ON TITCHMARSH-KODAIRA'S FORMULA CONCERNING WEYL-STONE'S EIGENFUNCTION EXPANSION

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1. Introduction. Let $q(x)$ be real and continuous in the infinite open interval $(-\infty, \infty)$ and let $y_1(x, \lambda)$, $y_2(x, \lambda)$ be the solutions of

(1.1) $y'' + \{\lambda - q(x)\}y = 0$[1]

with the initial conditions

(1.2) $y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1.$

For appropriate homogeneous real boundary conditions at $x = -\infty$, $x = \infty$ of the differential operator

(1.3) $$L_x = q(x) - \frac{d^2}{dx^2},$$

there corresponds real symmetric positive definite matrix

(1.4) $$P(u_2) - P(u_1) = (p_{jk}(u_2) - p_{jk}(u_1)), \quad (j, k = 1, 2),$$
$$-\infty < u_1 < u_2 < \infty,$$

such that we have Weyl[2]-Stone's[3] expansion (in the sense of L_2-convergence):

(1.5) for real-valued $f(x) \in L_2(-\infty, \infty)$,

$$f(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} d_u \left\{ \sum_{j,k=1}^{2} \int_0^u y_j(x, u) dp_{jk}(u) \int_{-n}^{n} f(s) y_k(s, u) ds \right\}.$$

Recently and independently of each other, E. C. Titchmarsh[4] and K. Kodaira[5]

Received December 20, 1949. (Added March 5, 1950). The result was communicated to Prof. K. Kodaira at Princeton, who informed to the author that a similar treatment may be carried on by Prof. N. Levinson. So a copy of the manuscript was sent to Prof. Levinson, who, in his letter of February 25, informed to the author that his work was submitted to the Duke Math. Journal in May, 1949. He says that his method is different from the pressent note; he proceeds in his proof from the Parseval relation of the Sturm-Liouville orthonormal functions.

1) The case of finite or half finite open interval may be treated exactly in the same manner. Moreover (apparently) general equation $(p(\xi)z_\xi)_\xi + \{\lambda r(\xi) - s(\xi)\}z = 0$ may be reduced to (1.1) by the Liouville Transformation $x = \int_0^\xi (p^{-1}r)^{1/2} d\xi$, $y = (pr)^{1/4}z$.

2) Über gewöhnlichc Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlichen Funktionen, Math. Ann., 68 (1910), 220-269.

3) Linear transformations in Hilbert space, Amer. Math. Soc. Coll. Publ. XV (1932).

4) Eigenfunction expansisons associated with second order differential equations, Oxford (1946).

5) The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices, Amer. J. of Math., **71** (1949), 921-945.

gave the explicite formula for the density matrix (1.4). The result is much important, since it enables us to unify the classical expansion theorem such as Fourier series and integrals, the Bessel-, Hermite- or Laguerre-function expansions etc. Titchmarsh's method makes use of Cauchy's calculus of residues and does not rely upon the theory of integral equations nor the theory of Hilbert space. Kodaira's method is a modernization, by the theory of Hilbert space, of Weyl's original method. The purpose of the present paper is to show that W-S-T-K's theory may be obtained, by making use of Weyl's analysis (see **2** below), as a natural limiting case of the classical expansion theorem due to Hilbert-Schmidt which concerns with the case of finite closed interval.

2. Preliminaries (Wey'ls analysis). For the sake of exposition and completeness we first develop, with or without proof, Weyl's analysis.[6]

Let, for $0 \leqq x \leqq b < \infty$,

(2.1) $$L_x F(x, \lambda) = \lambda F(x, \lambda), \quad L_x G(x, \lambda') = \lambda' G(x, \lambda').$$

We have Green's formula

(2.2) $$(\lambda' - \lambda)\int_0^x FGdx = \int_0^x \{FL_xG - GL_xF\}dx = \int_0^x \{-FG'' + F''G\}dx$$
$$= W_0(F, G) - W_x(F, G), \text{ where}$$
$$W_x(H, K) = H(x)K'(x) - H'(x)K(x).$$

We see, by putting $\lambda' = \lambda$, that $W_x(F(x, \lambda), G(x, \lambda))$ is independent of x. Let

(2.3) $$W_x(F(x, \lambda), \ G(x, \lambda)) = \omega(\lambda).$$

Next let

(2.4) $$F(x, \lambda) = y_2(x, \lambda), \quad G(b, \lambda) = -\sin\beta, \quad G'(b, \lambda) = \cos\beta,$$

then

(2.5) $$\omega(\lambda) = W_b(F(x, \lambda), G(x, \lambda)) = F(b, \lambda)\cos\beta + F'(b, \lambda)\sin\beta.$$

Thus the condition $\omega(\lambda_0) = 0$ is equivalent to the condition that λ_0 is an eigenvalue of

$$L_x y = \lambda y, \quad 1 \cdot y(0) + 0 \cdot y'(0) = 0, \quad y(b)\cos\beta + y'(b)\sin\beta = 0,$$

the corresponding eigenfunction being $y_2(x, \lambda)$. Hence the roots λ_0 of $\omega(\lambda) = 0$ must be real.

