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BOUNDED P.S.H. FUNCTIONS AND PSEUDOCONVEXITY IN KÄHLER MANIFOLD

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Abstract. It is proved that the C^2-smoothly bounded pseudoconvex domains in $\mathbb{P}^n$ admit bounded plurisubharmonic exhaustion functions. Further arguments are given concerning the question of existence of strictly plurisubharmonic functions on neighbourhoods of real hypersurfaces in $\mathbb{P}^n$.

Let $\Omega \Subset M$ be a pseudoconvex domain in a Kähler manifold M. When M is $\mathbb{P}^k$, Takeuchi [T], showed that the function $-\log \delta_\Omega$ is strictly plurisubharmonic (p.s.h.) in Ω. Here δ_Ω denotes the distance to the boundary for the standard Kähler metric on $\mathbb{P}^k$.

The result was extended by Elencwajg [E] to the case where M is Kähler with strictly positive holomorphic bisectional curvature. See also Suzuki [Su] and Green-Wu [G.W].

Based on their result we show that if $\Omega \Subset M$ is pseudoconvex with $\mathcal{C}^2$ boundary, then there is a bounded strictly p.s.h. function on Ω. When $M = \mathbb{C}^k$ the question was solved by Diedrich-Fornaess [D.F]. For a survey in this case see [S].

We give an example of a compact Kähler manifold M, containing a Stein domain $\Omega \Subset M$, with smooth boundary, however given any neighborhood U of $\partial\Omega$, there is no nonconstant bounded p.s.h. function on $U \cap \Omega$.

We show next that the existence of a strictly p.s.h. function near $\partial\Omega$ is equivalent to the nonexistence of a positive current T of bidimension $(1,1)$ supported on $\partial\Omega$ and satisfying the equation $\partial\bar{\partial}T = 0$. This result is inspired by a duality argument due to Sullivan [Su].

§1. Plurisubharmonic exhaustion function on smoothly bounded domains

Let (M, ω) be a Kähler manifold. Let $\Omega \Subset M$ be a pseudoconvex domain with smooth boundary. We consider first the question of existence of a strictly plurisubharmonic bounded exhaustion function for Ω.

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THEOREM 1.1. *Let $\Omega \Subset M$ be a pseudoconvex domain with $\mathcal{C}^2$ boundary in a complete Kähler manifold M. Assume the holomorphic bisectional curvature of M is strictly positive. Let $r(z) = -\mathrm{dist}(\mathrm{z}, \partial\Omega) =: \delta(\mathrm{z})$ where δ is computed with respect to the Kähler metric. Then there exists $\varepsilon > 0$ such that $\varphi = -(-r)^\varepsilon$ is strictly plurisubharmonic in Ω. More precisely there is a constant c_ε such that*

$$i\partial\overline{\partial}\varphi \geq c_\varepsilon|\varphi|\omega.$$

Proof. Under the above assumption on the curvature, Takeuchi [T] for the projective space, Elencwajg [E], in general, proved that $-\log\delta$ is strictly plurisubharmonic. More precisely there is a constant C depending on the lower bound for the curvature, such that

$$i\partial\overline{\partial}(-\log\delta) \geq C\omega.$$

So if $r = -\delta$ we get

$$-ri\partial\overline{\partial}r + i\partial r \wedge \overline{\partial}r \geq Cr^2\omega. \tag{1}$$

We can choose local coordinates near $p \in \partial\Omega$, such that $x_{2n} = r$, $e_i(r) = 0$, $i = 1, \ldots, n-1$, where (e_i) is an orthonormal basis for the complex tangent space to $\partial\Omega$ near p. Let (a_{ij}) denote the hermitian form corresponding to $i\partial\overline{\partial}r$. Inequality (1) gives in coordinates

$$-r\sum_{i,j=1}^{n} a_{ij}v_i\overline{v}_j + |\partial r|^2|v_n|^2 \geq Cr^2\sum_{j=1}^{n}|v_j|^2. \tag{2}$$

