ON THE CANONICAL FORM OF TURBULENCE

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§0. **Introduction.** In K. Itô's paper [1] on the theory of turbulence, the problem to determine the canonical form of the moment tensor of temporally homogeneous and isotropic turbulence, has not been solved. In the present paper, the author will solve the problem by making use of the result of his preceding paper [2]. We shall treat the turbulence in R^3, but the similar argument is possible in R^n.

§1. **Generalities.** In the theory of *turbulence*, the deviation of the velocity from its mean may be considered as a system of random vectors $\mathfrak{u}(t, \mathfrak{X}, \omega) = \langle u_p(t, \mathfrak{X}, \omega)/p = 1, 2, 3\rangle$, where $t \in R^1$ and $\mathfrak{X} \in R^3$ denote the time and the position respectively and $\omega \in (\Omega, P)$ is the probability parameter; we assume naturally that $u_p(t, \mathfrak{X}, \omega)$ is B-measurable in $\langle t, \mathfrak{X}, \omega\rangle$ and belongs to $L^2(\Omega, \mathrm{P})$ for any fixed $\langle p, t, \mathfrak{X}\rangle$.[1] We have clearly

$$(1.1) \qquad \mathbf{E}_\omega[u_p(t, \mathfrak{X}, \omega)] = 0 \quad (\mathbf{E}_\omega[.] \text{ denotes the expectation}).$$

Now we define the moment tensor of the turbulence by

$$(1.2) \qquad R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}) = \mathbf{E}_\omega[u_p(t, \mathfrak{X}, \omega)\, u_q(s, \mathfrak{Y}, \omega)];$$

then

$$(1.3) \qquad R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}) = R_{qp}(s, \mathfrak{Y}; t, \mathfrak{X}), \text{ and}$$

$$(1.4) \qquad \sum_{i,j} \alpha_i \bar{\alpha}_j R_{p_i p_j}(t_i, \mathfrak{X}_i; t_j, \mathfrak{X}_j) \geqq 0^{2)} \quad (\alpha_i: \text{complex number}).$$

We consider the turbulence satisfying the following three conditions:

$$(1.5) \qquad R_{pq}(t+\tau, \mathfrak{X}; s+\tau, \mathfrak{Y}) = R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}) \quad (\textit{temporally homogeneous});$$

$$(1.6) \qquad R_{pq}(t, \mathfrak{X}+\mathfrak{a}; s, \mathfrak{Y}+\mathfrak{a}) = R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}) \quad (\textit{spatially homogeneous});$$

and

$$(1.7) \qquad \sum_{p',q'} k_{p'p} k_{q'q} R_{p'q'}(t, \mathfrak{X}; s, \mathfrak{X}+K(\mathfrak{Y}-\mathfrak{X})) = R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y})$$

for any orthogonal transformation $K \equiv (K_{pq}/p, q = 1, 2, 3)$ (*isotropic*). We can easily prove by (1.3) that the isotropism implies the homogenuity (1.6). Consequently we get

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[1] See [1].

[2] If $R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y})$ is real-valued and satisfies (1.4), then it satisfies (1.3) automatically.

(1.8) $\qquad \sum_{p',q'} k_{p'p} k_{q'q} R_{p'q'}(t, K\mathfrak{X};\ s, K\mathfrak{Y}) = R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y})$.

We shall determine a canonical form of $R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y})$ which satisfies (1.4), (1.5) and (1.7).

Suppose that $R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y})$ satisfies (1.4), (1.5) and (1.7) (and consequently all of (1.3) — (1.8) above); then there exists $\mathfrak{u}(t, \mathfrak{X}, \omega)$ satisfying (1.1) and (1.2).[3] Define

(1.9) $$\begin{cases} u(\mathfrak{l};\ t, \mathfrak{X}, \omega) = (\mathfrak{l}, \mathfrak{u}(t, \mathfrak{X}, \omega))_3{}^{4)}; \text{ and} \\ R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{m}, s, \mathfrak{Y}) = \mathbf{E}_\omega[u(\mathfrak{l};\ t, \mathfrak{X}, \omega)\, u(\mathfrak{m};\ s, \mathfrak{Y}, \omega)] \end{cases}$$

for $\mathfrak{l} \equiv \langle l_1, l_2, l_3 \rangle \in S$ (= the unit sphere whose center is the origin of R^3), and put $\mathfrak{l}^p = \langle \delta_{p1}, \delta_{p2}, \delta_{p3} \rangle$ ($\in S$) for $p = 1, 2, 3$ (δ_{pq} is the Kronecker's delta). Then we have, by (1.2) and (1.8),

$$\begin{aligned} &\mathbf{E}_\omega[(K\mathfrak{l}^p, \mathfrak{u}(t, K\mathfrak{X}, \omega))_3 \cdot (K\mathfrak{l}^q, \mathfrak{u}(s, K\mathfrak{Y}, \omega))_3] \\ &\qquad = \mathbf{E}_\omega[(\mathfrak{l}^p, \mathfrak{u}(t, \mathfrak{X}, \omega))_3 \cdot (\mathfrak{l}^q, \mathfrak{u}(s, \mathfrak{Y}, \omega))_3]. \end{aligned}$$

From this equality and by simple calculation, we obtain

(1.8′) $\qquad R(K\mathfrak{l}, t, K\mathfrak{X};\ K\mathfrak{m}, s, K\mathfrak{Y}) = R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{m}, s, \mathfrak{Y})$.

