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SOME FUNCTION SPACES RELATIVE TO MORREY-CAMPANATO SPACES ON METRIC SPACES

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Abstract. In this paper, the author introduces the Morrey-Campanato spaces $L^s_p(X)$ and the spaces $C^s_p(X)$ on spaces of homogeneous type including metric spaces and some fractals, and establishes some embedding theorems between these spaces under some restrictions and the Besov spaces and the Triebel-Lizorkin spaces. In particular, the author proves that $L^s_p(X) = B^s_{\infty,\infty}(X)$ if $0 < s < \infty$ and $\mu(X) < \infty$. The author also introduces some new function spaces $A^s_p(X)$ and $B^s_p(X)$ and proves that these new spaces when $0 < s < 1$ and $1 < p < \infty$ are just the Triebel-Lizorkin space $F^s_{p,\infty}(X)$ if X is a metric space, and the spaces $A^1_p(X)$ and $B^1_p(X)$ when $1 < p \le \infty$ are just the Hajłasz-Sobolev spaces $W^1_p(X)$. Finally, as an application, the author gives a new characterization of the Hajłasz-Sobolev spaces by making use of the sharp maximal function.

§1. Introduction

On metric spaces including fractals, how to reasonably introduce some well-known functions on the Euclidean spaces is the main subject of a lot of recent papers and books; see [26], [21], [29], [30], [16], [17]. The main purpose of this paper is to introduce the Morrey-Campanato spaces $L^s_p(X)$ and the spaces $C^s_p(X)$ on spaces of homogeneous type including metric spaces and some fractals, whose versions on $\mathbb{R}^n$ and its domains are studied by DeVore and Sharpley in [6], Christ in [3] and Miyachi in [24], [25]; see also [28] for more references. Moreover, $L^0_1(\Omega)$ is just the usual space $bmo(\Omega)$ if Ω is a bounded C^∞ domain in $\mathbb{R}^n$; see [28, p. 49]. We will establish some embedding theorems between these spaces under some restrictions and the Besov spaces and the Triebel-Lizorkin spaces in [12], [13], [16], [17]. In particular, we will prove that $L^s_p(X) = B^s_{\infty,\infty}(X)$ if $0 < s < \infty$ and $\mu(X) < \infty$ (see Theorem 2.1 below), which is known if X is a bounded C^∞

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domain in $\mathbb{R}^n$; see [28, pp. 50, 247–248]. Motivated by [21], we also introduce some new function spaces $A_p^s(X)$ and $B_p^s(X)$, which can be regarded as the fractional versions of the function spaces studied in [21]. However, the metric spaces studied in [21] have the segment property and we do not need this property by assuming some other properties. It is easy to find a metric space satisfying our assumptions (see Definition 2.1 below), which has no segment property. For example, consider $X = [0,1] \cup [2,3]$ with the euclidean distance and the 1-dimensional Lebesgue measure. We will prove that these new spaces $A_p^s(X)$ and $B_p^s(X)$, when $0 < s < 1$, $1 < p < \infty$ and X is a metric space, are just the Triebel-Lizorkin space $F_{p,\infty}^s(X)$ (see Theorem 2.4 below), and that the spaces $A_p^1(X)$ and $B_p^1(X)$, when $1 < p \le \infty$ and X is a metric space, are just the Hajłasz-Sobolev spaces $W_p^1(X)$ in [9] (see Theorem 3.3 below). Finally, as an application of this result, we will establish a new characterization of the Sobolev space $W_p^1(X)$ by means of the sharp maximal function introduced by Triebel in [28, p. 246] (see Theorem 3.4 below), which is also known if X is a bounded C^∞ domain in $\mathbb{R}^n$; see [28, pp. 50, 247–248].

We remark that although some of our results are known if X is a bounded C^∞ domain in $\mathbb{R}^n$, some new ideas and techniques are needed to obtain their counterparts on spaces of homogeneous type. In particular, we need the Calderón reproducing formulae established by Han in [11] and we will also use some ideas from [21].

Section 2 is devoted to the study of the Morrey-Campanato spaces and the $C_p^s(X)$ spaces, and the new characterization of the Hajłasz-Sobolev spaces $W_p^1(X)$ is given in Section 3.

§2. Morrey-Campanato spaces and C_p^s spaces

Let us first recall some definitions and notation on spaces of homogeneous type. A quasi-metric ρ on a set X is a function $\rho\colon X \times X \to [0,\infty)$ satisfying that

(i) $\rho(x,y) = 0$ if and only if $x = y$;

(ii) $\rho(x,y) = \rho(y,x)$ for all x, $y \in X$;

(iii) there exists a constant $A \in [1,\infty)$ such that for all x, y and $z \in X$,

$$\rho(x,y) \le A[\rho(x,z) + \rho(z,y)].$$

Any quasi-metric defines a topology, for which the balls

$$B(x, r) = \{y \in X : \rho(y, x) < r\}$$

for all $x \in X$ and all $r > 0$ form a basis.