The homogeneous real boundary condition at $x = b$ of the solution of $L_x y = \lambda y$:

(2.6) $$\{y_1(b, \lambda) + h_b(\lambda)y_2(b, \lambda)\}\cos\beta + \{y_1'(b, \lambda) + h_b(\lambda)y_2'(b, \lambda)\sin\beta = 0$$

defines

(2.7) $$h_b(\lambda) = -\frac{y_1(b, \lambda)\cos\beta + y_1'(b, \lambda)\sin\beta}{y_2(b, \lambda)\cos\beta + y_2'(b, \lambda)\sin\beta}.$$

6) H. Weyl or E. C. Titchmarsh, loc. cit.

Since the denominator is $\omega(\lambda)$, $h_b(\lambda)$ is a meromorphic function in λ whose poles are all real. The numerator is also an $\omega(\lambda)$ and thus the zeros of $h_b(\lambda)$ are also all real. Thus, if $\mathfrak{J}(\lambda) = v \neq 0$,

$$h_b(\lambda, z) = -\frac{y_1(b, \lambda)z + y_1'(b, \lambda)}{y_2(b, \lambda)z + y_2'(b, \lambda)} \tag{2.8}$$

describes a finite circle $C_b(\lambda)$ in the complex plane when z describes the real axis. Since

$$h_b\left(\lambda, -\frac{y_2'(b, \lambda)}{y_2(b, \lambda)}\right) = \infty, \tag{2.9}$$

the centre of the circle $C_b(\lambda)$ is given by

$$h_b\left(\lambda, -\overline{\frac{y_2'(b, \lambda)}{y_2(b, \lambda)}}\right) = -\frac{W_b(y_1, \bar{y}_2)}{W_b(y_2, \bar{y}_2)}.$$

The radius $r_b(\lambda)$ of the circle $C_b(\lambda)$ is given by

$$\left| h_b(\lambda, 0) + \frac{W_b(y_1, \bar{y}_2)}{W_b(y_2, \bar{y}_2)} \right| = \left| \frac{y_1'(b, \lambda)}{y_1(b, \lambda)} - \frac{W_b(y_1, \bar{y}_2)}{W_b(y_2, \bar{y}_2)} \right|$$

$$= \left| \frac{W_b(y_1, y_2)}{W_b(y_2, \bar{y}_2)} \right| = \left(2|v|\int_0^b |y_2(x, \lambda)|^2 dx\right)^{-1}, \quad (\lambda = u + iv),$$

since $W_b(y_1, y_2) = W_0(y_1, y_2) = 1$,

$$y_j(x, \bar{\lambda}) = \overline{y_j(x, \lambda)}, \quad y_j'(x, \bar{\lambda}) = \overline{y_j'(x, \lambda)} \tag{2.10}$$

and hence, by Green's formula,

$$2v\int_0^b |y_2(x, \lambda)|^2 dx = 2v\int_0^b y_2(x, \lambda) y_2(x, \bar{\lambda}) dx \tag{2.11}$$

$$= iW_0(y_2(x, \lambda), y_2(x, \bar{\lambda})) - iW_b(y_2(x, \lambda), y_2(x, \bar{\lambda}))$$

$$= -iW_b(y_2, \bar{y}_2).$$

Thus, by (2.9) and

$$\mathfrak{J}\left(-\frac{y_2'(b, \lambda)}{y_2(b, \lambda)}\right) = \frac{i}{2}\left(\frac{y_2'(b, \lambda)}{y_2(b, \lambda)} - \overline{\frac{y_2'(b, \lambda)}{y_2(b, \lambda)}}\right) = \frac{-i}{2}\frac{W_b(y_2, \bar{y}_2)}{|y_2(b, \lambda)|^2}$$

$$= \frac{v\int_0^b |y_2(x, \lambda)|^2 dx}{|y_2(b, \lambda)|^2},$$

we see that the interior of $C_b(\lambda)$ corresponds to the lower (upper) z-plane if $v > 0$ ($v < 0$). Thus, since

$$z = -\frac{y_2'(b, \lambda)h_b + y_1'(b, \lambda)}{y_2(b, \lambda)h_b + y_1(b, \lambda)},$$

the two conditions $v > 0$, $h \in$ the interior of the circle $C_b(\lambda)$ is equivalent to

$$i\left(-\frac{y_2'(b, \lambda)h + y_1'(b, \lambda)}{y_2(b, \lambda)h + y_1(b, \lambda)} + \overline{\frac{y_2'(b, \lambda)h + y_1'(b, \lambda)}{y_2(b, \lambda)h + y_1(b, \lambda)}}\right) > 0.$$