And (1.6), (1.5), (1.4) (and (1.9)) imply following relations:

(1.6′) $\qquad R(\mathfrak{l}, t, \mathfrak{X} + \mathfrak{a};\ \mathfrak{m}, s, \mathfrak{Y} + \mathfrak{a}) = R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{m}, s, \mathfrak{Y})$,

(1.5′) $\qquad R(\mathfrak{l}, t + \tau, \mathfrak{X};\ \mathfrak{m}, s + \tau, \mathfrak{Y}) = R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{m}, s, \mathfrak{Y})$,

(1.4′) $\qquad \sum_{i,j} \alpha_i \bar{\alpha}_j R(\mathfrak{l}_i, t_i, \mathfrak{X}_i;\ \mathfrak{l}_j, t_j, \mathfrak{X}_j) \geqq 0$.

Now we put $\mathbf{R} = S \times R^1 \times R^3$ and define the transformations T_τ, U_K, $V_\mathfrak{a}$ on $\mathbf{R}$ by $T_\tau \langle \mathfrak{l}, t, \mathfrak{X} \rangle = \langle \mathfrak{l}, t + \tau, \mathfrak{X} \rangle$; $U_K \langle \mathfrak{l}, t, \mathfrak{X} \rangle = \langle K\mathfrak{l}, t, K\mathfrak{X} \rangle$; $V_\mathfrak{a} \langle \mathfrak{l}, t, \mathfrak{X} \rangle = \langle \mathfrak{l}, t, \mathfrak{X} + \mathfrak{a} \rangle$. Let G be the group of transformations on $\mathbf{R}$ generated by T_τ's, U_K's and $V_\mathfrak{a}$'s defined above. We easily see that $U_K V_\mathfrak{a} = V_{K\mathfrak{a}} U_K$ and that T_τ commutes with every element of G. Then $\mathbf{R}$ is a homogeneous space with the locally compact group G of homeomorphisms which is transitive on $\mathbf{R}$; for $P_1 \equiv \langle \mathfrak{l}^1, 0, 0 \rangle \in \mathbf{R}$, $\{g \in G / gP_1 = P_1\}$ is identical with $\left\{U_K / K = K_\theta \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix};\ 0 \leqq \theta < 2\pi\right\}$, which is compact subgroup of G; and $R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{m}, s, \mathfrak{Y})$ is a positive definite function on $\mathbf{R}^2$ in the sense of [2] by (1.4′), (1.5′),

3) See [1].

4) We denote the inner product and the norm in R^n by $(\cdot, \cdot)_n$ and $\|\cdot\|_n$ respectively.

(1.6′) and (1.8′). Hence there exists a cyclic unitary representation[5] $\{\mathfrak{H}, \mathfrak{U}, f_1\}$ of $\{\mathbf{R}, G, P_1\}$ such that

$$R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{m}, s, \mathfrak{Y}) = (T_t V_{\mathfrak{X}} U_{K(\mathfrak{l})} f_1,\ T_s V_{\mathfrak{Y}} U_{K(\mathfrak{m})} f_1),^{6)} \tag{1.10}$$

where $K(\mathfrak{l})$ is an orthogonal transformation such as $K(\mathfrak{l})\mathfrak{l}^1 = \mathfrak{l}$ and we denote the unitary operators $(\in \mathfrak{U})$ corresponding to T_t, $V_{\mathfrak{X}}$, $U_K (\in G)$ by the same notations respectively.

We shall denote by f_p the element of $\mathfrak{H}$ corresponding to the point $P_p \equiv \langle \mathfrak{l}^p, 0, 0\rangle \in \mathbf{R}$ $(p = 1, 2, 3)$.[7] Then by (1.10) and (1.9) we have

$$\begin{aligned}(T_t V_{\mathfrak{X}} U_{K(\mathfrak{l})} f_1,\ U_K f_p) &= R(\mathfrak{l}, t, \mathfrak{X};\ K\mathfrak{l}^p, 0, 0)\\ &= \textstyle\sum_{p'} k_{pp'} R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{l}^{p'}, 0, 0) = (T_t V_{\mathfrak{X}} U_{K(\mathfrak{l})} f_1,\ \sum_{p'} k_{p'p} f_{p'})\end{aligned}$$

$(K = (k_{pq}))$. Since $\{T_t V_{\mathfrak{X}} U_{K(\mathfrak{l})} f_1 / t \in R^1,\ \mathfrak{X} \in R^3,\ \mathfrak{l} \in S\}$ spans $\mathfrak{H}$,[7] we have $U_K f_p = \sum_{p'} k_{p'p} f_{p'}$; and hence

(1.11) $U_K f = \sum_{p'} (\sum_p k_{p'p} \alpha_p) f_{p'}$, for $\mathfrak{f} = \sum_p \alpha_p f_p$ (α_p: complex number).