In what follows, we set $\operatorname{diam} X = \sup\{\rho(x, y) : x, y \in X\}$. We also make the following conventions. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Throughout the paper, we will denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. We denote $\mathbb{N} \cup \{0\}$ simply by $\mathbb{Z}_+$ and for any $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. If X_1 and X_2 are two quasi-Banach spaces, $B_1 \subset B_2$ means that there is a constant $C > 0$ such that for all $f \in B_1$,

$$\|f\|_{B_2} \le C\|f\|_{B_1}.$$

Definition 2.1. ([16]) Let $d > 0$ and $0 < \theta \le 1$. A space of homogeneous type, $(X, \rho, \mu)_{d,\theta}$, is a set X together with a quasi-metric ρ and a nonnegative Borel regular measure μ on X with $\operatorname{supp} \mu = X$ such that for some constant $C_0 > 0$ and for all $0 < r < \operatorname{diam} X$ and all x, x', $y \in X$,

$$\mu(B(x, r)) \sim r^d \tag{2.1}$$

and

$$|\rho(x, y) - \rho(x', y)| \le C_0 \rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta}. \tag{2.2}$$

Obviously, d can be regarded as the Hausdorff dimension of X (see [23]). Moreover, if ρ is a metric, then θ in (2.2) can be 1; and, if $X = \mathbb{R}^n$, ρ is the usual Euclidean metric and μ is the n-dimensional Lebesgue measure, then $d = n$ and $\theta = 1$.

Space of homogeneous type defined above is a variant of space of homogeneous type introduced by Coifman and Weiss in [4]. In [22], Macias and Segovia have proved that one can replace the quasi-metric ρ of space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\bar{\rho}$ which yields the same topology on X as ρ such that $(X, \bar{\rho}, \mu)$ is the space defined by Definition 2.1 with $d = 1$.

Moreover, the spaces of homogeneous type in Definition 2.1 include the Euclidean space, the C^∞-compact Riemannian manifolds, the boundaries of Lipschitz domains and, in particular, the Lipschitz manifolds introduced recently by Triebel in [31] and the isotropic and anisotropic d-sets in $\mathbb{R}^n$. It has been proved by Triebel in [29] that the isotropic and anisotropic d-sets in $\mathbb{R}^n$ include various kinds of self-affine fractals, for example, the Cantor set (see also [23]), the generalized Sierpinski carpet, the fern-like fractals, Picasso-Xmas-Tree fractals and Oval-Ferny fractals; see [30], [1], [2] and [8]. We particularly point out that the spaces of homogeneous type in Definition 2.1 also include the post critically finite self-similar fractals studied by Kigami in [20] and by Strichartz in [26], and the metric spaces with heat kernel studied by Grigor'yan, Hu and Lau in [8].

We now recall the definition of the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. To do so, let us first recall the definition of the spaces of test functions on X in [15]; see also [11].

Definition 2.2. Fix $\gamma > 0$ and $\theta \geq \beta > 0$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the following conditions:

(i) $|f(x)| \leq C_1 \dfrac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$;

(ii) $|f(x)-f(y)| \leq C_1 \left(\dfrac{\rho(x,y)}{r+\rho(x,x_0)}\right)^\beta \dfrac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$

$$\text{for } \rho(x,y) \leq \frac{1}{2A}[r+\rho(x,x_0)],$$

where $C_1 > 0$ is independent of x, y and r. If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C_1 : \text{(i) and (ii) hold}\}.$$

Here and in what follows, θ is the same as in (2.2).

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta,\gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with the equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta,\gamma)$.

Also, let the dual space $(\mathcal{G}(\beta,\gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a finite constant $C>0$ such that for all $f\in\mathcal{G}(\beta,\gamma)$,

$$|\mathcal{L}(f)|\le C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h,f\rangle$ the natural pairing of elements $h\in(\mathcal{G}(\beta,\gamma))'$ and $f\in\mathcal{G}(\beta,\gamma)$. It is easy to see that, for all $h\in(\mathcal{G}(\beta,\gamma))'$, $\langle h,f\rangle$ is well defined for all $f\in\mathcal{G}(x_0,r,\beta,\gamma)$ with $x_0\in X$ and $r>0$. Moreover, in what follows, we will denote by $\mathring{\mathcal{G}}(\beta,\gamma)$, for $0<\beta,\ \gamma<\theta$, the completion of $\mathcal{G}(\theta,\theta)$ in $\mathcal{G}(\beta,\gamma)$.

To state the definition of the inhomogeneous Besov spaces $B^s_{p,q}(X)$ and the inhomogeneous Triebel-Lizorkin spaces $F^s_{p,q}(X)$ studied in [12], we need the following approximations to the identity which were first introduced in [11].

