NOTE ON p-GROUPS

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In connection with the class field theory a problem concerning p-groups was proposed by W. Magnus [1]: Is there any infinite tower of p-groups $G_1, G_2, \ldots, G_n, G_{n+1}, \ldots$ such that G_1 is abelian and each G_n is isomorphic to $G_{n+1}/\theta_n(G_{n+1})$, $\theta_n(G_{n+1}) \neq 1$, $n = 1, 2, \ldots$, where $\theta_n(G_{n+1})$ denotes the n-th commutator subgroup of G_{n+1}? The present note [2] is, firstly, to construct indeed such a tower, to settle the problem, and also to refine an inequality for p-groups of P. Hall.[3]

1. Let p be an odd prime number and let M_i be the principal congruence subgroup of "stufe" (p^i) of the homogeneous modular group in the rational p-adic number field R_p, that is, the totality of matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that a_{11}, $a_{12}, a_{21}, a_{22} \in R_p$, $a_{11} \equiv a_{22} \equiv 1 \pmod{p^i}$, and $a_{12} \equiv a_{21} \equiv 0 \pmod{p^i}$. Let $\theta_r(M_i)$ denote the r-th commutator subgroup of M_i.

LEMMA 1. $\theta_s(M_i) \leqq M_{2s}$ for $s = 0, 1, 2, \ldots$.

Proof. The case $s = 0$ is trivial. Assume $s > 0$ and that $\theta_{s-1}(M_1) \leqq M_{2s-1}$. Then $\theta_s(M_1) \leqq \theta_1(M_{2s-1})$. We shall prove $\theta_1(M_{2s-1}) \leqq M_{2s}$.
Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be any two elements of M_{2s-1}. Then

$$A^{-1}B^{-1}AB = |A|^{-1} \cdot |B|^{-1} \begin{pmatrix} (a_{22}b_{22} + a_{12}b_{21})(a_{11}b_{11} + a_{12}b_{21}) - (a_{22}b_{12} + a_{12}b_{11})(a_{21}b_{11} + a_{22}b_{21}) & (a_{22}b_{22} + a_{12}b_{21})(a_{11}b_{12} + a_{12}b_{22}) - (a_{22}b_{12} + a_{12}b_{11})(a_{21}b_{12} + a_{22}b_{22}) \\ -(a_{21}b_{22} + a_{11}b_{21})(a_{11}b_{11} + a_{12}b_{21}) + (a_{21}b_{12} + a_{11}b_{11})(a_{21}b_{11} + a_{22}b_{21}) & -(a_{21}b_{22} + a_{11}b_{21})(a_{11}b_{12} + a_{12}b_{22}) + (a_{21}b_{12} + a_{11}b_{11})(a_{21}b_{12} + a_{22}b_{22}) \end{pmatrix}$$

where $|A|$, $|B|$ are the determinants of A, B respectively, and therefore $|A|^{-1}a_{11}a_{22} \equiv |B|^{-1}b_{11}b_{22} \equiv 1 \pmod{p^{2s}}$. Now $a_{11} \equiv a_{22} \equiv b_{11} \equiv b_{22} \equiv 1 \pmod{p^{2s-1}}$, $a_{12} \equiv a_{21} \equiv b_{12} \equiv b_{21} \equiv 0 \pmod{p^{2s-1}}$. Then $(1,1)$- and $(2,2)$-elements of $A^{-1}B^{-1}AB$ are obviously $\equiv 1 \pmod{p^{2s}}$. Since

$$a_{22}b_{22}(a_{11}b_{12} + a_{12}b_{22}) - (a_{22}b_{12} + a_{12}b_{11})a_{22}b_{22} = a_{22}b_{22}\{b_{12}(a_{11} - a_{22}) + a_{12}(b_{22} - b_{11})\},$$
$$-(a_{21}b_{22} + a_{11}b_{21})a_{11}b_{11} + a_{11}b_{11}(a_{21}b_{11} + a_{22}b_{21}) = a_{11}b_{11}\{a_{21}(b_{11} - b_{22}) + b_{21}(a_{22} - a_{11})\},$$

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[1] W. Magnus, Beziehung zwishen Gruppen und Idealen in einem speziellen Ring, Math. Annalen **111** (1935).

[2] An impulse was given to the present work by Dr. K. Iwasawa, through a communication by Mr. M. Suzuki.

(1, 2)- and (2, 1)-elements of $A^{-1}B^{-1}AB$ are also $\equiv 0 \pmod{p^{2^s}}$.
Thus induction proves the lemma.

Remark. More generally it can easily be seen that $(M_k, M_l) \leqq M_{k+l}$; we shall use this fact later.