Thus we see that, in order to determine a canonical form of $R(\mathfrak{l}, t, \mathfrak{X};\ \mathfrak{m}, s, \mathfrak{Y})$, it is sufficient to consider cyclic unitary representations $\{\mathfrak{H}, \mathfrak{U}, f_1\}$ satisfying (1.11).

§ 2. Preliminary lemmas.

LEMMA 1. *Let $F(.)$ be an additive set function on R^n.*[8] *If $\int_{R^n} e^{i(\mathfrak{X}, X)_n} F(dX) = 0$ for any $\mathfrak{X} \in R^n$, then $F(.) \equiv 0$.*

The proof can be achieved by simple calculations and so will be omitted.

Let $\mathfrak{G}$ be a locally compact group, and let $\mathfrak{g}$ be a compact subgroup of $\mathfrak{G}$,

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5) See [2]; —— In the paper [2], if we use the notations following after the paper [2], the positive-definiteness of $f(p, q)$ is defined as follows: $f(p, q)$ is measurable in $\langle p, q\rangle$ and essentially bounded and satisfies.

(1) $\qquad f(\sigma p, \sigma q) = f(p, q)\ (\langle p, q\rangle \in \Omega^2,\ \sigma \in G)$, and

(2) $\qquad \iint_{\Omega^2} f(p, q)\, x(p)\, \bar{x}(q)\, dp dq \geqq 0\ (x \in L^1(\Omega))$;

and it is noted that if $f(p, q)$ is continuous in $\langle p, q\rangle$ then (2) is equivalent to

(3) $\qquad \sum_{i,j} \alpha_i \bar{\alpha}_j f(p_i, p_j) \geqq 0$

and (3) implies that $f(p, q)$ is bounded $(|f(p, q)| \leqq f(p, p) = f(p_0, p_0))$. But it may be proved without any use of the continuity of $f(p, q)$ that (3) implies (2) (cf. [5], pp. 56–57). So we can make use of the results of the paper [2] directly from (1.4′) (and (1.5′), (1.6′), (1.8′)) in the present paper.

6) We denote the inner product and the norm in $\mathfrak{H}$ by $(.,.)$ and $\|.\|$ respectively.

7) See [2], § 2.

8) Under an *additive set function* on a topological space $\mathbf{S}$ we understand a (generally complex-valued) countably additive Borel-set function on $\mathbf{S}$ such that the total variation on $\mathbf{S}$ is finite (following after S. Saks [6]), where Borel sets mean such sets as belongs to the minimal countably additive class including all open sets in the space $\mathbf{S}$.

we introduce a $\mathfrak{G}$-invariant measure ds on the homogeneous space $\mathfrak{S} \equiv \mathfrak{G}/\mathfrak{g}$ in the natural way so that $ds \cdot dK' = dK$[9] where dK denotes a left-invariant Haar-measure on the locally compact group $\mathfrak{G}$ and dK' denotes the Haar-measure on the compact group $\mathfrak{g}$ such as $\int_{\mathfrak{g}} dK' = 1$. Let $F(.)$, $F_1(.)$, ..., $F_n(.)$ be additive set functions on $\mathfrak{S}$, and $|F|(.)$, $|F_1|(.)$, ..., $|F_n|(.)$ be their total variations respectively. Then we have the following

LEMMA 2. *Suppose that* $F(K\Delta)(K \in \mathfrak{G})$ *is expressible in the form:*

$$F(K\Delta) = \sum_{\nu=1}^{n} \int_{\Delta} \varphi_\nu(K;\ s)\, F_\nu(ds),$$

$\varphi_\nu(K;\ s)$ *being summable on* $\mathfrak{S}$ *with respect to the measure* $|F_\nu|(.)$, $\nu = 1, 2, \ldots, n$. *Then* $F(.)$ *is absolutely continuous with respect to the* $\mathfrak{G}$*-invariant measure* ds.

Proof. Let $s_0 (\in \mathfrak{S})$ be the image of $\mathfrak{g}$ by the natural mapping $\mathfrak{G} \longrightarrow \mathfrak{S} \equiv \mathfrak{G}/\mathfrak{g}$, K_s be an element of $\mathfrak{G}$ such that $K_s s_0 = s$, and $\rho(K)$ be the Neumann's function with respect to the Haar-measure dK on $\mathfrak{G}$; it is easy to see that $\rho(K) \equiv 1$ on the compact subgroup $\mathfrak{g}$ of the group $\mathfrak{G}$, and hence $\rho(K_s)$ depends only on s. Suppose that Δ is a Borel set in $\mathfrak{S}$ such that $\int_{\Delta} ds = 0$, and let $C_\Delta(s)$ be the characteristic function of Δ. Then for all $s \in \mathfrak{S}$

$$\int_{\mathfrak{G}} C_\Delta(K^{-1}s)\, dK = \int_{\mathfrak{G}} C_\Delta(K^{-1}K_s s_0)\, dK = \int_{\mathfrak{G}} C_\Delta(K^{-1}s_0)\, dK = \int_{\mathfrak{G}} C_\Delta(K s_0)\, \rho(K)\, dK$$
$$= \int_{\mathfrak{S}} \rho(K_s)\, ds \int_{\mathfrak{g}} \rho(K')\, C_\Delta(K_s K' s_0)\, dK' = \int_{\mathfrak{S}} \rho(K_s)\, C_\Delta(s)\, ds = 0.$$