If $v_n = 0$ we obtain the estimate

$$\sum_{i,j=1}^{n-1} a_{ij}v_i\overline{v}_j \geq C|r|\sum_{j=1}^{n-1}|v_j|^2.$$

Expanding (2) we get

$$-r\sum_{i,j=1}^{n-1} a_{ij}v_i\overline{v}_j + 2\mathrm{Re}(-r)\sum_{k=1}^{n-1} a_{nk}v_n\overline{v}_k - ra_{nn}|v_n|^2 + |\partial r|^2|v_n|^2$$
$$\geq Cr^2\sum_{j=1}^{n}|v_j|^2.$$

Replacing v, by $v_j/(-r)$ for $j \leq n-1$ we obtain

$$\sum_{i,j=1}^{n-1} \frac{a_{ij}}{(-r)} v_i \bar{v}_j + 2\mathrm{Re} \sum_{k=1}^{n-1} a_{nk} v_n \bar{v}_k - r a_{nn} |v_n|^2 + |\partial r|^2 |v_n|^2 \geq C \sum_{j=1}^{n-1} |v_j|^2. \tag{3}$$

We write the left hand side of this inequality as

$$Q(z,v) + |\partial r|^2 |v_n|^2.$$

Let $\tilde{Q}(\zeta, v) := \liminf_{\substack{z\to\zeta \\ z\in\Omega}} Q(z,v) = \lim_{s\to 0} \inf_{\substack{|z-\zeta|<s \\ z\in\Omega}} Q(z,v)$. From (3) we obtain

$$\tilde{Q}(\zeta, v) + |\partial r|^2(\zeta) |v_n|^2 \geq C \sum_{j=1}^{n-1} |v_j|^2. \tag{4}$$

Observe that $\tilde{Q}(p,(0,v_n)) \geq 0$. So by the lower semicontinuity of $\tilde{Q}$, for c small enough

$$\tilde{Q}(\zeta, v) + |\partial r|^2(\zeta)|v_n|^2 > c|v_n|^2 \tag{5}$$

in a neighborhood of p. Inequality (5) remains valid in a neighborhood of $v' = 0$, i.e. for $|v'| \leq \alpha$, on the sphere $|v| = 1$, where $v = (v', v_n)$.

We get then that

$$Q(z,v) + |\partial r|^2(z)|v_n|^2 \geq \frac{c}{2}|v_n|^2$$

for $\delta(z) < \beta$, $|v'| \leq \alpha$. But, when $|v'| > \alpha$ and $|v| = 1$ we have $|v'|^2 \geq \varepsilon_0 |v_n|^2$, where $\varepsilon_0 = \alpha^2(1-\alpha^2)^{-1}$.

So using (4) we get that

$$Q(z,v) + |\partial r|^2 |v_n|^2 \geq \varepsilon' |v_n|^2 \quad \text{for some } \varepsilon' > 0$$

and for $\delta(z) < \beta$. This implies

$$Q(z,v) + |\partial r|^2 |v_n|^2 \geq \frac{\varepsilon'}{2}|v_n|^2 + \frac{c}{2}\sum_{j=1}^{n-1} |v_j|^2.$$

Rescaling this we obtain

$$-r \sum_{i,j=1}^{n} a_{ij} v_i \bar{v}_j + |\partial r|^2 |v_n|^2 \geq \frac{\varepsilon}{2}|v_n|^2 + \frac{c}{2}\sum_{j=1}^{n-1} |v_j|^2$$

which can be read as

$$-i\partial\bar{\partial}(-r)^{\varepsilon} = i\varepsilon(-r)^{\varepsilon}\left(\frac{\partial\bar{\partial}r}{-r} + (1-\varepsilon)\frac{\partial r\wedge\bar{\partial}r}{r^2}\right) \geq \frac{c}{2}\varepsilon|r|^{\varepsilon}\omega.$$

□

The condition of positivity of holomorphic sectional curvature in order to construct a strictly p.s.h. bounded exhaustion function seems quite sharp. Indeed we have the following result.