Definition 2.3. A sequence $\{S_k\}_{k=0}^\infty$ of linear operators is said to be an approximation to the identity of order $\epsilon\in(0,\theta]$ if there exist C_2, $C_3>0$ such that for all $k\in\mathbb{Z}_+$ and all x, x', y and $y'\in X$, $S_k(x,y)$, the kernel of S_k is a function from $X\times X$ into $\mathbb{C}$ satisfying

(i) $S_k(x,y)=0$ if $\rho(x,y)\ge C_2 2^{-k}$ and $\|S_k\|_{L^\infty(X\times X)}\le C_3 2^{dk}$;

(ii) $|S_k(x,y)-S_k(x',y)|\le C_3 2^{k(d+\epsilon)}\rho(x,x')^\epsilon$;

(iii) $|S_k(x,y)-S_k(x,y')|\le C_3 2^{k(d+\epsilon)}\rho(y,y')^\epsilon$;

(iv) $$|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]| \le C_3 2^{k(d+2\epsilon)}\rho(x,x')^\epsilon\rho(y,y')^\epsilon;$$

(v) $\displaystyle\int_X S_k(x,y)\,d\mu(y)=1$;

(vi) $\displaystyle\int_X S_k(x,y)\,d\mu(x)=1$.

Here, that $S_k(x,y)$ is the kernel of S_k means that for suitable functions f,

$$S_kf(x)=\int_X S_k(x,y)f(y)\,d\mu(y).$$

We point out that by a similar Coifman's construction to that in [5], one can construct an approximation to the identity with compact supports as in Definition 2.3 for those spaces of homogeneous type in Definition 2.1.

Now, we can introduce the spaces $B^s_{p,q}(X)$ and $F^s_{p,q}(X)$ via the approximations to the identity defined above, which were first studied in [12].

Definition 2.4. Suppose $s \in (-\theta, \theta)$ and that $\{S_k\}_{k=0}^{\infty}$ is an approximation to the identity of order θ and let $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. The inhomogeneous Besov space $B^s_{p,q}(X)$ for $1 \le p,\ q \le \infty$ is the collection of $f \in \big(\mathring{\mathcal{G}}(\beta, \gamma)\big)'$ for $|s| < \beta < \theta$ and $0 < \gamma < \theta$ such that

$$\|f\|_{B^s_{p,q}(X)} = \left\{\sum_{k=0}^{\infty}\Big[2^{ks}\|D_k(f)\|_{L^p(X)}\Big]^q\right\}^{1/q} < \infty.$$

The inhomogeneous Triebel-Lizorkin space $F^s_{p,q}(X)$ for $1 < p < \infty$ and $1 < q \le \infty$ is the collection of $f \in \big(\mathring{\mathcal{G}}(\beta, \gamma)\big)'$ for $|s| < \beta < \theta$ and $0 < \gamma < \theta$ such that

$$\|f\|_{F^s_{p,q}(X)} = \left\|\left\{\sum_{k=0}^{\infty}\big[2^{ks}|D_k(f)|\big]^q\right\}^{1/q}\right\|_{L^p(X)} < \infty.$$

It was proved in [12] that the above definitions of the spaces $B^s_{p,q}(X)$ and $F^s_{p,q}(X)$ are independent of the choices of approximations to the identity and the pair (β, γ) with $\max(0, -s) < \beta < \theta$ and $0 < \gamma < \theta$. Moreover, in [16], it was also proved that the above definitions are also independent of the equivalent quasi-metrics satisfying (2.2). We say that a quasi-metric ρ is equivalent to another quasi-metric ρ' if there is a constant $C > 0$ such that for all $x,\ y \in X$,

$$C^{-1}\rho'(x, y) \le \rho(x, y) \le C\rho'(x, y).$$

Moreover, it was proved in [32] that the Besov spaces on d-sets in $\mathbb{R}^n$ defined by two different and equivalent methods, namely, traces and quarkonial decompositions in the sense of Triebel in [29], [30] are the same spaces as those introduced in [12], [17] by regarding the d-set as a space of homogeneous type when $0 < s < 1$, $1 < p < \infty$ and $1 \le q \le \infty$.

For $s \in \mathbb{R}$, $C_4 > 0$, $u \in (0, \infty]$ and $x \in X$, we introduce the sharp maximal function

$$f^s_{u,C_4}(x) = \sup_{0<t<C_4} t^{-s}\left(⨍_{B(x,t)}\left|f(y) - ⨍_{B(x,t)} f(z)\, d\mu(z)\right|^u d\mu(y)\right)^{1/u},$$

where $⨍_{B(x,t)} f(y)\, d\mu(y)$ means the average on $B(x, t)$ of f, that is,

$$⨍_{B(x,t)} f(y)\, d\mu(y) = \frac{1}{\mu(B(x, t))}\int_{B(x,t)} f(y)\, d\mu(y).$$

Using this sharp maximal function, we can now introduce the so-called Morrey-Campanato spaces and the spaces $C_p^s(X)$, whose versions on $\mathbb{R}^n$ and its domains have been introduced by DeVore and Sharpley in [6] and Christ in [3]; see also [24], [25] and [28, pp. 48–49, 246] for some applications of these spaces, the detailed references, further explanations and a short history of the versions on $\mathbb{R}^n$ and its domains of these spaces.