Hence for every ν we have by Fubini's theorem

$$0 = \int_{\mathfrak{S}} |F_\nu|(ds) \int_{\mathfrak{G}} C_\Delta(K^{-1}s)\, dK = \int_{\mathfrak{G}} dK \int_{\mathfrak{S}} C_\Delta(K^{-1}s)\, |F_\nu|(ds) = \int_{\mathfrak{G}} |F_\nu|(K\Delta)\, dK.$$

Therefore

$$|F_\nu|(K\Delta) = 0, \qquad \nu = 1, 2, \ldots, n,$$

for almost all K with respect to dK; and for such a K we obtain by the assumption that

$$F(\Delta) = F(K^{-1}K\Delta) = \sum_{\nu=1}^{n} \int_{K\Delta} \varphi_\nu(K^{-1};\ s)\, F_\nu(ds) = 0, \quad q.e.d.$$

§ 3. **Canonical form of isotropic turbulence.** In this paragraph as well as in the next, we represent every point $X \in R^3$ except the origin by the couple

[9] See [5], pp. 43-45.

$\langle\rho, \lambda\rangle$ of $\rho = \|X\|_3 (>0)$ and $\lambda = X/\|X\|_3$ ($\in$ the unit sphere S with the center $\langle 0, 0, 0\rangle \in R^3$) : $X \equiv \rho\lambda = \langle\rho, \lambda\rangle$; and in order to clarify that a set Δ is a subset of the set $\{X/X \in R^3\}$, we shall use the notation with the subscript X: Δ_X ; similarly Γ_τ and B_ρ denote subsets of $\{\tau/-\infty<\tau<\infty\}$ and $\{\rho/0<\rho<\infty\}$ respectively.

The purpose of this paragraph is to prove the following

THEOREM 1. *A necessary and sufficient condition for* $(R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}))/p, q = 1, 2, 3)$ *to be a moment tensor of a temporally homogeneous and isotropic turbulence is that* $R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}))$ *is expressible in the form:*

$$(3.1)\qquad \begin{aligned} R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}) &= \int_{[0,\infty)_\tau\times(0,\infty)_\rho\times S} \cos[(t-s)\tau + (\mathfrak{X}-\mathfrak{Y}, \rho\lambda)_3] \\ &\qquad \times \Big(K(\lambda)^{-1}\mathfrak{l}^p, \mu(d\tau d\rho) K(\lambda)^{-1}\mathfrak{l}^q\Big)_3 d\lambda \\ &\qquad + \delta_{pq}\int_{-0}^{\infty} \cos(t-s)\tau \cdot \nu(d\tau), \end{aligned}$$ [10]

where $\mathfrak{l}^p = \langle\delta_{p1}, \delta_{p2}, \delta_{p3}\rangle (\in S)(p = 1, 2, 3)$ *and* $\{K(\lambda)/\lambda \in S\}$ *denotes a system of orthogonal transformations such that* $K(\lambda)$ *is an arbitrarily fixed one such as* $K(\lambda)\langle 1, 0, 0\rangle = \lambda$ *for every* $\lambda \in S$,[11] and $\mu(A)$ *is written as*

$$(3.2)\qquad \mu(A) = \begin{pmatrix} \mu_1(A) & & 0 \\ & \mu_2(A) & \\ 0 & & \mu_2(A) \end{pmatrix}$$

and $\mu_p(A)(p=1, 2)$ *and* $\nu(\Gamma_\tau)$ *are measures defined on* $[0, \infty)_\tau \times (0, \infty)_\rho$ *and on* $[0, \infty)_\tau$ *respectively such that* $\mu_p([0, \infty)_\tau \times (0, \infty)_\rho) < \infty$ $(p = 1, 2)$ and $\nu([0, \infty)_\tau) < \infty$.

The sufficiency of this condition is easily proved by simple calculations.

To prove the necessity, we consider the cyclic unitary representation $\{\mathfrak{H}, \mathrm{U}, f_1\}$ of $\{R, G, P_1\}$ defined in §1. Since $\{T_t V_{\mathfrak{X}}/t \in R^1, \mathfrak{X} \in R^3\}$ is a unitary representation of the 4-dimensional vecter space, we have by Stone's theorem (in the form generalized by W. Ambrose [4])

$$(3.3)\qquad T_t V_{\mathfrak{X}} = \int_{R^4} e^{it\tau} e^{i(\mathfrak{X}, X)_3} E(d\tau dX)$$

where $E(.)$ is a resolution of the identity on $R^4 (= R_\tau^1 \times R_X^3)$. The function

10) The second term of the right-hand side corresponds to the case mentioned in [1] (see [1], Theorem 1).

