A THEOREM ON THE CLUSTER SETS OF PSEUDO-ANALYTIC FUNCTIONS

KIYOSHI NOSHIRO

1. Let D be an arbitrary connected domain and $w = f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, be an interior transformation in the sense of Stoïlow in D. Denote by γ a set, in D, such that D and the derived set γ' of γ have no point in common. We suppose that

(i) $u_x,\ u_y,\ v_x,\ v_y$ exist and are continuous in $D^* = D - \gamma$;

(ii) $J(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0$ at every point in D^*;

(iii) the function $q(z)$ defined as the ratio of the major and minor axes of an infinitesimal ellipse with centre $f(z)$, into which an infinitesimal circle with centre at each point z of D^* is transfomed by $w = f(z)$, is bounded in D^*: $q(z) \leqq A$.

$f(z)$ is then called pseudo-meromorphic (A) in D.[1]

Next, suppose that $w = f(z)$ is pseudo-meromorphic (A) in D. Let C be the boundary of D, E be a closed set of capacity[2] zero, included in C, and z_0 be a point in E. We can associate with z_0 three cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ as follows: $S_{z_0}^{(D)}$ is the set of all values α such that $\lim_{\nu\to\infty} f(z_\nu) = \alpha$ with a sequence $\{z_\nu\}$ of points tending to z_0 inside D. $S_{z_0}^{*(C)}$ is the intersection $\bigcap_r M_r$, where M_r denotes the closure of the union $\bigcup_{\zeta'} S_{\zeta'}^{(D)}$ for all ζ' belonging to the common part of $C - E$ and $U(z_0, r)$: $|z - z_0| < r$. In the particular case when E consists of a single point z_0, we denote $S_{z_0}^{*(C)}$ by $S_{z_0}^{(C)}$ for simplicity. Obviously $S_{z_0}^{(D)}$ and $S_{z_0}^{*(C)}$ are closed sets such that $S_{z_0}^{*(C)} \subset S_{z_0}^{(D)}$ and $S_{z_0}^{(D)}$ is always non-empty while $S_{z_0}^{*(C)}$ becomes empty if and only if there exists a positive number r such that $C - E$ and $U(z_0, r)$ have no point in common.

In the particular case where $w = f(z)$ is single-valued meromorphic in D, the following theorems concerning the cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ are known:

Received February 20, 1950.

[1] For the definition of pseudo-meromorphic functions, Cf. S. Kakutani: Applications to the theory of pseudo-regular functions to the type-problem of Riemann surfaces, Jap. Journ. of Math. Vol. **13** (1937), pp. 375-392. R. Nevanlinna: Eindeutige analytische Funktionen, Berlin, 1936, p. 343.

[2] "Capacity" means logarithmic capacity in this note.

Theorem I. (Iversen-Beurling-Kunugui) [3] $B(S_{z_0}^{(D)}) \subset S_{z_0}^{(C)}$, where $B(S_{z_0}^{(D)})$ denotes the boundary of $S_{z_0}^{(D)}$, or what is the same, $\Omega = S_{z_0}^{(D)} - S_{z_0}^{(C)}$ is an open set.

Theorem II. (Beurling-Kunugui) [4] Suppose that $\Omega = S_{z_0}^{(D)} - S_{z_0}^{(C)}$ is not empty and denote by Ω_n any component of Ω. Then $w = f(z)$ takes every value, with two possible exceptions, belonging to Ω_n infinitely often in any neighbourhood of z_0.

Theorem I*. (Tsuji) [5] $B(S_{z_0}^{(D)}) \subset S_{z_0}^{*(C)}$, that is, $\Omega = S_{z_0}^{(D)} - S_{z_0}^{*(C)}$ is an open set.

Theorem II*. (Kametani-Tsuji) [6] Suppose that $\Omega = S_{z_0}^{(D)} - S_{z_0}^{*(C)}$ is not empty. Then $w = f(z)$ takes every value, except a possible set of w-values of capacity zero, belonging to Ω infinitely often in any neighbourhood of z_0.

The object of the present note is to propose the following

THEOREM 1. *Suppose that E is included in a single boundary-component C_0 of C and $w = f(z)$ is pseudo-meromorphic (A) in D. Then $\Omega = S_{z_0}^{(D)} - S_{z_0}^{*(C)}$ is an open set. Suppose further that Ω is not empty. Then $w = f(z)$ takes every value, with two possible exceptions, belonging to any component Ω_n of Ω infinitely often in any neighbourhood of z_0.*

Remark. It is obvious that Theorem 1 contains Theorems I and II [7] and holds good provided that D is simply connected.[8] There is an anticipation that Theorems I* and II* may be probably true when $w = f(z)$ be pseudo-meromor-

3) F. Iversen: Sur quelques propriétés des fonctions monogènes au voisinage d'un point singulier, Öfv. af Einska Vet-Soc. Förh. 58 (1916).