By (2.10) and by a simple calculation we see that it is equivalent to

$$iW_b(y_1 + hy_2, \bar{y}_1 + \bar{h}\bar{y}_2) > 0.$$

Hence, if $v = \mathfrak{J}(\lambda) > 0$, we have, by Green's formula and $W_0(y_2, \bar{y}_1) = -1$, $W_0(y_2, \bar{y}_2) = 0$, $W_0(y_1, \bar{y}_1) = 0$, $W_0(y_1, \bar{y}_2) = 1$,

$$2v\int_0^b |y_1 + hy_2|^2 dx = i(W_0(y_1 + hy_2, \bar{y}_1 + \bar{h}\bar{y})$$
$$- W_b(y_1 + hy_2, \bar{y}_1 + \bar{h}\bar{y}_2)) < iW_0(y_1 + hy_2, \bar{y}_1 + \bar{h}\bar{y}_2) = 2\mathfrak{J}(h).$$

Therefore if $v = \mathfrak{J}(\lambda) > 0$, the two conditions $h \in$ the interior of $C_b(\lambda)$ and $h \in$ the perimeter of $C_b(\lambda)$ are respectively equivalent to

(2.12) $$\int_0^b |y_1(x, \lambda) + hy_2(x, \lambda)|^2 dx < v^{-1}\mathfrak{J}(h) \text{ and } = v^{-1}\mathfrak{J}(h).$$

Thus, if $\mathfrak{J}(\lambda) \neq 0$,

(2.13) $b < b'$ implies $C_{b'}(\lambda) \subseteqq C_b(\lambda)$,

and hence

(2.14) $$C_\infty(\lambda) = \bigcap_{b>0} C_b(\lambda)$$

is either i) a circle with a positive radius

$$r_\infty(\lambda) = \lim_{b\to\infty} r_b(\lambda) = (2|v|\int_0^\infty |y_2(x, \lambda)|^2 dx)^{-1} \text{ (the limit circle case)},$$

or ii) a point = a circle with radius 0:

$$r_\infty(\lambda) = \lim_{b\to\infty} r_b(\lambda) = (2|v|\int_0^\infty |y_2(x, \lambda)|^2 dx)^{-1} = 0 \text{ (the limit point case)}.$$

We see from (2.12), that in the limit circle case (for λ_0, $\mathfrak{J}(\lambda_0) \neq 0$), all the solutions of $L_x y = \lambda_0 y$ belong to $L_2(0, \infty)$. And this implies, as will be seen easily,[7] that all the solution of $L_x y = \lambda y$ belong to $L_2(0, \infty)$ for every (real or complex) λ. Conversely, if all the solutions of $L_x y = \lambda_0 y$ (for $\mathfrak{J}(\lambda_0) \neq 0$) are

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[7] Let $K(x, s) = y_1(x, \lambda_0)y_2(s, \lambda) - y_1(s, \lambda_0)y_2(x, \lambda)$, then $u(x) = (Kv)(x) = \int_0^x K(x, s)v(s)ds$ satisfies $L_x u - \lambda_0 u = v$, $u(0) = u'(0) = 0$. Hence the solution of $L_x\hat{y} = \lambda\hat{y}$, $\hat{y}(0, \lambda) = \gamma$, $\hat{y}'(0, \lambda) = \delta$, when expanded as

$$\hat{y}(x, \lambda) = u_0(x) + (\lambda - \lambda_0)u_1(x) + (\lambda - \lambda_0)^2 u_2(x) + \ldots,$$

gives us

$$u_n(x) = (Ku_{n-1})(x), \quad n = 1, 2, \ldots.$$

For, we must have

$$(L_x - \lambda_0)u_0 = 0, \quad u_0(0) = \gamma, \quad u_0'(0) = \delta, \quad (L_x - \lambda_0)u_n = u_{n-1}, \quad u_n(0) = u_n'(0) = 0$$

by comparing the coefficients of $(\lambda - \lambda_0)^n$ on both sides of $L_x\hat{y} = \lambda\hat{y}$.

Thus we obtain, by induction with respect to n,

$$|u_n(x)|^2 \leqq \frac{1}{(n-1)!}\int_0^\infty |u_0(x)|^2 dx \times \frac{d}{dx}\Big(\int_0^x k(s)ds\Big)^n, \quad k(x) = \int_0^x |K(x, s)|^2 ds.$$

Hence, by $k(x) \in L_1(0, \infty)$, we obtain $\hat{y}(x.\lambda) \in L_2(0, \infty)$ easily. This proof is due to Kodaira.