11) It is true that the above system $\{K(\lambda)\}$ is not uniquely determined, but we may see from (3.2) that $\Big(K(\lambda)^{-1}\mathfrak{l}^p, \mu(A)K(\lambda)^{-1}\mathfrak{l}^q\Big)_3$ is independent of the special choice of this system and is continuous in λ for any fixed A.

$F_{pq}(\Gamma_\tau \times \Delta_X) = \big(E(\Gamma_\tau \times \Delta_X) f_p, f_q\big) (\Gamma_\tau \subseteqq R^1, \Delta_X \subseteqq R^3)$ is an additive set function on R^4 and $F(\Gamma_\tau \times \Delta_X) = \big(F_{pq}(\Gamma_\tau \times \Delta_X)/p, q = 1, 2, 3\big)$ is a hermitian 3×3-matrix. In considering (1.11), we obtain from (1.10) and (3.3)

$$\text{(3.4)} \qquad R(\mathfrak{l}, t, \mathfrak{X}; \mathfrak{m}, s, \mathfrak{Y}) = (T_t V_{\mathfrak{X}} U_{K(\mathfrak{l})} f_1, T_s V_{\mathfrak{Y}} U_{K(\mathfrak{m})} f_1) = \int_{R^4} e^{i(t-s)\tau} e^{i(\mathfrak{X}-\mathfrak{Y}, X)_3} (\mathfrak{l}, F(d\tau dX)\mathfrak{m})_3$$

for $\mathfrak{l}, \mathfrak{m} \in S$, and hence, putting $K = (k_{pq})$, we get for any t and $\mathfrak{X}$

$$\int_{R^4} e^{it\tau} e^{i(\mathfrak{X}, X)_3} F_{pq}(d\tau\, d_X(KX))$$
$$= \int_{R^4} e^{it\tau} e^{i(\mathfrak{X}, K^{-1}X)_3} (\mathfrak{l}^p, F(d\tau dX)\mathfrak{l}^q)_3 = (T_t V_{K\mathfrak{X}} f_p, f_q)$$
$$= (T_t V_{\mathfrak{X}} U_{K^{-1}} f_p, U_{K^{-1}} f_q) = \int_{R^4} e^{it\tau} e^{i(\mathfrak{X}, X)_3} \Big[\sum_{p',q'} k_{pp'} k_{qq'} F_{p'q'}(d\tau dx)\Big].$$

Therefore by Lemma 1

$$\text{(3.5)} \qquad F_{pq}(\Gamma_\tau \times K\Delta_X) = \textstyle\sum_{p',q'} k_{pp'} k_{qq'} F_{p'q'}(\Gamma_\tau \times \Delta_X).$$

If we put $R_\tau = (-\infty, \infty)_\tau \times \{0\}_X (\subseteqq R^4)$ and define $H(M)$ and $H'(M)$ for every Borel set $M \subseteqq R^4$ by

$$\text{(3.6)} \qquad \begin{cases} H(M) \equiv (H_{pq}(M)) = (F_{pq}(M - R_\tau)) \cdot \\ H'(M) \equiv (H'_{pq}(M)) = (F_{pq}(M \cap R_\tau)), \end{cases}$$

then $H'(M)$ is a hermitian matrix and satisfies $H'(M) = K \cdot H'(M) \cdot K^{-1}$ for any orthogonal transformation $K \equiv (k_{pq})$ by (3.5). Therefore we see that $H'(M)$ is specialized as

$$\text{(3.7)} \qquad H'(M) = \begin{pmatrix} H_1'(M) & & 0 \\ & H_1'(M) & \\ 0 & & H_1'(M) \end{pmatrix}.$$

Next we consider the matrix $H(M)$. For any Borel set $A \subseteqq (-\infty, \infty)_\tau \times (0, \infty)_\rho$, the function $H_{pq}(A \times \Lambda)$ is an additive function of a Borel set $\Lambda \subseteqq S$, and we have by (3.5) and (3.6)

$$\text{(3.8)} \qquad H_{pq}(A \times K\Lambda) = \textstyle\sum_{p',q'} k_{pp'} k_{qq'} H_{p'q'}(A \times \Lambda).$$