THEOREM 1.2. *There is a compact Kähler surface M which has the following property. There is $\Omega \Subset M$ a Stein domain with real analytic boundary with $\partial\Omega$ Levi-flat, such that for every neighborhood U of $\partial\Omega$ there is no nonconstant bounded p.s.h. function on $U\cap\Omega$.*

Proof. M will be given as the quotient of $\mathbb{C}\times\mathbb{P}^1$ under a $\mathbb{Z}^2$ action. For $(a,b)\in\mathbb{Z}^2$ let $f_{a,b}(z,\omega) = (z+a+b\omega, w+a+\alpha b)$ where $\omega\in\mathbb{C}$ $\mathrm{Im}\omega > 0$, and $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ are fixed. Here w denotes an inhomogeneous coordinate on $\mathbb{P}^1$. It is clear that M is a compact surface. M is also a $\mathbb{P}^1$-bundle on the torus $A = \mathbb{C}/\mathbb{Z}^2$. Hence M is Kähler. We can also observe that M is homogeneous in the sense that the tangent bundle is generated by global holomorphic vector fields.

We observe that M is foliated by complex leaves. Let $\pi:\mathbb{C}\times\mathbb{P}^1\to M$ be the canonical projection. For w_0 fixed π is injective on $\mathbb{C}\times w_0$, because $\alpha\notin\mathbb{Q}$. We also have that $\pi(\mathbb{C}\times w_0) = \pi(\mathbb{C}\times w_1)$ iff $w_1 = w_0+a_0+\alpha b_0$. It follows that for any $y_0\in\mathbb{R}$ $L_{y_0} := \pi(\mathbb{C}\times y_0)\supset\pi(\mathbb{C}\times\mathrm{Im}w = y_0)\cup A_1$, where A_1 denote the torus $\pi(\mathbb{C}\times\infty)$. It is then clear that the closure of each leaf (except for A_1) contains a Levi-flat hypersurface which is a real analytic three dimensional torus. Define $\Omega := \pi(\mathrm{Im}w > 0)$. Then $\partial\Omega$ is real analytic and Levi-flat. Let U be a neighborhood of $\partial\Omega$. Assume $\varphi\, U\cap\Omega\to[-c_1,0[$ is p.s.h. For $0<y_0<\varepsilon_0$, $\varepsilon_0\ll 1$ L_{y_0} is contained in U. Since φ is bounded above it is constant on $L_{y_0} = \pi(\mathrm{Im}w = y_0)$. Fix $p\in\overline{L_{y_0}}\cap U$. Choose $\varepsilon>0$ small enough so that $B(p,\varepsilon)\subset U$. Let $c := \max\varphi_{\overline{B(p,\varepsilon)}\cap\overline{L_{y_0}}}$. The closed set $(\varphi\geq c)$ is invariant under the foliation. So $\varphi = c$ on $\overline{L_{y_0}}$. As a consequence φ is just a function of y, i.e. $\varphi = h(y)$, h defined for $0<y<\varepsilon_0$. The plurisubharmonicity of φ implies that $w\to h(y)$ is subharmonic so h is convex with respect to y. The function is defined for $y>0$ bounded hence constant.

The domain Ω is Stein. Indeed the function $\pi(z,w) \to \sup(-\text{Log}|y|, |y|)$ is a p.s.h. exhaustion function on Ω. Since M is homogeneous and Ω does not contain a relatively compact leaf, it follows form a theorem of Hirshowitz [H] that Ω is Stein. □

§2. Strictly p.s.h. functions near $\partial\Omega$

Let $\Omega \Subset M$ be a pseudoconvex domain with $\mathcal{C}^2$ boundary in the complex manifold M. We are interested in the existence of a strictly p.s.h. function in a neighborhood of $\partial\Omega$. The examples of in the previous paragraph show that this is not always the case, even when Ω is Stein. Our result is inspired by the duality principle from Sullivan [Su]. Recall that currents of bidimension $(1,1)$ act on forms of bidegree $(1,1)$. Let X be a closed subset of M. Assume $x \to \alpha_x$ is a continuous map on X with values in complex linear maps, α_x is allowed to be zero on some subset of X.