K. Kunugui: Sur un théorème de M. M. Seidel-Beurling, Proc. Acad. Tokyo, 15 (1939); Sur un problème de M. A. Beurling, Proc, Acad. Tokyo, 16 (1940); Sur l'allure d'une fonction analytique uniform au voisinage d'un point frontière de son domaine de définition, Jap. Journ. of Math. 18 (1942), pp. 1-39.

A. Beurling: Études sur un problème de majoration, Thèse de Upsal, 1933; Cf. pp. 100-103.

4) Beurling: l. c. 3); Kunugui: l. c. 3).

5) M. Tsuji: On the cluster set of a meromorphic function, Proc. Acad. Tokyo, 19 (1943); On the Riemann surface of an inverse function of a meromorphic function in the neighbourhood of a closed set of capacity zero, Proc. Acad. Tokyo, 19 (1943).

6) Tsuji: l. c. 5)、S. Kametani: The exceptional values of functions with the set of capacity zero of essential singularities, Proc. Acad. Tokyo, 17 (1941), pp. 429-433.

7) Recently E. Sakai has obtained some interesting results concerning pseudo-meromorphic functions. Theorem 1 answers affirmatively a problem represented by him. Cf. E. Sakai: Note on pseudo-analytic functions, forthcoming Proc. Acad. Tokyo.

8) The special case where D is simply connected and $w = f(z)$ is single-valued meromorphic in D has been treated by the writer in another note. Cf. K. Noshiro: Note on the cluster sets of analytic functions, forthcoming Journ. Math. Soc. Japan.

phic (A) in (D). But the writer has not yet succeeded in proving it.

2. To prove Theorem 1 we use two lemmas.

LEMMA 1. *Let $w = f(z)$ be pseudo-regular (A) in a bounded domain D and E be a closed set of capacity zero, included in the boundary C of D. If*

$$\overline{\lim_{z\to\zeta}}|f(z)| \leqq M$$

for every point ζ of $C - E$ and $f(z)$ is bounded in a neighbourhood of every point ζ of E, then $|f(z)| \leqq M$ for all points z in D.

Proof. We suppose, contrary to the assertion, that there exists a point z_0 in D such that $|f(z_0)| > M$. Let Φ be the Riemannian image of D by $w = f(z)$ and denote by P_0 the point on Φ which corresponds to z_0. Consider the star-region H in Gross' sense formed by the sum of segments from P_0 with projection $w_0 = f(z_0)$ to singular points along all rays: $\arg(w - w_0) = \varphi$ on Φ, whose projections lie in the half-plane $\Re[e^{-i\arg w_0}\cdot(w - w_0)] > 0$. We shall show that the linear measure of the set Γ of arguments φ of singular rays (by which we understand rays meeting singular points in finite distances) is equal to zero. Denote by H_R the common part of H and a circular disc $|w - w_0| < R$ and by Δ_R the image of H_R by the inverse transformation of $w = f(z)$. Then, Δ_R is a simply connected domain included in D. Since E is a closed set of capacity zero, Evans' theorem [9] shows that there exists a distribution of positive mass $d\mu(a)$ entirely on E such that

$$u(z) = \int_E \log\left|\frac{1}{z-a}\right| d\mu(a)\,, \quad \mu(E) = 1 \tag{1}$$

is harmonic outside E, excluding $z = \infty$, and has boundary value $+\infty$ at any point of E. Let $v(z)$ be its conjugate harmonic function and put

$$t = \chi(z) = e^{u(z)+iv(z)} = \rho(z)e^{iv(z)}\,. \tag{2}$$

For the sake of convenience, we call the function $t = \chi(z)$ "Evans' function." Let C_λ be the niveau curve: $\rho(z) = \text{const.} = \lambda$ $(0 < \lambda < +\infty)$. Then C_λ consists of a finite number of simple closed curves surrounding E. Further, Evans' function has the property

$$\int_{C_\lambda} dv(z) = \int_{C_\lambda} \frac{\partial u}{\partial n}\, ds = 2\pi\,, \tag{3}$$

where s denotes the arc length of C_λ and n is the inner normal of C_λ. Now

[9] G. C. Evans: Potentials and positively infinite singularities of harmonic functions, Monatsheft für Math. und Phys. **43** (1936), pp. 419-424.