$\in L_2(0,\infty)$, then we are in the limit circle case. This we see from

$$r_\infty(\lambda) = (2|\mathfrak{J}|(\lambda_0)|\int_0^\infty |y_2(x,\lambda_0)|^2 dx)^{-1} \neq 0 .$$

Therefore the limit point case is characterised, independently of the parameter λ, by the fact that, for at least one λ (real or complex), $L_x y = \lambda y$ admits solution $\overline{\in} L_2(0,\infty)$. That, in this case, the equation $L_x y = \lambda y$ for $\mathfrak{J}(\lambda) \neq 0$ admits solution $\in L_2(0,\infty)$ will be seen from (2.12):

$$\int_0^\infty |y_1(x,\lambda) + C_\infty(\lambda) y_2(x,\lambda)|^2 dx \leqq v^{-1}\mathfrak{J}(C_\infty(\lambda)) . \tag{2.15}$$

Since the centre and the radius of the circle $C_b(\lambda)$ depend continuously on b, λ we see, from (2.13), that in the limit circle case there exists a sequence $\{b_n\}$ with $b_n \uparrow \infty$ such that $h_{b_n}(\lambda)$ converges to a function $m_2(\lambda)$ regular for $\mathfrak{J}(\lambda) \neq 0$, uniformly in any bounded closed λ-domain not containing the real numbers. In the limit point case we may replace $\lim_{n\to\infty} h_{b_n}(\lambda)$ by $\lim_{b\to\infty} h_b(\lambda)$ $(= m_2(\lambda) = C_\infty(\lambda))$.

Similarly we may define, for $-\infty < a \leqq 0$,

$$h_a(\lambda) = -\frac{y_1(a,\lambda)\cos\alpha + y_1'(a,\lambda)\sin\alpha}{y_2(a,\lambda)\cos\alpha + y_2'(a,\lambda)\sin\alpha}, \quad (\alpha = \text{real}) \tag{2.16}$$

and the finite circle $C_a(\lambda)$. We have, if $v = \mathfrak{J}(\lambda) \neq 0$,

$$\int_a^0 |y_1(x,\lambda) + h y_2(x,\lambda)|^2 dx < -v^{-1}\mathfrak{J}(h) \text{ or } = -v^{-1}\mathfrak{J}(h) \tag{2.17}$$

according as $h \in$ the interior of $C_a(\lambda)$ or $\in$ the perimeter of $C_a(\lambda)$. There exists, as above, a sequence $\{a_n\}$ with $a_n \downarrow -\infty$ such that $h_{a_n}(\lambda)$ converges to a function $m_1(\lambda)$ regular for $\mathfrak{J}(\lambda) \neq 0$, uniformly in any bounded closed λ-domain not containing real numbers. In the limit point case we may replace $\lim_{n\to\infty} h_{a_n}(\lambda)$ by $\lim_{a\to-\infty} h_a(\lambda)$ $(= m_1(\lambda) = C_{-\infty}(\lambda) = \bigcap_{a<0} C_a(\lambda))$.

3. Weyl-Stone's expansion for the finite closed interval [a, b]. Let a real-valued function $f(x)$ in $(-\infty,\infty)$ be such that $f''(x)$ is continuous and $f(x) \equiv 0$ for $-\infty < x \leqq a'$, $b' \leqq x < \infty$, where $a' > a$, $b' < b$. By Hilbert-Schmidt's expansion theorem we have absolutely and uniformly convergent expansion:

$$f(x) = \sum_n f_{n,a,b} y_{n,a,b}(x), \quad a \leqq x \leqq b, \tag{3.1}$$

$$f_{n,a,b} = (f, y_{n,a,b}) = \int_{-\infty}^\infty f(x)\overline{y_{n,a,b}(x)}dx = \int_a^b f(x)\overline{y_{n,a,b}(x)}dx ,$$

where $\{y_{n,a,b}(x)\}$ is a complete system of normed orthogonal eigen-functions of

$$L_x y_{n,a,b} = \lambda_{n,a,b} y_{n,a,b}, \tag{3.2}$$

$$y_{n,a,b}(a)\cos\alpha + y'_{n,a,b}(a)\sin\alpha = 0,$$

$$y_{n,a,b}(b)\cos\beta + y'_{n,a,b}(b)\sin\beta = 0.$$

Also, by Hilbert-Schmidt's theorem, the unique solution $y(x,\lambda)$ of

$$(3.3)\qquad L_x y - \lambda y = f(x)\,, \quad \mathfrak{Z}(\lambda) \neq 0\,,$$
$$y(a,\lambda)\cos\alpha + y'(a,\lambda)\sin\alpha = 0\,,\quad y(b,\lambda)\cos\beta + y'(b,\lambda)\sin\beta = 0$$

may be expanded in absolutely and uniformly convergent series

$$(3.4)\qquad y(x,\lambda) = \sum_n -(\lambda - \lambda_{n,a,b})^{-1} f_{n,a,b}\, y_{n,a,b}(x)\,,\quad a \leqq x \leqq b\,.$$

