - (2\alpha - 2) & (0 \leq \alpha \leq \frac{3}{2}) \\ g\alpha - (\alpha - \frac{1}{2})^2 & (\alpha \geq \frac{3}{2}). \end{cases}$$

Observe that in the range $0 \leq \alpha \leq \frac{g+1}{4}$ the function $\alpha \mapsto \kappa_\alpha$ is positive and non-decreasing.

In the same way as before one deduces

COROLLARY. Under the hypothesis (8) one has

$$(10) \quad \lambda_p \ll_F p^{\frac{gk}{2}-\kappa'_\alpha} \quad (p \rightarrow \infty)$$

where

$$\kappa'_\alpha := \begin{cases} g\alpha & (0 \leq \alpha \leq 1) \\ \kappa_\alpha & (\alpha \geq 1). \end{cases}$$

Note that if one should know (8) for some “large” α (i.e. α close to $\frac{g+1}{4}$), then by (10) one could considerably improve upon (3). It also follows from (10) that the bound (1) implies (2) if $g = 3$.

In practice, however, as is clear from the discussion in sect.2 the bounds for λ_p so far truly provable using (10) are much weaker than those obtained previously by other tools.

§4. Proof

We use the notation

$$A[B] = B'AB$$

for matrices A and B of appropriate sizes (B' = transpose of B).

We abbreviate $\Gamma \equiv \Gamma_g$. We let $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ with 0 resp. E the zero resp. unit matrix of size g . We denote by

$$\mathcal{O}_p := \{\gamma \in \mathbf{Z}^{(2g,2g)} \mid J[\gamma] = pJ\}$$

the set of symplectic similitudes of scale p . For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{O}_p$ we put

$$(F|_k \gamma)(Z) := (\det \gamma)^{\frac{k}{2}} \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1})$$

(Z in the Siegel upper half-space of genus g).

Then by definition

$$T_p F = p^{\frac{gk}{2} - \frac{g(g+1)}{2}} \sum_{\gamma \in \Gamma \backslash \mathcal{O}_p} F|_k \gamma.$$

As is well-known, a set of representatives for $\Gamma \backslash \mathcal{O}_p$ can be chosen in upper triangular form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. More precisely, when D runs over a fixed set of left non-associated (w.r.t. $GL_g(\mathbf{Z})$) right-divisors of pE and one sets $A = pD'^{-1}$, and moreover B runs over a set of representatives of matrices in $\mathbf{Z}^{(g,g)}$ satisfying $B'D = D'B$ modulo the equivalence relation $B_1 \sim B_2$ if $B_1 - B_2 = SD$ with $S \in \mathbf{Z}^{(g,g)}$, $S' = S$, then $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ runs over a set of representatives for $\Gamma \backslash \mathcal{O}_p$ [6, chap.IV, sects.2,3]. Therefore

$$(T_p F)(Z) = p^c \sum_{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma \backslash \mathcal{O}_p} (\det D)^{-k} F((pD'^{-1}Z + B)D^{-1})$$

where

$$(11) \quad c := gk - \frac{g(g+1)}{2}.$$

Inserting the Fourier expansion

$$F(Z) = \sum_{T > 0} a(T) e^{2\pi i \text{tr}(TZ)}$$

we obtain

$$(12) \quad (T_p F)(Z) = p^c \sum_{T > 0, \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma \backslash \mathcal{O}_p} a(T) (\det D)^{-k} e^{2\pi i \text{tr}(TBD^{-1})} e^{2\pi i \text{tr}(pT[D'^{-1}]Z)}.$$

We now follow [11] (one could also consult [6, chap.IV, sect.3]). According to [11, Satz 6] we can choose

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} p\Delta^{-1} & G \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} U^{-1} & 0 \\ 0 & U' \end{pmatrix}$$

with Δ , U and G as follows:

- $\Delta = \text{diag}[d_1 d_2 \dots d_g]$ is an integral diagonal matrix with $d_1 | d_2 | \dots | d_g | p$, $d_\mu > 0$ ($\forall \mu$);
- for Δ fixed, U runs over a set of representatives for $GL_g(\mathbf{Z})/GL_{g,\Delta}(\mathbf{Z})$ where $GL_{g,\Delta}(\mathbf{Z}) := GL_g(\mathbf{Z}) \cap \Delta GL_g(\mathbf{Z}) \Delta^{-1}$;
- $G = (g_{\mu\nu})$ runs over all matrices in $\mathbf{Z}^{(g,g)}$ such that $g_{\mu\nu} = \frac{d_\nu}{d_\mu} g_{\nu\mu}$ ($\mu < \nu$) and $0 \leq g_{\mu\nu} < d_\nu$ ($\mu \geq \nu$).