DEFINITION 2.1. A positive current T, of bidimension $(1,1)$, is directed by $\ker \alpha_X$ iff $T \wedge i\alpha_x \wedge \overline{\alpha}_x = 0$ on X.

The positivity of T implies that $T \wedge i\alpha_x \wedge \overline{\alpha}_x$ is a positive measure, so we are asking that this measure vanishes on X.

If we assume that T is supported on X, this is equivalent to the fact that T belongs to the closure of the convex cone generated by the currents $\varepsilon_x(i\xi_n \otimes \overline{\xi}_n)$ where $\alpha_x(\xi_x) = 0$ and ε_x denote the Dirac mass at x. We will consider M as a hermitian manifold, which allows one to give a norm T to positive currents, i.e. $\|T\| = \langle T, \omega\rangle$ where ω is a fixed strictly positive $(1,1)$-form.

THEOREM 2.2. *Let $\Omega \Subset M$ be a pseudoconvex domain with $\mathcal{C}^2$ boundary. The following are equivalent.*

i) *There is a smooth strictly plurisubharmonic function near $\partial\Omega$.*

ii) *There is no, nontrivial, positive current T, of bidimension $(1,1)$ supported on $\partial\Omega$ and directed by the complex tangent spaces to $\partial\Omega$, satisfying the equation $i\partial\overline{\partial}T = 0$.*

Proof. Assume i). Let T be positive $(1,1)$ and supported on $\partial\Omega$. Let φ be a strictly p.s.h. function near $\partial\Omega$. Then if T is non-zero,

$$0 < \langle T, i\partial\overline{\partial}\varphi\rangle = \langle i\partial\overline{\partial}T, \varphi\rangle = 0,$$

a contradiction. We now show ii) implies i). Let ρ be a $\mathcal{C}^2$ defining function for $\partial\Omega$. Define $C = \{T \mid T \geq 0, (1,1), \|T\| = 1, T$ directed by $\ker \partial\rho.\}$ The set is convex and compact for the topology of currents. If i) does not hold then $C \cap \{i\partial\overline{\partial}u\}^{\perp} = \emptyset$, here $\{i\partial\overline{\partial}u\}^{\perp}$ denote the orthogonal space of $\{i\partial\overline{\partial}u\}$ when u is a test function on M, i.e. a smooth function on M, so the space is closed. Using Hahn-Banach and reflexivity for the space of test functions we get the existence of $\psi \in \text{closure}\{i\partial\overline{\partial}u\}$ such that $\langle T, \psi\rangle > 0$ for every T in C. Since C is compact we can assume that $\psi = i\partial\overline{\partial}u$.

If $T = \varepsilon_x i\xi \otimes \overline{\xi}$ we get that $\langle i\partial\overline{\partial}u(x)\xi \wedge \overline{\xi}\rangle > 0$, for $x \in \partial\Omega$ and ξ complex tangent.

Define $\varphi_\lambda = u + \frac{e^{\lambda\rho}-1}{\lambda}$. For $\lambda \gg 1$, the pseudoconvexity of $\partial\Omega$ implies that φ_λ is strictly p.s.h. near $\partial\Omega$. □

Without assuming $\partial\Omega$ smooth we get easily the following.

THEOREM 2.3. *Let $\Omega \Subset M$. The following are equivalent.*

i) *There is a smooth strictly p.s.h. function near $\partial\Omega$.*

ii) *There is no, nontrivial, positive $(1,1)$ current T supported on $\partial\Omega$ such that $i\partial\overline{\partial}T = 0$.*

It is of interest to localize the support of such pluriharmonic currents i.e. positive currents satisfying $i\partial\overline{\partial}T = 0$. Assume $\Omega \Subset M$. We define $J \subset \partial\Omega$ as the set of $x \in \partial\Omega$ such that there exists a Stein neighborhood $U \ni x$, and a p.s.h. function φ_x defined near $\overline{U}$ with $\varphi_x(x) > 0$ and $\sup_{\partial\Omega\cap\partial U} \varphi_x < 0$.