It is well-known (and it may be verified easily), that we have another expression for $y(x,\lambda)$:

$$(3.5)\qquad y(x,\lambda) = \int_a^b G_{a,b}(x,s,\lambda) f(s)\,ds\,,\quad a \leqq x \leqq b\,,$$

where the Green's function $G_{a,b}(x,s,\lambda)$ is given by

$$(3.6)\qquad \begin{aligned} G_{a,b}(x,s,\lambda) &= -W_x(y_a,y_b)^{-1} y_b(x,\lambda) y_a(s,\lambda)\,,\quad x \geqq s\,,\\ &\quad\; -W_x(y_a,y_b)^{-1} y_a(x,\lambda) y_b(s,\lambda)\,,\quad x < s\,,\\ y_a(x,\lambda) &= y_1(x,\lambda) + h_a(\lambda) y_2(x,\lambda)\,,\\ y_b(x,\lambda) &= y_1(x,\lambda) + h_b(\lambda) y_2(x,\lambda)\,. \end{aligned}$$

We have, by (1.2),

$$(3.7)\qquad W_x(y_a,y_b) = W_0(y_a,y_b) = -h_a(\lambda) + h_b(\lambda)\,.$$

Hence we have, by (3.4) and (3.5),

$$(3.8)\qquad \begin{aligned} &\sum_n (\lambda - \lambda_{n,a,b})^{-1} f_{n,a,b}\, y_{n,a,b}(x)\\ &= (h_b(\lambda) - h_a(\lambda))^{-1} y_b(x,\lambda) \int_a^x y_a(s,\lambda) f(s)\,ds\\ &+ (h_a(\lambda) - h_a(\lambda))^{-1} y_a(x,\lambda) \int_x^b y_b(s,\lambda) f(s)\,ds\,. \end{aligned}$$

Thus we have, from (3.1),

$$(3.9)\qquad \begin{aligned} f(x) &= \sum_n f_{n,a,b}\, y_{n,a,b}(x) = \text{residue sum of } \sum_n (\lambda - \lambda_{n,a,b})^{-1} f_{n,a,b}\, y_{n,a,b}(x)\\ &= \sum_n (-\nu_n)^{-1} (y_1(x,\lambda_n) + \mu_n y_2(x,\lambda_n))(y_1(s,\lambda_n) + \mu_n y_2(s,\lambda_n), f(s))\\ &+ \sum_m (\mu'_{2m} - \mu'_{1m})^{-1} y_1(x,\lambda'_m)(y_1(s,\lambda'_m), f(s))\\ &+ \sum_k \mu''_{1k}\mu''_{2k}(\mu''_{2k} - \mu''_{1k})^{-1} y_2(x,\lambda''_k)(y_2(s,\lambda''_k), f(s))\,, \end{aligned}$$

where

$$(3.10)\qquad \begin{aligned} &h_a(\lambda_n) = h_b(\lambda_n) = \mu_n \neq 0\,,\quad h_a(\lambda) - h_b(\lambda) \sim (\lambda - \lambda_n)\nu_n\,;\\ &h_a(\lambda) \sim \mu'_{1m}(\lambda - \lambda'_m)\,,\quad h_b(\lambda) \sim \mu'_{2m}(\lambda - \lambda'_m)\,;\\ &h_a(\lambda) \sim \mu''_{1k}(\lambda - \lambda''_k)^{-1},\quad h_b(\lambda) \sim \mu''_{2k}(\lambda - \lambda''_k)^{-1}. \end{aligned}$$

That the zeros and poles of $h_a(\lambda)$, $h_b(\lambda)$ are all simple may be proved as follows. Let, for example, $h_b(\lambda)$ have multiple zero u_0. Then we obtain a contradiction from (2.12), by putting $h = h_b(\lambda)$, $\lambda = u_0 + iv$ and letting $v \downarrow 0$.