Hence by Lemma 2 (in putting $\mathfrak{S} = S$, $\varphi_{p'q'}(K; \lambda) \equiv k_{pp'} k_{qq'}$) and by Radon-Nikodym's theorem, we obtain a function $h_{pq}(A; \lambda)$ such that

$$\text{(3.9)} \qquad H_{pq}(A \times \Lambda) = \int_\Lambda h_{pq}(A; \lambda)\, d\lambda,$$

where $d\lambda$ is *the K-invariant measure on S such as* $\int_S d\lambda = 4\pi$ (*i.e.*, the ***area*** in the usual sense). And for any fixed A, we can determine $h_{pq}(A; \lambda)$ for ***all*** λ so that

$$h_{pq}(A\,;\;K\lambda) = \textstyle\sum_{p',q'} k_{pp'}k_{qq'}h_{p'q'}(A\,;\;\lambda)\,, \tag{3.10}$$

in making use of (3.8) and the fact that the group $\{K\}$ of all orthogonal transformations is transitive on S. Consequently from $H_{pp}(A\times\Lambda)=(E(A\times\Lambda-R_\tau)f_p,\,f_p)$ $=\|E(A\times\Lambda-R_\tau)f_p\|^2\geqq 0$ (see (3.6)) we get

$$h_{pp}(A\,;\;\lambda)\geqq 0\,. \tag{3.11}$$

The matrix $h(A\,;\;\lambda)=(h_{pq}(A\,;\;\lambda)/p,\,q=1,2,3)$ is hermitian for every A and λ, and by (3.10) we have

$$h(A\,;\;K\lambda)=K\cdot h(A\,;\;\lambda)\cdot K^{-1}. \tag{3.10'}$$

If we put $\lambda=\lambda^1\equiv\langle 1,0,0\rangle$ and $K=K(\lambda)$ in (3.10′) and define $h(A)\equiv(h_{pq}(A))$ $=(h_{pq}(A\,;\;\lambda^1))$, then we have

$$h(A\,;\;\lambda)=K(\lambda)\,h(A)\,K(\lambda)^{-1}. \tag{3.12}$$

And if we put $\lambda=\lambda^1$ and $K=K_0\equiv\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$ in (3.10′), then we have

$$K_0\,h(A)\,K_0^{-1}=h(A\,;\;K_0\lambda^1)=h(A)\,.$$

Thus we see that $h(A)$ is a hermitian matrix which is invariant under any transformation K_0 of the above form. This fact implies that $h(A)$ is written as

$$h(A)=\begin{pmatrix}h_1(A) & & 0\\ & h_2(A) & \\ 0 & & h_2(A)\end{pmatrix}, \tag{3.13}$$

consequently $K(\lambda)\,h(A)\,K(\lambda)^{-1}$ depends only on λ and is independent of the choice of $K(\lambda)$, and hence $h(A\,;\;\lambda)$ is continuous in λ by (3.12). From this and (3.11) we get $0\leqq h_p(A)(\equiv h_{pp}(A))<\infty$ $(p=1,2)$.

Now we shall show that the set functions $h_p(A)=h_{pp}(A)\,(p=1,2)$ have the properties of *measure*. Obviously they are defined for all Borel sets $A\subseteqq(-\infty,\infty)_\tau\times(0,\infty)_\rho$. We shall prove the countable additivity of $h_p(A)$. If $A=\sum_{n=1}^{\infty}A_n$, then we have by (3.11) and Lebesgue's theorem

$$\int_\Lambda\sum_{n=1}^{\infty}h_{pp}(A_n;\;\lambda)\,d\lambda=\sum_{n=1}^{\infty}\int_\Lambda h_{pp}(A_n\,;\;\lambda)\,d\lambda=\sum_{n=1}^{\infty}H_{pp}(A_n\times\Lambda)$$
$$=H_{pp}(A\times\Lambda)=\int_\Lambda h_{pp}(A\,;\;\lambda)\,d\lambda$$

for any Borel set $\Lambda\subseteqq S$. Hence

$$\sum_{n=1}^{\infty}h_{pp}(A_n\,;\;\lambda)=h_{pp}(A\colon\;\lambda) \tag{3.14}$$

for almost all λ; since the group $\{K\}$ of all orthogonal transformations is transitive on S, we see from (3.10) that (3.14) is true for *all* λ, especially putting $\lambda = \lambda^1$, we get $\sum_{n=1}^{\infty} h_p(A_n) = h_p(A)$, which proves the countable additivity of $h_p(A)$.

Thus if we put $h'(\Gamma_\tau) = H_1'(\Gamma_\tau \times \{0\}_X)$ (see (3.7)) and apply (3.6), (3.7), (3.9) and (3.12) to (3.4), we obtain

$$R(\mathfrak{l}, t, \mathfrak{X}; \mathfrak{m}, s, \mathfrak{Y}) = \int_{R^4 - R_\tau} e^{i(t-s)\tau} e^{i(\mathfrak{X}-\mathfrak{Y}, X)_3} \left(\mathfrak{l}, K(\lambda)\, h(d\tau d\rho)\, K(\lambda)^{-1} \mathfrak{m}\right)_3 d\lambda + \int_{-\infty}^{\infty} e^{i(t-s)\tau} (\mathfrak{l}, \mathfrak{m})_3 h'(d\tau), \quad X = \langle \rho, \lambda \rangle.$$