K. Noshiro: Contributions to the theory of the singularities of analytic functicns, Jap. Journ. of Math. **19** (1948), pp. 299-327.

we consider the Riemannian image $\tilde{\Delta}_R$ of Δ_R by $t = \chi(z)$ and the function $w = W(t) = f[z(t)]$ defined on $\tilde{\Delta}_R$. Let $\tilde{\Theta}_\lambda$ be the set of cross-cuts of $\tilde{\Delta}_R$ above the circle $|t| = \lambda$. We denote by $\lambda\theta(\lambda)$ the total length of $\tilde{\Theta}_\lambda$ and $L(\lambda)$ that of the image of $\tilde{\Theta}_\lambda$ by $w = W(t)$. Then, applying a well-known method in proving Gross' theorem, we get

$$(4)\quad \int_{\lambda_0}^{\lambda} \frac{[L(\lambda)]^2}{\lambda\theta(\lambda)} d\lambda \leqq (A + \sqrt{A^2 - 1}) \int_{\lambda_0}^{\lambda} \int_{\tilde{\Theta}_\lambda} J(t) \lambda d\lambda d\theta \leqq \pi A R^2, \quad (0 < \lambda_0 \leqq \lambda).$$

Since $\theta(\lambda) \leqq 2\pi$, we have

$$\underline{\lim_{\lambda \to \infty}} L(\lambda) = 0.$$

Accordingly, we see that the set Γ of arguments φ of singular rays is of linear measure zero. Consequently there exists at least one asymptotic path Λ inside D reaching a point ζ in E, along which $w = f(z)$ converges to ∞ as z tends to ζ. But this is a contradiction, since $f(z)$ is bounded in a neighbourhood of ζ.

Remark. Lemma 1 is an immediate consequence from R. Nevanlinna's theorem [10] in the case when $w = f(z)$ is single-valued regular in D.

By a similar argument as in Lemma 1, we obtain, without difficulty,

LEMMA 2. (*An extension of Iversen's theorem*) [11] *Let D be an arbitrary domain, C being its boundary, and let E be a closed set of capacity zero included in C. Suppose that $f(z)$ is pseudo-meromorphic (A) in D and $S_{z_0}^{(D)} - S_{z_0}^{*(C)}$ is not empty. If $w = f(z)$ does not take a value α, contained in $S_{z_0}^{(D)} - S_{z_0}^{*(C)}$, infinitely often, then α is either an asymptotic value of $w = f(z)$ at z_0 or there is a sequence of accessible boundary points ζ_n in E tending to z_0 such that α is an asymptotic value at each ζ_n.*

3. Proof to Theorem 1. Let w_0 be an arbitrary value belonging to $S_{z_0}^{(D)} - S_{z_0}^{*(C)}$. By hypothesis, there exists a circle $K: |z - z_0| = r$, arbitrarily small, such that $K \cdot E = 0$ and $f(z) \neq w_0$ on $K \cdot D$. We may suppose that w_0 does not belong to the closure M_r of the union $\bigcup_{\zeta'} S_{\zeta'}^{(D)}$ for all ζ' belonging to the common part of $C - E$ and $|z - z_0| \leqq r$. We denote by ρ_1 the distance of M_r from w_0. Let ρ_2 be a positive number such that $|f(z) - w_0| \geqq \rho_2 > 0$ on $K \cdot D$. We denote by ρ a positive number less than $\min(\rho_1, \rho_2)$. Since w_0 is a cluster value of $w = f(z)$ at z_0, there exists a sequence of points z_μ $(\mu = 1, 2, \ldots)$ inside $(K) \cdot D$, (K) denoting the interior of K, tending to z_0 such that $w_\mu = f(z_\mu)$ tends to w_0.

[10] R. Nevanlinna: l. c. 1), pages 132 and 134.

[11] K. Noshiro: On the theory of the cluster sets of analytic functions, Journ. Fac. of Sci., Hokkaido Imp. Univ. **6** (1938), pp. 217–231; Cf. theorem 4.

We keep hereafter the sequence z_μ ($\mu = 1, 2, \ldots$) fixed. Consider the open set D_0 of points z inside $(K)\cdot D$ whose images $w = f(z)$ lie in (c): $|w - w_0| < \rho$. Then D_0 consists of a finite or an enumerable number of connected domains Δ. Denote by Δ_μ the component containing z_μ; some Δ_μ may coincide with one other.