LEMMA 2.

$$M_{2s} = \left\{\begin{pmatrix} 1+p^{2^s} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p^{2^s} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p^{2^s} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1+p^{2^s} \end{pmatrix}, M_{2s+t}\right\}$$

for $s, t = 0, 1, 2, \ldots$.

Proof. The case $t = 0$ is trivial. Assume $t > 0$ and

$$M_{2s} = \left\{\begin{pmatrix} 1+p^{2^s} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p^{2^s} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p^{2^s} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1+p^{2^s} \end{pmatrix}, M_{2s+t-1}\right\}.$$

We shall prove

$$M_{2s+t-1} \leqq \left\{\begin{pmatrix} 1+p^{2^s} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p^{2^s} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p^{2^s} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1+p^{2^s} \end{pmatrix}, M_{2s+t}\right\}.$$

Let $\begin{pmatrix} 1+a'_{11} & a_{12} \\ a_{21} & 1+a'_{22} \end{pmatrix}$ be any element of M_{2s+t-1}. Then

$$\begin{pmatrix} 1+a'_{11} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1+a'_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_{21} & 1 \end{pmatrix}\begin{pmatrix} 1 & a_{12} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+a'_{11} & a_{12}+a'_{11}a_{12} \\ a_{21}+a'_{22}a_{21} & 1+a'_{22}+a_{21}a_{12}+a'_{22}a_{21}a_{12} \end{pmatrix} \equiv \begin{pmatrix} 1+a'_{11} & a_{12} \\ a_{21} & 1+a'_{22} \end{pmatrix} \bmod M^{2^{s+t}}.$$

And $\begin{pmatrix} 1+a'_{11} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1+a'_{22} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_{21} & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_{12} \\ 0 & 1 \end{pmatrix}$ are respectively contained in $\left\{\begin{pmatrix} 1+p^{2^s} & 0 \\ 0 & 1 \end{pmatrix}, M_{2s+t}\right\}, \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1+p^{2^s} \end{pmatrix}, M_{2s+t}\right\}, \left\{\begin{pmatrix} 1 & 0 \\ p^{2^s} & 1 \end{pmatrix}, M_{2s+t}\right\}$, $\left\{\begin{pmatrix} 1 & p^{2^s} \\ 0 & 1 \end{pmatrix}, M_{2s+t}\right\}$, because $p > 2$. Now the lemma is proved by induction.

Remark. More generally it can again easily be seen that

$$M_n = \left\{\begin{pmatrix} 1+p^n & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p^n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p^n & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1+p^n \end{pmatrix}, M_{n+q}\right\}$$

for $n = 1, 2, \ldots$; $q = 0, 1, 2, \ldots$.

LEMMA 3. The centrum $C_1(M_1)$ of M_1 is $\left\{\begin{pmatrix} 1+a & 0 \\ 0 & 1+a \end{pmatrix}, a \equiv 0 \pmod{p}\right\}$.

Proof. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be in $C_1(M_1)$, and let $B = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ or $= \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$. Then $B^{-1}AB = A = \begin{pmatrix} a_{11} - pa_{21}, & pa_{11} - p^2a_{21} + a_{12} - pa_{22} \\ a_{21}, & pa_{21} + a_{22} \end{pmatrix}$ or $= \begin{pmatrix} a_{11} + pa_{12} & a_{12} \\ -pa_{11} + a_{21} - p^2a_{12} + pa_{22}, & -pa_{12} + a_{22} \end{pmatrix}$. Therefore $a_{12} = a_{21} = 0$, $a_{11} = a_{22}$.

LEMMA 4. $\theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1) = M_{2s} \cdot C_1(M_1)$ for $s, t = 0, 1, 2, \ldots$.

Proof. The case $s = 0$ is trivial. Assume $s > 0$ and $\theta_{s-1}(M_1) \cdot M_{2s-1+t} \cdot C_1(M_1) = M_{2s-1} \cdot C_1(M_1)$ for $t = 0, 1, 2, \ldots$.
Put $q = p^{2^{s-1}}$. Then $\frac{1}{1+q}\begin{pmatrix} 1+q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1+q \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q^2 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} 1+q^2+q^4 & q^3 \\ -q^3 & -q^2+1 \end{pmatrix}$ are elements of $\theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1)$, because $\theta_1\{\theta_{s-1}(M_1) \cdot M_{2s-1+t} \cdot C_1(M_1)\}$ $\subseteqq \theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1)$. Now $\begin{pmatrix} 1 & q^2 \\ 0 & 1 \end{pmatrix}$ is contained in $\theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1)$. Symmetrically the same is the case for $\begin{pmatrix} 1 & 0 \\ q^2 & 1 \end{pmatrix}$. Next $\begin{pmatrix} 1+q^2+q^4 & 0 \\ -q^3 & 1-q^2+\ldots \end{pmatrix}$ is contained in $\theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1)$, because $\begin{pmatrix} 1+q^2+q^4 & 0 \\ -q^3 & 1-q^2+\ldots \end{pmatrix}$ $\equiv \begin{pmatrix} 1+q^2+q^4 & q^3 \\ -q^3 & 1-q^2 \end{pmatrix}$ mod. $\theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1)$. Similarly $\begin{pmatrix} 1+q^2+q^4 & 0 \\ 0 & 1-q^2+\ldots \end{pmatrix}$ is contained in $\theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1)$.
Finally