Hence (3.9) may be written as

$$f(x) = \int_{-\infty}^{\infty} du \Big(\sum_{j,k=1}^{2} \int_0^u y_j(x, u) dp_{jk}^{(a,b)}(u) (f(s), y_k(s, u)) \Big) \tag{3.11}$$

where

$$p_{11}^{(a,b)}(u_2) - p_{11}^{(a,b)}(u_1) = (2\pi i)^{-1} \int_{C(u_1,u_2)} (h_b(\lambda) - h_a(\lambda))^{-1} d\lambda, \tag{3.12}$$

$$p_{12}^{(a,b)}(u_2) - p_{12}^{(a,b)}(u_1) = p_{21}^{(a,b)}(u_2) - p_{21}^{(a,b)}(u_1) = (2\pi i)^{-1} \frac{1}{2} \int_{C(u_1,u_2)} (h_a(\lambda) + h_b(\lambda))(h_b(\lambda) - h_a(\lambda))^{-1} d\lambda,$$

$$p_{22}^{(a,b)}(u_2) - p_{22}^{(a,b)}(u_1) = (2\pi i)^{-1} \int_{C(u_1,u_2)} h_a(\lambda) h_b(\lambda) (h_b(\lambda) - h_a(\lambda))^{-1} d\lambda.$$

Here the path of integration $C(u_1, u_2)$ is a polygonal line connecting $u_1 - iv$, $u_2 - iv$, $u_2 + iv$, $u_1 + iv$, $u_1 - iv$ in this order, v being any positive number. Since

$$h_a(\bar{\lambda}) = \overline{h_a(\lambda)}, \quad h_b(\bar{\lambda}) = \overline{h_b(\lambda)} \tag{3.13}$$

by (2.10), we see that

$$p_{jk}^{(a,b)}(u_2) - p_{jk}^{(a,b)}(u_1) = \lim_{v \downarrow 0} \pi^{-1} \int_{u_1}^{u_2} f_{jk}^{(a,b)}(u + iv) du, \tag{3.14}$$

$$f_{11}^{(a,b)}(\lambda) = \mathfrak{I}(h_a(\lambda) - h_b(\lambda))^{-1},$$

$$f_{12}^{(a,b)}(\lambda) = f_{21}^{(a,b)}(\lambda) = \frac{1}{2} \mathfrak{I}(h_a(\lambda) + h_b(\lambda))(h_a(\lambda) - h_b(\lambda))^{-1},$$

$$f_{22}^{(a,b)}(\lambda) = \mathfrak{I} h_a(\lambda) h_b(\lambda) (h_a(\lambda) - h_b(\lambda))^{-1}.$$

4. Titchmarsh-Kodaira's formula.

LEMMA 1. Let a harmonic function $h(z)$ in $|z| < 1$ be such that

$$(2\pi)^{-1} \int_{-\pi}^{\psi} |h(re^{i\theta})| d\theta = q_r(\psi) \leqq C < \infty \text{ for } -\pi \leqq \psi \leqq \pi, 0 \leqq r < 1. \tag{4.1}$$

Then we have Poisson's representation:

$$h(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathfrak{R} \frac{e^{i\psi} + re^{i\theta}}{e^{i\psi} - re^{i\theta}} dq(\psi), \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |dq(\psi)| \leqq C. \tag{4.2}$$

Proof. See, for exmple, Stone's book, loc, cit., p. 570.

LEMMA 2. Let $v = \mathfrak{I}(\lambda) > 0$. Then we have

$$f_{11}^{(a,b)}(\lambda) = v \int_a^b |G_{a,b}(0, s, \lambda)|^2 ds, \tag{4.3}$$

$$f_{12}^{(a,b)}(\lambda) = f_{21}^{(a,b)}(\lambda) = v \int_a^b G_{a,b}(0, s, \lambda) \overline{G'_{a,b}(0, s, \lambda)} ds,$$

$$f_{22}^{(a,b)}(\lambda) = v \int_a^b |G'_{a,b}(0, s, \lambda)|^2 ds.$$

In particular we have, for $\mathfrak{J}(\lambda) > 0$,

(4.4) $$f_{11}^{(a,b)}(\lambda) \geqq 0, \quad f_{22}^{(a,b)}(\lambda) \geqq 0, \quad |f_{12}^{(a,b)}(\lambda)| \leqq (f_{11}^{(a,b)}(\lambda) f_{22}^{(a,b)}(\lambda))^{1/2}.$$

Proof is easy from (1.2), (2.12), (2.17), (3.6) and (3.7).