Definition 2.5. Let $C_4 > 0$.

(i) Let $1 \le p < \infty$ and $-d/p \le s < 1$. Then

$$L_p^s(X) = \Big\{ f \in L^p(X) : \|f\|_{L_p^s(X)} = \|f\|_{L^p(X)} + \sup_{x \in X} f_{p,C_4}^s(x) < \infty \Big\}.$$

(ii) Let $0 < s < 1$, $0 < p \le \infty$ and $\bar{p} = \max(1, p)$. Then

$$C_p^s(X) = \Big\{ f \in L^{\bar{p}}(X) : \|f\|_{C_p^s(X)} = \|f\|_{L^p(X)} + \|f_{p,C_4}^s\|_{L^p(X)} < \infty \Big\}.$$

Remark 2.1. It is easy to see that the definition of the spaces $L_p^s(X)$ and $C_p^s(X)$ are independent of the choice of $C_4 > 0$.

The following is one of the main theorems of this section.

Theorem 2.1. *Let* $1 \le p < \infty$.

(i) $L_p^{-d/p}(X) = L^p(X)$ *with equivalent norms, and for* $\min(-\theta, -d/p) < s < \theta$, $L_p^s(X) \subset B_{\infty,\infty}^s(X)$, *that is, there is a constant* $C > 0$ *such that for all* $f \in L_p^s(X)$,

$$\|f\|_{B_{\infty,\infty}^s(X)} \le C \|f\|_{L_p^s(X)}.$$

(ii) *If* $0 < s < \theta$ *and* $\mu(X) < \infty$, *then* $B_{\infty,\infty}^s(X) \subset L_p^s(X)$, *that is, there is a constant* $C > 0$ *such that for all* $f \in B_{\infty,\infty}^s(X)$,

$$\|f\|_{L_p^s(X)} \le C \|f\|_{B_{\infty,\infty}^s(X)}.$$

Remark 2.2. The space $B_{\infty,\infty}^s(X)$ for $0 < s < \theta$ is usually called the Hölder-Zygmund space; see [28], [27].

To establish Theorem 2.1, we need the following inhomogeneous Calderón reproducing formulae established in [11].

LEMMA 2.1. *Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity of order $\epsilon_1 \in (0,\theta]$ as defined in Definition 2.3. Let $D_k = S_k - S_{k-1}$ for $k\in\mathbb{N}$ and $D_0 = S_0$. Then there exist a family of linear operators $\widetilde{D}_k$ for $k\in\mathbb{Z}_+$ such that for $f\in\mathcal{G}(\beta_1,\gamma_1)$ with $0<\beta_1,\ \gamma_1<\epsilon_1$,*

$$f=\sum_{k=0}^{\infty} D_k\widetilde{D}_k(f), \tag{2.3}$$

where the series converge in the norm of $\mathcal{G}(\beta_1',\gamma_1')$ for $0<\beta_1'<\beta_1$ and $0<\gamma_1'<\gamma_1$. Moreover, the kernel, $\widetilde{D}_k(x,y)$, of the operator $\widetilde{D}_k$ for $k\in\mathbb{Z}_+$ satisfies the conditions

(i) $|\widetilde{D}_k(x,y)|\le C\dfrac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$,

(ii) $|\widetilde{D}_k(x,y)-\widetilde{D}_k(x,y')|\le C\left(\dfrac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\dfrac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$

$$\textit{for}\ \ \rho(y,y')\le\frac{1}{2A}(2^{-k}+\rho(x,y)),$$

and

(iii) $\displaystyle\int_X \widetilde{D}_k(x,y)\,d\mu(y)=\int_X \widetilde{D}_k(x,y)\,d\mu(x)=\begin{cases}1, & k=0;\\ 0, & k\in\mathbb{N},\end{cases}$

where $\epsilon\in(0,\epsilon_1)$.

The following lemma can be found in [12]; see also [15].

LEMMA 2.2. *Let $\{\widetilde{D}_k\}_{k\in\mathbb{Z}_+}$ be as in Lemma 2.1 with $\theta>|s|$.*

(i) *For $1\le p,\ q\le\infty$ and all $f\in B^s_{p,q}(X)$,*

$$\left\{\sum_{k=0}^{\infty}2^{ksq}\|\widetilde{D}_kf\|^q_{L^p(X)}\right\}^{1/q}\le C\|f\|_{B^s_{p,q}(X)},$$

where C is independent of f.

(ii) *For $1<p<\infty$, $1<q\le\infty$ and all $f\in F^s_{p,q}(X)$,*

$$\left\|\left\{\sum_{k=0}^{\infty}2^{ksq}|\widetilde{D}_kf|^q\right\}^{1/q}\right\|_{L^p(X)}\le C\|f\|_{F^s_{p,q}(X)},$$

where C is independent of f.