Since $R(\mathfrak{l}, t, \mathfrak{X}; \mathfrak{m}, s, \mathfrak{Y})$ is a *real-valued* positive definite function, we can easily prove that

$$R(\mathfrak{l}, t, \mathfrak{X}; \mathfrak{m}, s, \mathfrak{Y}) = \int_{[0, \infty)_\tau \times (0, \infty)_\rho \times S} \cos\left[(t-s)\tau + (\mathfrak{X} - \mathfrak{Y}, \rho\lambda)_3\right] \times \left(\mathfrak{l}, K(\lambda)\, \mu(d\tau d\rho)\, K(\lambda)^{-1} \mathfrak{m}\right)_3 d\lambda + \int_{-\infty}^{\infty} \cos(t-s)\tau \cdot (\mathfrak{l}, \mathfrak{m})_3 \cdot \nu(d\tau), \tag{3.15}$$

where

$$\mu(A) = \begin{pmatrix} \mu_1(A) & & 0 \\ & \mu_2(A) & \\ 0 & & \mu_2(A) \end{pmatrix} \tag{3.2}$$

and $\mu_p(A)$ $(p = 1, 2)$ and $\nu(\Gamma_\tau)$ are measures defined on $[0, \infty)_\tau \times (0, \infty)_\rho$ and on $[0, \infty)_\tau$ respectively such that $\mu_p([0, \infty)_\tau \times (0, \infty)_\rho) < \infty$ $(p = 1, 2)$ and $\nu([0, \infty)_\tau) < \infty$; it is true that the transformation $K(\lambda)$ such as $K(\lambda)\lambda^1 = \lambda$ is not uniquely determined, but the measure matrix $K(\lambda)\, \mu(d\tau d\rho)\, K(\lambda)^{-1}$ depends only on λ and is independent of special choice of $K(\lambda)$, as may be verified from the form of (3.2). Putting $\mathfrak{l} = \mathfrak{l}^p$, $\mathfrak{m} = \mathfrak{l}^q$ in (3.15) and making use of (1.2) and (1.9), we obtain (3.1), which completes the proof of Theorem 1.

§ 4. **Inversion formula.** In this paragraph, we shall show the formulas which express the measures $\mu_p(A)$ $(p = 1, 2)$ and $\nu(\Gamma_\tau)$ in Theorem 1 by means of corresponding $(R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y}))$, from which we may conclude the uniqueness of the measures $\mu_p(A)$ and $\nu(\Gamma_\tau)$ for the functions $R_{pq}(t, \mathfrak{X}; s, \mathfrak{Y})$; $p, q = 1, 2, 3$. By definition we shall term *continuity points of* μ such ρ and τ as $\mu_p([0, \infty)_\tau \times \{\rho\}) = 0$ and $\mu_p(\{\tau\} \times (0, \infty)_\rho) = 0$ for $p = 1, 2$, and *continuity point of* ν such τ as $\nu(\{\tau\}) = 0$.

THEOREM 2. *The measures* $\mu_p(A)$ *and* $\nu(\Gamma_\tau)$ *in Theorem 1 are expressible* *by* $R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y})$ *as follows: if we put* $\Lambda_n = \{\lambda \in S/ \|\lambda - \lambda^1\|_3 \leqq 1/n\ (\lambda^1 = \langle 1, 0, 0 \rangle)$ *and* $I_c = \{\langle t, \mathfrak{X} \rangle / |t|, |x_1|, |x_2|, |x_3| \leqq c\}\ (\mathfrak{X} = \langle x_1, x_2, x_3 \rangle)$, *then we have for any continuity points* $\rho, \rho'\ (0 < \rho < \rho')$ *and* τ *of* μ

(4.1)
$$\begin{aligned} &\mu_p([0, \tau] \times (\rho, \rho']) \\ &= \lim_{n\to\infty} \left(\frac{n^2}{\pi^5}\right) \lim_{c\to\infty} \int_{I_c} R_{pp}(t, \mathfrak{X};\ 0, 0)\, dt\, d\mathfrak{X} \\ &\qquad \times \int_{(-1, \tau] \times (\rho, \rho'] \times \Lambda_n} \cos[t\tau + (\mathfrak{X}, X)_3]\, d\tau\, dX, \text{[12]} \\ &\qquad X = \langle \rho, \lambda \rangle, \end{aligned}$$

and for any continuity point τ *of* ν

(4.2)
$$\begin{aligned} \nu([0, \tau]) &= \lim_{\rho \downarrow 0} \left(\frac{1}{\pi^4}\right) \lim_{c\to\infty} \int_{I_c} R_{11}(t, \mathfrak{X};\ 0, 0)\, dt\, d\mathfrak{X} \\ &\qquad \times \int_{(-1, \tau] \times (\|X\|_3 \leqq \rho)} \cos[t\tau + (\mathfrak{X}, X)_3]\, d\tau\, dX. \end{aligned}$$