First we consider the case in which there are infinitely many distinct components Δ_μ. For the sake of simplicity, we suppose that $\Delta_\mu \neq \Delta_\nu$ if $\mu \neq \nu$. Then, we easily show that Δ_μ ($\mu = 1, 2, \ldots$) converges to z_0. For, if otherwise there exists a circle K': $|z - z_0| = r'$ ($< r$) such that $K'\cdot E = 0$ and $K'\cdot\Delta_{\mu_n} \neq 0$ ($n = 1, 2, \ldots$), where Δ_{μ_n} denotes a sub-sequence of Δ_μ. Let ζ_n be any boundary point of Δ_{μ_n}, lying on the circle K' and ζ_0 be a point of accumulation of the sequence ζ_n ($n = 1, 2, \ldots$). Since $f(\zeta_n)$ lies on the circle c: $|w - w_0| = \rho$, ζ_0 must belong to either $C - E$ or D. However, either of two cases leads to a contradiction, because either the set M_r intersects the circle $|w - w_0| = \rho$ or infinitely many niveau curves: $|f(z) - w_0| = \rho$ intersect any neighbourhood of ζ_0, while $w = f(z)$ is pseudo-regular (A) in D. If Δ_μ is compact in D, then it is evident that $w = f(z)$ takes every value in (c): $|w - w_0| < \rho$. If Δ_μ is not compact in D, its boundary consists of a closed subset E_μ of E and a finite or an enumerable number of analytic curves inside D; by Lemma 1, the value-set $\mathfrak{D}_\mu$ of $w = f(z)$ in Δ_μ is everywhere dense in (c): $|w - w_0| < \rho$, what is the same, the closure $\overline{\mathfrak{D}}_\mu$ coincides with $|w - w_0| \leqq \rho$. Considering that Δ_μ ($\mu = 1, 2, \ldots$) converges to z_0, we see that the cluster set $S_{z_0}^{(D)}$ includes the closed circular disc $|w - w_0| \leqq \rho$.

Next, let r_n and ρ_n be two decreasing sequences of positive numbers tending to zero, such that, for each n, r_n and ρ_n are selected as stated above, and consider two sequences of circles K_n: $|z - z_0| = r_n$ and c_n: $|w - w_0| = \rho_n$ ($n = 1, 2, \ldots$). Denote by $\Delta_\mu^{(n)}$ the component with an interior point z_μ, which is an inverse image of (c_n): $|w - w_0| < \rho_n$. If the sequence $\Delta_\mu^{(n)}$ ($\mu \geqq N_\mu$) consists of infinitely many distinct domains for at least one n, then the reasoning used above shows that $S_{z_0}^{(D)}$ includes the closed disc $|w - w_0| \leqq \rho_n$. Thus, we have only to consider the case in which the sequence $\Delta_\mu^{(n)}$ consists of only a finite number of distinct domains for every n. Denote by Δ_1 any $\Delta_\mu^{(1)}$ containing a sub-sequence $\{z_\mu^{(1)}\}$ of $\{z_\mu\}$, and by Δ_2 any $\Delta_\mu^{(2)}$ containing a sub-sequence $\{z_\mu^{(2)}\}$ of $\{z_\mu^{(1)}\}$ and so on. Thus, we obtain a new sequence of domains $\{\Delta_n\}$ such that $\Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_n \supset \ldots$ and each Δ_n has a boundary point z_0 in common. Accordingly, since the value-set of $w = f(z)$ in Δ_n is included in (c_n): $|w - w_0| < \rho_n$ and the diameter of Δ_n tends to zero as $n \to \infty$, there exists an asymptotic path Λ of $w = f(z)$ reaching z_0 along which $w = f(z)$ converges to w_0. Denote by Ω_0 the component containing w_0 of the complementary set of $S_{z_0}^{*(C)}$ with respect to the w-plane. We shall now show that $w = f(z)$ takes every value, except two possible exceptions, belonging to Ω_0 infinitely often in any neighbourhood of z_0. Without loss of generality, we may suppose that Ω_0 does not contain $w = \infty$. Suppose, contrary to the assertion, that there are three exceptional values w_1, w_2, w_3 in Ω_0. Then, there exists a positive number η_1 such that $f(z) \neq w_1, w_2, w_3$ in the common part of D and $U(z_0, \eta_1)$: $|z - z_0| < \eta_1$. Inside Ω_0 we draw a simple closed regular analytic curve Γ which surrounds w_0, w_1, w_2 and passes through w_3, and whose interior consists only of interior points of Ω_0. By hypothesis, we can select a positive number η ($< \eta_1$), arbitrarily small, such that, K' denoting the circle $|z - z_0| = \eta$, $K' \cdot (C - E) = 0$ and the closure M_η of the union $\bigcup_{\zeta'} S_{\zeta'}^{(D)}$ for all ζ' belonging to the common part of $C - E$ and $|z - z_0| \leqq \eta$ lies outside Γ. We may assume that the image of Λ by $w = f(z)$ is a curve lying completely in the interior of Γ. Consider the set D_η of points z inside the intersection of D and $U(z_0, \eta)$ such that $w = f(z)$ lies in the interior of Γ. Then the open set D_η consists of at most an enumerable number of connected components. We shall denote by Δ the component which contains the asymptotic path Λ. It is easily seen that the boundary of Δ consists of a finite number of arcs of the circle K', a finite or an enumerable number of analytic contours inside D and a closed subset E_0 of E. Further it should be noticed that Δ is simply connected. For, by hypothesis, E is included in a single boundary-component C_0 of the boundary C of D and the frontier of Δ contains no closed analytic contour, since every analytic contour of Δ is transformed by $w = f(z)$ into a curve lying on the simple closed curve Γ passing through an exceptional value w_3. Denote by Φ the Riemannian image of Δ transformed by $w = f(z)$ in a one-one manner and by Φ_0 the domain obtained by excluding two points w_1 and w_2 from the interior of Γ. Then, Φ is a simply connected covering surface of basic surface Φ_0 whose Euler's characteristic is equal to 1. With an aid of Evans' theorem stated before, we can prove, without difficulty, that Φ satisfies the condition of regular exhaustion (with a slightly modified form) in Ahlfors' sense. But this will lead to a contradiction by Ahlfors' main theorem on covering surfaces.[12] Thus, it is proved that $S_{z_0}^{(D)} - S_{z_0}^{*(C)}$ is an open set.