Proof of Theorem 2.1. We first show (i). Let $f \in L_p^{-d/p}(X)$. From Definition 2.5, we know that $f \in L^p(X)$ and

$$\|f\|_{L^p(X)} \le \|f\|_{L_p^{-d/p}(X)}. \tag{2.4}$$

We now suppose that $f \in L^p(X)$ and the Hölder inequality and $\mu(B(x,t)) \sim t^d$ tell us that for all $x \in X$,

$$\begin{aligned} f_{p,C_4}^{-d/p}(x) &= \sup_{0<t<C_4} t^{d/p} \left(\fint_{B(x,t)} \left| f(y) - \fint_{B(x,t)} f(z)\, d\mu(z) \right|^p d\mu(y) \right)^{1/p} \\ &\le C\|f\|_{L^p(X)}. \end{aligned}$$

Thus, $f \in L_p^{-d/p}(X)$ and

$$\|f\|_{L_p^{-d/p}(X)} = \|f\|_{L^p(X)} + \sup_{x\in X} f_{p,C_4}^{-d/p}(x) \le C\|f\|_{L^p(X)}. \tag{2.5}$$

The estimates (2.4) and (2.5) show that $L_p^{-d/p}(X) = L^p(X)$ with equivalent norms.

Let $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 2.4. Let $\min(-\theta, -d/p) < s < \theta$ and $f \in L_p^s(X)$. We denote $B_1 = B(x, 2C_1 2^{-k})$. The Hölder inequality yields that

$$\begin{aligned} |D_0 f(x)| &= \left| \int_X D_0(x,y) f(y)\, d\mu(y) \right| \\ &\le \|f\|_{L^p(X)} \left\{ \int_X |D_0(x,y)|^{p'} d\mu(y) \right\}^{1/p'} \\ &\le C\|f\|_{L^p(X)}, \end{aligned} \tag{2.6}$$

and, by the fact that

$$\int_X D_k(x,y) f(y)\, d\mu(y) = 0 \tag{2.7}$$

and $\operatorname{supp} D_k(x,\,\cdot\,) \subset B_1$ for $k \in \mathbb{N}$, we have

$$\begin{aligned} |D_k f(x)| &= \left| \int_X D_k(x,y) f(y)\, d\mu(y) \right| \\ &= \left| \int_X D_k(x,y) \left[f(y) - \fint_{B_1} f(z)\, d\mu(z) \right] d\mu(y) \right| \end{aligned} \tag{2.8}$$

$$\leq C \fint_{B_1} \left| f(y) - \fint_{B_1} f(z)\,d\mu(z) \right| d\mu(y)$$
$$\leq C \left\{ \fint_{B_1} \left| f(y) - \fint_{B_1} f(z)\,d\mu(z) \right|^p d\mu(y) \right\}^{1/p}.$$

Thus, the estimates (2.6) and (2.8) tell us that

$$\begin{aligned}\|f\|_{B^s_{\infty,\infty}(X)} &= \sup_{k\in\mathbb{Z}_+} \sup_{x\in X} 2^{ks}|D_k f(x)| \\ &\leq C\|f\|_{L^p(X)} + C \sup_{x\in X} f^s_{p,2C_1}(x) \\ &\leq C\|f\|_{L^s_p(X)}.\end{aligned}$$

This proves (i).

We now turn to the proof of (ii). Let $f \in B^s_{\infty,\infty}(X)$. By the Hölder inequality, we have that

$$\begin{aligned}\|f\|_{L^p(X)} &= \left\{ \int_X \left| \sum_{k=0}^{\infty} D_k f(x) \right|^p d\mu(x) \right\}^{1/p} \\ &\leq \left\{ \int_X \left[\sum_{k=0}^{\infty} 2^{-ks} \sup_{k\in\mathbb{Z}_+,\, x\in X} 2^{ks}|D_k f(x)| \right]^p d\mu(x) \right\}^{1/p} \\ &\leq C\mu(X)\|f\|_{B^s_{\infty,\infty}(X)}.\end{aligned} \tag{2.9}$$

On the other hand, let $C_4 = 1$ and $2^{-k_0-1} \leq t < 2^{-k_0}$ for some $k_0 \in \mathbb{Z}_+$. Let y, $z \in B(x, 2^{-k_0})$. By Lemma 2.1, we decompose $f(y) - f(z)$ into

$$\begin{aligned}f(y) - f(z) &= \sum_{k=0}^{\infty} \left[D_k \widetilde{D}_k f(y) - D_k \widetilde{D}_k f(z) \right] \\ &= \sum_{k=0}^{k_0} \left[D_k \widetilde{D}_k f(y) - D_k \widetilde{D}_k f(z) \right] \\ &\quad + \sum_{k=k_0+1}^{\infty} \left[D_k \widetilde{D}_k f(y) - D_k \widetilde{D}_k f(z) \right] \\ &= I_1 + I_2.\end{aligned} \tag{2.10}$$

For I_1, noting that $\rho(y,z) \leq C2^{-k_0}$ if $\rho(y,x) \leq C2^{-k_0}$ and $\rho(z,x) \leq C2^{-k_0}$, by Lemma 2.2, we have