Proof. Define the measures $\sigma_p^1(M)$, $\sigma^2(M)$ and $\sigma_p(M)$ on $R^4 (= R_\tau^1 \times R_X^3)$ by

$$\begin{cases} \sigma_p^1(\Gamma_\tau \times B_\rho \times \Lambda) = \int_\Lambda \left(K(\lambda)^{-1}\mathfrak{l}^p, \mu(\Gamma_\tau \times B_\rho) K(\lambda)^{-1}\mathfrak{l}^p\right)_3 d\lambda, \\ \sigma_p^1(M) = 0 \text{ if } M \cap ([0, \infty)_\tau \times (0, \infty)_\rho \times S) \text{ is empty}; \end{cases}$$

$$\begin{cases} \sigma^2(\Gamma_\tau \times \{0\}_X) = \nu(\Gamma_\tau), \\ \sigma^2(M) = 0 \text{ if } M \cap ([0, \infty)_\tau \times \{0\}_X) \text{ is empty; and} \end{cases}$$

$$\sigma_p(M) = \sigma_p^1(M) + \sigma^2(M).$$

Then it follows from Theorem 1 that

$$R_{pp}(t, \mathfrak{X};\ 0, 0) = \int_{R^4} \cos[t\tau + (\mathfrak{X}, X)_3]\, \sigma_p(d\tau\, dX).$$

Hence, by Lévy-Haviland's inversion formula [3], for any continuity points $\rho, \rho'\ (0 < \rho < \rho')$ and τ of μ, we have

(4.3)
$$\begin{aligned} &\int_{\Lambda_n} \left(K(\lambda)^{-1}\mathfrak{l}^p, \mu([0, \tau] \times (\rho, \rho']) K(\lambda)^{-1}\mathfrak{l}^p\right)_3 d\lambda \\ &= \sigma_p\left((-1, \tau] \times (\rho, \rho'] \times \Lambda_n\right) \\ &= \frac{1}{\pi^4} \lim_{c\to\infty} \int_{I_c} R_{pp}(t, \mathfrak{X};\ 0, 0)\, dt\, dX \\ &\qquad \times \int_{(-1, \tau] \times (\rho, \rho'] \times \Lambda_n} \cos[t\tau + (\mathfrak{X}, X)_3]\, d\tau\, dX, \\ &\qquad X = \langle \rho, \lambda \rangle. \end{aligned}$$

[12] " -1 " in the domain $(-1, \tau]$ of the integral in the right-hand side has no particular sense, and may be replaced by an arbitrary negative number, as will easily be seen from the proof of this theorem.

Since $\left(K(\lambda)^{-1}\mathfrak{f}^{p},\ \mu([0,\tau]\times(\rho,\rho'])\,K(\lambda)^{-1}\mathfrak{f}^{p}\right)_3$ is continuous in λ, we get

$$(4.4)\qquad \mu_p([0,\tau]\times(\rho,\rho']) = \left(\mathfrak{f}^{p},\ \mu([0,\tau]\times(\rho,\rho'])\,\mathfrak{f}^{p}\right)_3$$
$$= \left(K(\lambda^1)^{-1}\mathfrak{f}^{p},\ \mu([0,\tau]\times(\rho,\rho'])\,K(\lambda^1)^{-1}\mathfrak{f}^{p}\right)_3$$
$$= \lim_{n\to\infty}\frac{1}{|\Lambda_n|}\int_{\Lambda_n}\left(K(\lambda)^{-1}\mathfrak{f}^{p},\ \mu([0,\tau]\times(\rho,\rho'])\,K(\lambda)^{-1}\mathfrak{f}^{p}\right)_3 d\lambda$$

where $|\Lambda_n| = \int_{\Lambda_n} d\lambda$. From (4.4), (4.3) and the fact that $\lim_{n\to\infty}(|\Lambda_n|\,n^2/\pi) = 1$, we obtain (4.1)..

Next we put

$$(4.5)\qquad \psi(\tau,\rho) = \lim_{c\to\infty}\frac{1}{\pi^4}\int_{I_c} R_{11}(t,\mathfrak{X};0,0)\,dt\,d\mathfrak{X}$$
$$\times\int_{(-1,\tau]\times(\|X\|_3\leqq\rho)}\cos[t\tau+(\mathfrak{X},X)_3]\,d\tau\,dX;$$

then, by Lévy-Haviland's inversion formula, we have

$$(4.6)\qquad \sigma_1^1\left((-1,\tau)\times(0,\rho)\times S\right) + \sigma^2\left((-1,\tau)\times\{0\}_X\right) \leqq \psi(\tau,\rho)$$
$$\leqq \sigma_1^1\left((-1,\tau]\times(0,\rho]\times S\right) + \sigma^2\left((-1,\tau]\times\{0\}_X\right)$$

for any τ and ρ. If τ is a continuity point of ν, then $\sigma^2\left((-1,\tau)\times\{0\}_X\right)$ $= \sigma^2\left((-1,\tau]\times\{0\}_X\right) = \nu([0,\tau])$; and hence, if ρ tend to 0 in (4.6), we obtain

$$(4.7)\qquad \nu([0,\tau]) = \lim_{\rho\downarrow 0}\psi(\tau,\rho).$$

Thus (4.2) follows from (4.5) and (4.7), *q.e.d.*

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