Inserting into (12) one picks up the exponential sum

$$\begin{aligned} & \sum_G e^{2\pi i \text{tr}(TG\Delta^{-1})} \\ &= \begin{cases} \prod_{\nu=1}^g d_\nu^{g+1-\nu} & \text{if } d_\mu | t_{\mu\mu} \forall \mu, d_\nu | 2t_{\mu\nu} \forall \mu, \nu \text{ with } \mu > \nu \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

($T = (t_{\mu\nu})$). If $b(T)$ denotes the T -th Fourier coefficient of $T_p F$, one obtains from this easily

$$(13) \quad b(T) = p^c \sum_{\Delta, U} \left(\prod_{\nu=1}^g d_\nu^{g+1-\nu-k} \right) a\left(\frac{1}{p} T[U\Delta]\right)$$

with Δ as before and U for fixed Δ running over elements in $GL_g(\mathbf{Z})/GL_{g,\Delta}(\mathbf{Z})$ such that

$$(14) \quad \frac{1}{p} T[U\Delta] = \begin{pmatrix} d_1 s_{11} & d_1 s_{12} & d_1 s_{13} & \dots & d_1 s_{1g} \\ d_1 s_{21} & d_2 s_{22} & d_2 s_{23} & \dots & d_2 s_{2g} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_1 s_{g1} & d_2 s_{g2} & d_3 s_{g3} & \dots & d_g s_{gg} \end{pmatrix}$$

where $S = (s_{\mu\nu}) \in \mathbf{Z}^{(g,g)}$ is half-integral (and symmetric).

So far the above computations would have been valid with p replaced by an arbitrary positive integer. We shall now use that p is a prime. In this case we have

$$d_1 = \dots = d_r = 1, d_{r+1} = \dots = d_g = p$$

for some $r \in \{0, 1, \dots, g\}$. If $r = 0$ or $r = g$ one has $\Delta = \text{diag}[p, \dots, p]$ or $\Delta = E$, respectively, and $GL_{g,\Delta}(\mathbf{Z}) = GL_g(\mathbf{Z})$, $U = E$. If $r \in \{1, \dots, g-1\}$ then according to the arguments given in [11, pp.103–105] one can choose as a set of representatives for $GL_g(\mathbf{Z})/GL_{g,\Delta}(\mathbf{Z})$ the matrices (QR) where Q runs over a set of representatives for matrices in $(\mathbf{Z}/p\mathbf{Z})^{(g,r)}$ having maximal rank mod p , modulo right multiplication by $GL_r(\mathbf{Z}/p\mathbf{Z})$, and each such Q is completed in exactly one way to an element (QR) in $GL_g(\mathbf{Z})$. The condition (14) then is equivalent to $T[Q] \equiv 0 \pmod{p}$.

From the condition (14) we see, in particular, that the value

$$\det\left(2 \cdot \frac{1}{p} T[U\Delta]\right) = p^{g-2r} \det(2T)$$

is integral, hence since by assumption $p \nmid \det(2T)$, we conclude that only matrices $\Delta_r = \text{diag}[d_1 \dots d_r d_{r+1} \dots d_g]$ with $r \in \{0, 1, \dots, \lfloor \frac{g}{2} \rfloor\}$ can contribute to the sum on the right of (13). The term with $r = 0$ gives the contribution

$$\left(\prod_{\nu=1}^g p^{g+1-\nu-k}\right) a(pT),$$

hence its contribution to $b(T)$ is $a(pT)$.

Since $T_p F = \lambda_p F$ we therefore obtain

$$\begin{aligned} \lambda_p a(T) - a(pT) &= p^c \sum_{r=1}^{\lfloor \frac{g}{2} \rfloor} \sum_U \left(\prod_{\nu=r+1}^g p^{g+1-\nu-k} \right) a\left(\frac{1}{p} T[U\Delta_r]\right) \\ &\ll_F p^c \sum_{r=1}^{\lfloor \frac{g}{2} \rfloor} \sum_U \left(\prod_{\nu=r+1}^g p^{g+1-\nu-k} \right) \left(\det\left(\frac{1}{p} T[U\Delta_r]\right) \right)^{\frac{k}{2}-\alpha} \end{aligned}$$

where in the last line we have used (8).

Since

$$\sum_{\nu=r+1}^g (g+1-\nu-k) = \frac{g(g+1)}{2} - gk - r(g+1-k) + \frac{r(r+1)}{2}$$

and

$$\det\left(\frac{1}{p}T[U\Delta_r]\right) = p^{g-2r} \det T,$$

it follows (observing (11)) that

$$\lambda_p a(T) - a(pT) \ll_F (\det T)^{\frac{k}{2}-\alpha} p^{g(\frac{k}{2}-\alpha)} \sum_{r=1}^{\lfloor \frac{g}{2} \rfloor} p^{r(2\alpha-g+\frac{r-1}{2})} N_{r,T}$$

where

$$N_{r,T} := \#\{Q \in \mathbf{F}_p^{(g,r)} \mid \text{rk } Q = r, T[Q] = 0\} / GL_r(\mathbf{F}_p)$$

(here we have identified $T \in \frac{1}{2}\mathbf{Z}^{(g,g)}$ with its image in $\mathbf{F}_p^{(g,g)}$).