(2.11)

$$
\begin{aligned}
|I_1| &\le \sum_{k=0}^{k_0} \int_X |D_k(y,u) - D_k(z,u)| \big|\widetilde{D}_k f(u)\big| \, d\mu(u) \\
&\le C \sum_{k=0}^{k_0} 2^{k(d+\theta)} \int_{\{u\in X: \rho(y,u)\le C2^{-k} \text{ or } \rho(z,u)\le C2^{-k}\}} \rho(y,z)^\theta \big|\widetilde{D}_k f(u)\big| \, d\mu(u) \\
&\le C \sum_{k=0}^{k_0} 2^{-(k_0-k)\theta} \sup_{u\in X} \big|\widetilde{D}_k f(u)\big| \\
&= C 2^{-k_0 s} \left[\sum_{k=0}^{k_0} 2^{-(k_0-k)(\theta-s)} \right] \sup_{k\in\mathbb{Z}_+,\, u\in X} 2^{ks} \big|\widetilde{D}_k f(u)\big| \\
&\le C 2^{-k_0 s} \|f\|_{B^s_{\infty,\infty}(X)},
\end{aligned}
$$

and for I_2, we have

$$
\begin{aligned}
|I_2| &\le \sum_{k=k_0+1}^{\infty} \Big[\big|D_k \widetilde{D}_k f(y)\big| + \big|D_k \widetilde{D}_k f(z)\big| \Big] \\
&\le \sum_{k=k_0+1}^{\infty} 2^{-ks} \left[\int_X \big|D_k(y,u)\big| d\mu(u) + \int_X \big|D_k(z,u)\big| d\mu(u) \right] \\
&\quad \times \sup_{k\in\mathbb{Z}_+,\, u\in X} 2^{ks} \big|\widetilde{D}_k f(u)\big| \\
&\le C 2^{-k_0 s} \|f\|_{B^s_{\infty,\infty}(X)}.
\end{aligned} \tag{2.12}
$$

The definition of $f^s_{p,1}$ and the estimates (2.11) and (2.12) imply that

(2.13)

$$
\begin{aligned}
f^s_{p,1}(x) &= \sup_{0<t<1} t^{-s} \left(⨏_{B(x,t)} \left| f(y) - ⨏_{B(x,t)} f(z)\, d\mu(z) \right|^p d\mu(y) \right)^{1/p} \\
&\le \sup_{0<t<1} t^{-s} \left(⨏_{B(x,t)} \left[⨏_{B(x,t)} |f(y) - f(z)|\, d\mu(z) \right]^p d\mu(y) \right)^{1/p} \\
&\le C \sup_{k_0\in\mathbb{Z}_+} 2^{k_0 s} \left(⨏_{B(x,2^{-k_0})} \left[⨏_{B(x,2^{-k_0})} |f(y) - f(z)|\, d\mu(z) \right]^p d\mu(y) \right)^{1/p} \\
&\le C \|f\|_{B^s_{\infty,\infty}(X)}.
\end{aligned}
$$

Finally, (2.9) and (2.13) yield that

$$\|f\|_{L^s_p(X)} \le C\|f\|_{B^s_{\infty,\infty}(X)}.$$

This finishes the proof of Theorem 2.1. $\square$

On the relation between $C^s_p(X)$ and the Triebel-Lizorkin space $F^s_{p,q}(X)$, we have the following result.

THEOREM 2.2. *Let* $0 < s < \theta$.

(i) *If* $1 < p < \infty$, *then* $C^s_p(X) \subset F^s_{p,\infty}(X)$, *namely, there is a constant* $C > 0$ *such that for all* $f \in C^s_p(X)$,

$$\|f\|_{F^s_{p,\infty}(X)} \le C\|f\|_{C^s_p(X)}.$$

(ii) *If* $1 < p_2 < p_1 < \infty$ *and* $\mu(X) < \infty$, *then* $F^s_{p_1,\infty}(X) \subset C^s_{p_2}(X)$, *namely, there is a constant* $C > 0$ *such that for all* $f \in F^s_{p_1,\infty}(X)$,

$$\|f\|_{C^s_{p_2}(X)} \le C\mu(X)^{1/p_2-1/p_1}\|f\|_{F^s_{p_1,\infty}(X)}.$$

Proof. We first show (i). Let $f \in C^s_p(X)$ and $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 2.4. We first have

$$|D_0f(x)| = \left|\int_X S_0(x,y)f(y)\,d\mu(y)\right| \le CMf(x), \tag{2.14}$$

where M is the Hardy-Littlewood maximal function. Thus, for $1 < p < \infty$, by the $L^p(X)$-boundedness of M (see [4], [18]), we obtain

$$\|D_0f\|_{L^p(X)} \le C\|Mf\|_{L^p(X)} \le C\|f\|_{L^p(X)}. \tag{2.15}$$

Let $B_1 = B(x, 2C_1 2^{-k})$. The estimate (2.8) tells us that for $k \in \mathbb{N}$ and all $x \in X$,

$$2^{ks}|D_kf(x)| \le Cf^s_{p,C_1}(x), \tag{2.16}$$

where C is independent of x. The estimates (2.15) and (2.16) yield that

$$C^s_p(X) \subset F^s_{p,\infty}(X)$$

and

$$\|f\|_{F^s_{p,\infty}(X)} = \left\| \sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f| \right\|_{L^p(X)} \tag{2.17}$$
$$\le C\|f\|_{L^p(X)} + C\left\|f^s_{p,C_1}\right\|_{L^p(X)}$$
$$\le C\|f\|_{C^s_p(X)}.$$

This proves (i).