Shrinking U we can assume the existence of a strictly p.s.h. function ρ, on neighborhood of $\overline{U}$, $\varphi_x + \varepsilon\rho$, with $0 < \varepsilon \ll 1$, will be strictly p.s.h. near x and will have the same properties as φ_x otherwise. Composing with a convex increasing function, we can assume φ_x vanishes identically in a neighborhood of $\partial\Omega \cap \partial U$, with respect to $\partial\Omega$. We call J the weak Jensen boundary of $\partial\Omega$. Clearly J is open and contains the points of strict pseudoconvexity of $\partial\Omega$, when $\partial\Omega$ is of class $\mathcal{C}^2$.

THEOREM 2.4. *Assume $\Omega \Subset M$ is pseudoconvex with $\mathcal{C}^2$ boundary. Let T be a pluriharmonic positive current directed by the complex tangent space to $\partial\Omega$. Then the support of T is contained in the complement of J, the weak-Jensen boundary of $\partial\Omega$.*

Proof. Let $x \in J$. Choose φ a p.s.h. function in U, strictly p.s.h. near x, vanishing on a neighborhood in $\partial\Omega$, of $\partial U \cap \partial\Omega$. If T is a positive $(1,1)$ current directed by the complex tangent space to $\partial\Omega$ we get

$$\langle T, i\partial\overline{\partial}\varphi\rangle = \langle i\partial\overline{\partial}T, \varphi\rangle.$$

The integration by part is possible because we consider T as a current on $\partial\Omega$, and φ as a function with compact support on $U \cap \partial\Omega$. □

It is of interest to consider the possibility of existence of positive closed currents with support on the boundary of a pseudoconvex domain $\Omega \Subset M$. This is possible for domains in a $\mathbb{P}^1$ bundle over a Riemann surface or in a complex torus. However in $\mathbb{P}^2$, this is not possible.

THEOREM 2.5. *Let Σ be a hypersurface of class $\mathcal{C}^2$ in $\mathbb{P}^2$. Then there is no positive $(1,1)$ closed current T supported on Σ.*

Proof. Let Ω_1, Ω_2 be the components of $\mathbb{P}^2 \setminus \Sigma$. Let ω be the standard Kähler form in $\mathbb{P}^2$. Suppose there are 2-cycles $\sigma_1 \subset \Omega_1$, $\sigma_2 \subset \Omega_2$ such that $\langle \sigma_j, \omega\rangle = a_j \neq 0$. Since the second Betti number of $\mathbb{P}^2$ is 1, by Poincaré duality $\sigma_j \sim a_j\omega$. But $\sigma_1 \wedge \sigma_2 = 0$ and $a_1a_2\omega \wedge \omega \neq 0$ a contradiction. So we can assume that for every 2 cycle σ in a neighborhood of $\overline{\Omega}_1$ we have $\langle \sigma, \omega\rangle = 0$. We are using here that Ω_1 is smoothly bounded. By De Rham Theorem there is a smooth form φ such that $d\varphi = \omega$ in a neighborhood of Ω_1. Let T be a positive closed current of bidimension $(1,1)$ supported on Σ. Then if T is nonzero

$$0 < \langle T, \omega\rangle = \langle T, d\varphi\rangle = \langle dT, \varphi\rangle = 0.$$

So $T = 0$. □

Remark. Let Σ be a real hypersurface in $\mathbb{P}^k$. We prove similarly that there is no non-zero positive closed current of bidimension $(1,1)$ supported on Σ. We get

$$\langle T, \omega^{k-1}\rangle = \langle T, d(\varphi \wedge \omega^{k-2})\rangle = 0.$$

In particular there is no one dimensional complex curve on Σ.

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