$$\begin{pmatrix} 1+q^2+q^4 & 0 \\ 0 & 1-q^2+\ldots \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1-2q^2+\ldots \end{pmatrix} \text{ mod. } \theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1),$$

and $\begin{pmatrix} 1 & 0 \\ 0 & 1+q^2 \end{pmatrix}$ is contained in $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1-2q^2+\ldots \end{pmatrix}, M_{2s+t}\right\}$ because $p > 2$. Hence $\begin{pmatrix} 1 & 0 \\ 0 & 1+q^2 \end{pmatrix}$ and, symmetrically, $\begin{pmatrix} 1+q^2 & 0 \\ 0 & 1 \end{pmatrix}$ are contained in $\theta_s(M_1) \cdot M_{2s+t} \cdot C_1(M_1)$. Our induction argument is completed.

Remark. More generally it can be seen that

$$\theta_m(M_1) \cdot M_n \cdot C_1(M_1) = M_{2m} \cdot C_1(M_1) \quad \text{for } n = 2^m, 2^m+1, \ldots.$$

Besides it can be seen analogously that

$$H_m(M_1) \cdot M_n \cdot C_1(M_1) = M_n \cdot C_1(M_1) \quad \text{for } n = m, m+1, \ldots,$$

where H denotes the lower central series.

LEMMA 5. $\theta_s\left(\frac{M_1}{M_{2s+t}\cdot C_1(M_1)}\right) = \frac{M_{2s}\cdot C_1(M_1)}{M_{2s+t}\cdot C_1(M_1)}$ for $t = 0,1,2,\ldots$.

Proof. $\theta_s\left(\frac{M_1}{M_{2s+t}C_1(M_1)}\right) = \frac{\theta_s(M_1)\cdot M_{2s+t}\cdot C_1(M_1)}{M_{2s+t}\cdot C_1(M_1)} = \frac{M_{2s}\cdot C_1(M_1)}{M_{2s+t}\cdot C_1(M_1)}$ from Lemma 4.

Now we can construct actually in the following manner an infinite tower of p-groups satisfying the condition proposed by W. Magnus:

Designate $\frac{M_1}{M_{2n}\cdot C_1(M_1)}$ by G_n. Then $G_1 \neq 1$ is abelien, $\frac{G_n}{\theta_{n-1}(G_n)}$ is isomorphic to G_{n-1} by Lemma 5, and $\theta_{n-1}(G_n) \neq 1$. Therefore $\{G_1, G_2, \ldots, G_n, \ldots\}$ gives surely an infinite tower fulfilling the condition.

Remark. It is very likely that also for $p = 2$ we may start with M_2 to obtain a similar series in a little bit more complicated form.

For non p-groups such a construction is easier than for p-groups.

2. In his celebrated paper P. Hall[3] gave the following theorem: "Let G be a p-group ($p > 2$) of the smallest order p^n such that $\theta_m(G)$ be different from 1. Then

$$2^{m-1}(2^m - 1) \geqq n \geqq 2^m + m$$

Now we can refine the upper bound of this inequallity to be 3.2^m. To this we consider the group $G = \frac{M_1}{M_{2^m+1}\cdot C_1(M_1)}$ which was constructed above. Then $\theta_m(G)$ is obviously different from 1. The order of G is $p^{3\cdot 2^m}$ because $(M_1 : M_{2^m+1}) = (M_1 : M_{2^m+1}\cdot C_1(M_1))$. $(M_{2^m+1}\cdot C_1(M_1) : M_{2^m+1})$ and $(M_1 : M_{2^m+1}) = p^{4\cdot 2^m}$, $(M_{2^m+1}\cdot(C_1(M_1) : M_{2^m+1}) = p^{2^m}$.

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[3] P. Hall, A contribution to the theory of groups of prime power order, Proc. London Math. Soc. **36** (1934).