We now turn to prove (ii). By the properties of $F^s_{p,q}(X)$ in [12], [14] (see also [16]), we have that

$$F^s_{p_1,\infty}(X) \subset F^0_{p_1,2}(X) = L^{p_1}(X) \subset L^{p_2}(X),$$

since $s > 0$, $p_2 < p_1$ and $\mu(X) < \infty$. Thus,

$$\|f\|_{L^{p_2}(X)} \le \mu(X)^{1/p_2-1/p_1}\|f\|_{L^{p_1}(X)} \tag{2.18}$$
$$\le C\mu(X)^{1/p_2-1/p_1}\|f\|_{F^s_{p_1,\infty}(X)}.$$

Moreover, without loss of generality, we may assume that $C_4 = 1$ in the definition of $C^s_{p_2}(X)$ by Remark 2.1. Let $2^{-k_0-1} \le t < 2^{-k_0}$ for some $k_0 \in \mathbb{Z}_+$ and we decompose $f(y) - f(z)$ as in (2.10) of the proof of Theorem 2.1 by means of Lemma 2.1. Then, for y, $z \in B(x, 2^{-k_0})$,

(2.19)

$$|I_1| \le \sum_{k=0}^{k_0} \int_X |D_k(y,u) - D_k(z,u)|\left|\widetilde{D}_k f(u)\right| d\mu(u)$$
$$\le C\sum_{k=0}^{k_0} 2^{k(d+\theta)} \int_{\{u\in X:\rho(y,u)\le C2^{-k} \text{ or } \rho(z,u)\le C2^{-k}\}} \rho(y,z)^\theta \left|\widetilde{D}_k f(u)\right| d\mu(u)$$
$$\le C\sum_{k=0}^{k_0} 2^{(k-k_0)\theta}2^{kd} \int_{\{u\in X:\rho(y,u)\le C2^{-k} \text{ or } \rho(z,u)\le C2^{-k}\}} \left|\widetilde{D}_k f(u)\right| d\mu(u)$$
$$\le C2^{-k_0 s}\sum_{k=0}^{k_0} 2^{(k-k_0)(\theta-s)}$$
$$\times \left\{ 2^{kd} \int_{\{u\in X:\rho(y,u)\le C2^{-k}\}} \left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f(u)\right| \right) d\mu(u) \right.$$
$$\left. + 2^{kd} \int_{\{u\in X:\rho(z,u)\le C2^{-k}\}} \left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f(u)\right| \right) d\mu(u) \right\}$$

$$
\begin{aligned}
&\le C2^{-k_0 s}\left[\sum_{k=0}^{k_0} 2^{(k-k_0)(\theta-s)}\right]\\
&\qquad \times \left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(y)+M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(z)\right\}\\
&\le C2^{-k_0 s}\left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(y)+M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(z)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &\le \sum_{k=k_0+1}^{\infty}\int_X \big[|D_k(y,u)|+|D_k(z,u)|\big]\big|\widetilde{D}_k f(u)\big|\,d\mu(u)\\
&\le C\left[\sum_{k=k_0+1}^{\infty} 2^{-ks}\right]\\
&\qquad \times \left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(y)+M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(z)\right\}\\
&\le C2^{-k_0 s}\left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(y)+M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)(z)\right\}.
\end{aligned}
\tag{2.20}
$$

Then similarly to (2.13), the estimates (2.19) and (2.20) and the Minkowski inequality yield that

$$
\begin{aligned}
&f^s_{p_2,1}(x)\\
&\quad\le C\sup_{k_0\in\mathbb{Z}_+} 2^{k_0 s}\left(\fint_{B(x,2^{-k_0})}\left[\fint_{B(x,2^{-k_0})}|f(y)-f(z)|\,d\mu(z)\right]^{p_2} d\mu(y)\right)^{1/p_2}\\
&\quad\le CM^2\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f(u)\big|\right)(x)+C\left\{M\left[M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)\right]^{p_2}\right\}^{1/p_2},
\end{aligned}
\tag{2.21}
$$

where M is the Hardy-Littlewood maximal operator and M^2 means $M\circ M$, the composition of M. The estimate (2.21), the Hölder inequality, Lemma 2.2 and the $L^p(X)$-boundedness of M for $p\in(1,\infty]$ then imply that