LEMMA. *Let $r \in \{1, \dots, \lfloor \frac{g}{2} \rfloor\}$. Then $N_{r,T} \ll_g p^{r(g-r)-\frac{r(r+1)}{2}}$.*

Proof. Since $p \neq 2$, we may assume that T is diagonal. Note that T is regular by hypothesis.

Any $Q \in \mathbf{F}_p^{(g,r)}$ with $\text{rk } Q = r$ is equivalent modulo right-multiplication by elements in $GL_r(\mathbf{F}_p)$ and simultaneous left-multiplication by permutation matrices in $GL_g(\mathbf{F}_p)$ to a matrix of the form $\begin{pmatrix} E_r \\ X \end{pmatrix}$ with $X \in \mathbf{F}_p^{(g-r,r)}$. Let $P \in GL_g(\mathbf{F}_p)$ be a permutation matrix. If $T[P] = \tilde{T} = \begin{pmatrix} \tilde{T}_1 & 0 \\ 0 & \tilde{T}_2 \end{pmatrix}$ ($\tilde{T}_1 \in \mathbf{F}_p^{(r,r)}$, $\tilde{T}_2 \in \mathbf{F}_p^{(g-r,g-r)}$), then $\tilde{T} \left[\begin{pmatrix} E_r \\ X \end{pmatrix} \right] = 0$ means that $\tilde{T}_2[X] = -\tilde{T}_1$.

Therefore it is sufficient to prove that for $T_1 \in GL_r(\mathbf{F}_p)$, $T_2 \in GL_{g-r}(\mathbf{F}_p)$ one has

$$N_{r,T_1,T_2} := \#\{X \in \mathbf{F}_p^{(g-r,r)} \mid T_2[X] = -T_1\} \ll_g p^{r(g-r)-\frac{r(r+1)}{2}}.$$

According to [14, Hilfssatz 12], under the hypothesis T_1, T_2 invertible and $p \neq 2$ one has

$$(15) \quad p^{\frac{r(r+1)}{2}-(g-r)r} N_{r,T_1,T_2} = \begin{cases} (1 - \delta_1 p^{-\frac{g-r}{2}})(1 + \delta_2 p^{r-\frac{g}{2}}) \prod_{j=1}^{\frac{r}{2}-1} (1 - p^{2j-(g-r)}) & \text{if } g \text{ even, } r \text{ even} \\ (1 - \delta_1 p^{-\frac{g-r}{2}}) \prod_{j=1}^{\frac{r-1}{2}} (1 - p^{2j-(g-r)}) & \text{if } g \text{ odd, } r \text{ odd} \\ (1 + \delta_2 p^{r-\frac{g}{2}}) \prod_{j=1}^{\frac{r-1}{2}} (1 - p^{2j-(g-r)-1}) & \text{if } g \text{ even, } r \text{ odd} \\ \prod_{j=1}^{\frac{r}{2}} (1 - p^{2j-(g-r)-1}) & \text{if } g \text{ odd, } r \text{ even} \end{cases}$$

where

$$\begin{aligned}\delta_1 &:= \left(\frac{(-1)^{\frac{g-r}{2}} \det T_2}{p} \right) && (g-r \text{ even}), \\ \delta_2 &:= \left(\frac{(-1)^{\frac{g}{2}-r} \det T_1 \det T_2}{p} \right) && (g \text{ even}).\end{aligned}$$

Since $r \leq \frac{g}{2}$, each single factor on the right of (15) is ≤ 2 or ≤ 1 , respectively. This proves the Lemma. From the Lemma we now obtain

$$(16) \quad \lambda_p a(T) - a(pT) \ll_F (\det T)^{\frac{k}{2}-\alpha} p^{g(\frac{k}{2}-\alpha)} \sum_{r=1}^{\lfloor \frac{g}{2} \rfloor} p^{r(2\alpha-1-r)}.$$

If $0 \leq \alpha \leq 1$, then all terms in the sum on the right of (16) are ≤ 1 , and the largest term is $p^{2\alpha-2}$.

For $\alpha \geq 1$ we write

$$\sum_{r=1}^{\lfloor \frac{g}{2} \rfloor} p^{r(2\alpha-1-r)} = \sum_{1 \leq r < 2\alpha-1} p^{r(2\alpha-1-r)} + O_g(1),$$

and the right-hand side is

$$\ll_g \begin{cases} p^{2\alpha-2} & \text{if } \alpha \leq \frac{3}{2} \\ p^{(\alpha-\frac{1}{2})^2} & \text{if } \alpha \geq \frac{3}{2} \end{cases}$$

by some elementary considerations. This proves the theorem.

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Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
Germany
winfried@mathi.uni-heidelberg.DE