Suppose that the open set $\Omega = S_{z_0}^{(D)} - S_{z_0}^{*(C)}$ is not empty. Let Ω_n be any connected component of Ω. We shall now prove that $w = f(z)$ takes every value, with two possible exceptions, belonging to Ω_n infinitely often in any neighbourhood of z_0. We may suppose that Ω_n does not contain $w = \infty$. Contrary to the

---

[12] L. Ahlfors: Zur Theorie der Überlagerungsflächen, Acta Math. **65** (1935), pp. 157-194. R. Nevanlinna: l. c. 1), Cf. p. 323. K. Noshiro: l. c. 8).

assertion, we suppose that there are three exceptional values w_0, w_1 and w_2 in Ω_n. Then, there exists a positive number η_1 such that $f(z) \neq w_0, w_1, w_2$ in the common part of D and $U(z_0, \eta_1)$: $|z - z_0| < \eta_1$. Inside Ω_n we draw a simple closed regular analytic curve Γ which surrounds w_0, w_1 and passes through w_2, and whose interior consists only of interior points of Ω_n. We can select a positive number η $(< \eta_1)$, arbitrarily small, such that, K' denoting the circle $|z - z_0| = \eta$, $K' \cdot (C - E) = 0$ and the closure M_η of the union $\bigcup_{\zeta'} S_{\zeta'}^{(\eta)}$ for all ζ' belonging to the common part of $C - E$ and $|z - z_0| \leqq \eta$ lies outside Γ. Now, by Lemma 2 either w_0 is an asymptotic value of $w = f(z)$ at z_0 or there exists a sequence of ζ_n in E tending to z_0 such that w_0 is an asymptotic value at each ζ_n. Consequently it is possible to find a point ζ_0 (distinct from z_0 or not) belonging to $E \cdot U(z_0, \eta)$ such that w_0 is an asymptotic value of $w = f(z)$ at ζ_0. Let Λ be the asymptotic path with the asymptotic value w_0 at ζ_0. We may assume that the image of Λ by $w = f(z)$ is a curve lying completely inside Γ. Consider the set D_η of points z inside the intersection of D and $U(z_0, \eta)$ such that $w = f(z)$ lies inside Γ. Now, we denote by Δ the component, of D_η, which contains the asymptotic path Λ. Since Δ must be simply connected, we would arrive at a contradiction.[13]

Mathematical Institute, Nagoya University

[13] K. Noshiro: l. c. 8).