Therefore, by Lemma 1 and the transformation

(4.5) $$\lambda = u + iv = i(1-z)(1+z)^{-1}, \quad z = re^{i\theta}, \quad (0 \leqq r < 1)$$

we obtain

(4.6) $$f_{jk}^{(a,b)}(\lambda) = f_{jk}^{(a,b)}\left(i\frac{1-z}{1+z}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re \frac{e^{i\psi} + re^{i\theta}}{e^{i\psi} - re^{i\theta}} dq_{jk}^{(a,b)}(\psi),$$

where

(4.7) $q_{jj}^{(a,b)}(\psi)$ is monotone increasing and $q_{jj}^{(a,b)}(-\pi) = 0$, $(2\pi)^{-1}$

$$q_{jj}^{(a,b)}(\pi) = f_{jj}^{(a,b)}(i),$$

$q_{12}^{(a,b)}(\psi) = q_{21}^{(a,b)}(\psi)$ is of bounded variation and $q_{12}^{(a,b)}(-\pi) = 0$,

$$(2\pi)^{-1}\int_{-\pi}^{\pi} |dq_{12}^{(a,b)}(\pi)| \leqq (f_{11}^{(a,b)}(i) f_{22}^{(a,b)}(i))^{1/2}.$$

Putting

(4.5)′ $$s = i(1-e^{i\psi})(1+e^{i\psi})^{-1} = \tan(\psi/2), \; \lambda = u + iv = i(1-z)(1+z)^{-1}$$

we obtain

(4.8) $$\Re\frac{e^{i\psi} + re^{i\theta}}{e^{i\psi} - re^{i\theta}} = \Re\frac{(i-s)(i+s)^{-1} + (i-\lambda)(i+\lambda)^{-1}}{(i-s)(i+s)^{-1} - (i-\lambda)(i+\lambda)^{-1}} = \Re\frac{i(\lambda s + 1)}{\lambda - s}$$

$$= \Re\frac{i|\lambda|^2 s - i\lambda s + i(\bar{\lambda} - s)}{|\lambda - s|^2} = \frac{v(s^2+1)}{(u-s)^2 + v^2}$$

and thus

(4.9) $$f_{jk}^{(a,b)}(u+iv) = \int_{-\infty}^{\infty} v((u-s)^2 + v^2)^{-1}(s^2+1)\, dq_{jk}^{(a,b)}(2\tan^{-1}s).$$

Now $f_{jk}^{(a,b)}(\lambda)$ converges, when $a = a_n \downarrow -\infty$, $b = b_n \uparrow \infty$, to $f_{jk}(\lambda)$ uniformly in any bounded closed λ-doamin not containing the real numbers. Here

(4.10) $$f_{11}(\lambda) = \mathfrak{J}(m_1(\lambda) - m_2(\lambda))^{-1},$$

$$f_{12}(\lambda) = f_{21}(\lambda) = \frac{1}{2}\mathfrak{J}(m_1(\lambda) + m_2(\lambda))(m_1(\lambda) - m_2(\lambda))^{-1},$$

$$f_{22}(\lambda) = \mathfrak{J} m_1(\lambda) m_2(\lambda)(m_1(\lambda) - m_2(\lambda))^{-1}.$$

For the proof we must show that $m_1(\lambda) \neq m_2(\lambda)$ if $\mathfrak{J}(\lambda) \neq 0$. This may be proved as follows. Firstly $m_1(\lambda) \not\equiv m_2(\lambda)$. If, otherwise, we would obtain a contradiction

$$\int_{-\infty}^{\infty} |y_1(x,\lambda) + m_1(\lambda) y_2(x,\lambda)|^2 dx \leqq v^{-1}\mathfrak{J}(m_1(\lambda) - v^{-1}\mathfrak{J}(m_1(\lambda)) = 0$$

from (2.12) and (2.17). Secondly let $m_1(\lambda_0) - m_2(\lambda_0) = 0$ for $\mathfrak{J}(\lambda_0) \neq 0$. Then, by Hurwitz's theorem on the sequence of uniformly convergent regular functions, $h_{a_n}(\lambda) - h_{b_n}(\lambda)$ has zero in any vicinity of λ_0, if n is taken sufficiently large. Such (non-real) zero is an eigenvalue of the boundary value problem corresponding to the finite closed interval (a_n, b_n). This is a contradiction.

Hence, by applying Helly's theorem, we see that, when $a = a_{n'} \downarrow -\infty$, $b = b_{n'} \uparrow \infty$,

(4.11) finite limit $q_{jk}^{(a,b)}(2\tan^{-1}s)$ existe for all s, and

$$f_{jk}(u+iv) = \int_{-\infty}^{\infty} v((u-s)^2+v^2)^{-1}(s^2+1)dq_{jk}(2\tan^{-1}s). \tag{4.12}$$

By, (4.9),

$$p_{jk}^{(a,b)}(u_2) - p_{jk}^{(a,b)}(u_1) = \lim_{v\downarrow 0} \pi^{-1}\int_{-\infty}^{\infty} dq_{jk}^{(a,b)}(2\tan^{-1}s)\int_{u_1}^{u_2} v((u-s)^2+v^2)^{-1} (s^2+1)du$$
$$= \int_{u_1}^{u_2}(s^2+1)dq_{jk}^{(a,b)}(2\tan^{-1}s),$$

if $q_{jk}^{(a,b)}(2\tan^{-1}s)$ is continuous at $s = u_1, u_2$.

Hence, by (4.11) and (4.12), we see that, when $a = a_{n'} \downarrow -\infty$ and $b = b_{n'} \uparrow \infty$

$$\lim(p_{jk}^{(a,b)}(u_2) - p_{jk}^{(a,b)}(u_1)) = \int_{u_1}^{u_2}(s^2+1)dq_{jk}(2\tan^{-1}s) \tag{4.13}$$
$$= \lim_{v\downarrow 0}\pi^{-1}\int_{-\infty}^{\infty} dq_{jk}(2\tan^{-1}s)\int_{u_1}^{u_2} v((u-s)^2+v^2)^{-1}(s^2+1)du$$
$$= \lim_{v\downarrow 0}\pi^{-1}\int_{u_1}^{u_2} f_{jk}(u+iv)du = p_{jk}(u_2) - p_{jk}(u_1)$$

if $q_{jk}(2\tan^{-1}s)$ is continuous at $s = u_1, u_2$. Here we again make use of Helly's theorem, and assume that $s = u_1, u_2$ are not discontinuous points of the functions $q_{jk}^{(a,b)}(2\tan^{-1}s)$, $(j,k=1,2;\ a = a_{n'},\ b = b_{n'},\ n = 1,2,\ldots)$.

From (3.11) we obtain

$$(g,f) = \int_{-\infty}^{\infty} du\Big\{\sum_{j,k=1}^{2}\int_0^u (g(x), y_j(x,u))dp_{jk}^{(a,b)}(u)(f(s), y_k(s,u))\Big\} \tag{4.14}$$

if the real-valued function $g(x)$ is, like $f(x)$, such that $g''(x)$ is continuous and $g(x) \equiv 0$ for $x \overline{\in} [a', b']$. By the positive definiteness of the density matrix

$$(p_{jk}^{(a,b)}(u_2) - p_{jk}^{(a,b)}(u_1)),\quad j,k = 1,2,\quad (u_2 > u_1),$$

we easily obtain

$$\sum_m \Big|\sum_{j,k}\int_{u_{m-1}}^{u_m} (g(x), y_j(x,u))dp_{jk}^{(a,b)}(u)(f(s), y_k(s,u))\Big| \leqq$$
$$\sum_m\Big[\sum_{j,k}\int_{u_{m-1}}^{u_m} (g(x), y_j(x,u))dp_{jk}^{(a,b)}(u)(g(s), y_k(s,u)\Big]^{1/2} \times$$
$$\Big[\int_{u_{m-1}}^{u_m} (f(x), y_j(x,u)dp_{jk}^{(a,b)}(u)(f(s), y_k(s,u))\Big]^{1/2} \leqq (g,g)^{1/2}(f,f)^{1/2}.$$

Thus, letting $a = a_{n'} \downarrow -\infty$ and $b = b_{n'} \uparrow \infty$ in (4.14) we obtain

$$(g,f) = \int_{-\infty}^{\infty} d_u \Big\{ \sum_{j,k=1}^{2} \int_0^u (g(x), y_j(x,u))\, dp_{jk}(u)\, (f(s), y_k(s,u)) \Big\} . \tag{4.15}$$

This is W-S-T-K's *theorem* for real-valued functions $f(x)$, $g(x)$ if $f''(x)$, $g''(x)$ are continuous and $f(x) \equiv g(x) \equiv 0$ for sufficiently large $|x|$. The general case: $f(x), g(x) \in L_2(-\infty, \infty)$, may be obtained from this special case by the customary limiting process.

Remark. As was shown in 2, the condition that the boundary point $x = \infty$ $(x = -\infty)$ is in the limit point case is independent of λ and $m_2(\lambda)$ is independent of β (λ and $m_1(\lambda)$ is independent of α). Thus, we may obtain Titchmarsh-Kodaira's formula

$$m_2(\lambda) = \lim_{b\to\infty} -y_1(b,\lambda)/y_2(b,\lambda)\,, \quad (m_1(\lambda) = \lim_{a\to-\infty} -y_1(a,\lambda)/y_2(a,\lambda))\,. \tag{4.17}$$

In the limit point case the follwing formula may also be of use:

(4.18) $m_2(\lambda) = y'(0)/y(0)$ for any solution of

$$y'' + \{\lambda - q(x)\}y = 0\,, \quad 0 < \int_0^\infty |y(x)|^2 dx < \infty\,,$$

and similarly for $m_1(\lambda)$. The proof is easy, since, in the limit point case of $x = \infty$, the above solution y must be a constant multiple of

$$y_\infty(x,\lambda) = y_1(x,\lambda) + m_2(\lambda)y_2(x,\lambda)\,. \tag{4.19}$$

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