$$
\|f^s_{p_2,1}\|_{L^{p_2}(X)} \le C\left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f(u)\big|\right\|_{L^{p_2}(X)}
\tag{2.22}
$$

$$
\begin{aligned}
&+ C\mu(X)^{1/p_2-1/p_1}\left\|M\left[M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)\right]^{p_2}\right\|_{L^{p_1/p_2}(X)}^{1/p_2}\\
&\le C\mu(X)^{1/p_2-1/p_1}\left\{\left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f(u)\big|\right\|_{L^{p_1}(X)}\right.\\
&\qquad \left. + \left\|M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)\right\|_{L^{p_1}(X)}\right\}\\
&\le C\mu(X)^{1/p_2-1/p_1}\left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f(u)\big|\right\|_{L^{p_1}(X)}\\
&\le C\mu(X)^{1/p_2-1/p_1}\|f\|_{F^s_{p_1,\infty}(X)}.
\end{aligned}
$$

The estimates (2.18) and (2.22) imply (ii) and we finish the proof of Theorem 2.2. ∎

Motivated by [21], we now introduce some new function spaces on spaces of homogeneous type. These spaces under some restrictions will be proved to be the special cases of the Triebel-Lizorkin spaces.

Definition 2.6. Let $1 \le p \le \infty$ and $s > 0$. The space $B^s_p(X)$ is the set of functions $f \in L^p(X)$ satisfying that there exists a function $g \in L^p(X)$ such that

$$
\left|f(x) - \fint_B f(z)\,d\mu(z)\right| \le r(B)^s g(x) \tag{2.23}
$$

for μ-a. e. $x \in B$ and any ball $B \subset X$, where $r(B)$ is the radius of the ball B. Moreover, if $f \in B^s_p(X)$, we define its norm by

$$
\|f\|_{B^s_p(X)} = \|f\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},
$$

where the infimum is taken over all functions g satisfying (2.23).

We remark that in some sense, the function g in Definition 2.6 behaves like the fractional derivative of f, which is also the main subject of [7].

The following theorem will indicate the relation between the space $B^s_p(X)$ and the Triebel-Lizorkin space.

Theorem 2.3. *Let $1 < p < \infty$ and $0 < s < \theta$. Then $B^s_p(X) = F^s_{p,\infty}(X)$ with equivalent norms.*

Proof. Let $f \in F^s_{p,\infty}(X)$. If $r(B) \geq 1$, then, for $y \in B$, we have

$$\begin{aligned}
(2.24) \qquad & \left| f(y) - \frac{1}{\mu(B)} \int_B f(z)\, d\mu(z) \right| \\
&= \left| \frac{1}{\mu(B)} \int_B [f(y) - f(z)]\, d\mu(z) \right| \\
&= \left| \frac{1}{\mu(B)} \int_B \sum_{k=0}^{\infty} \Big[D_k \widetilde{D}_k f(y) - D_k \widetilde{D}_k f(z) \Big] d\mu(z) \right| \\
&\leq C \sum_{k=0}^{\infty} \frac{1}{\mu(B)} \int_B \Big[M\big(\widetilde{D}_k f\big)(y) + M\big(\widetilde{D}_k f\big)(z) \Big] d\mu(z) \\
&\leq C \sum_{k=0}^{\infty} M^2\big(\widetilde{D}_k f\big)(y) \\
&\leq C M^2 \Big(\sup_{k \in \mathbb{Z}_+} 2^{ks} \big|\widetilde{D}_k f\big| \Big)(y) \sum_{k=0}^{\infty} 2^{-ks} \\
&\leq C M^2 \Big(\sup_{k \in \mathbb{Z}_+} 2^{ks} \big|\widetilde{D}_k f\big| \Big)(y) \\
&\leq C r(B)^s M^2 \Big(\sup_{k \in \mathbb{Z}_+} 2^{ks} \big|\widetilde{D}_k f\big| \Big)(y),
\end{aligned}$$

since $s > 0$, where we used the facts that

$$(2.25) \qquad |f(x)| \leq Mf(x)$$

and

$$(2.26) \qquad |D_k f(x)| \leq Mf(x).$$

If $2^{-(k_0+1)} \leq r(B) < 2^{-k_0}$ for some $k_0 \in \mathbb{Z}_+$, then for $y, z \in B$, we write $f(y) - f(z)$ as in (2.10) and the estimates (2.19) and (2.20) tell us that

$$|f(y) - f(z)| \leq C r(B)^s \left\{ M \Big(\sup_{k \in \mathbb{Z}_+} 2^{ks} \big|\widetilde{D}_k f\big| \Big)(y) + M \Big(\sup_{k \in \mathbb{Z}_+} 2^{ks} \big|\widetilde{D}_k f\big| \Big)(z) \right\}.$$

Therefore, this estimate and (2.25) yield

$$(2.27) \qquad \left| f(y) - \frac{1}{\mu(B)} \int_B f(z)\, d\mu(z) \right| \leq C r(B)^s M^2 \Big(\sup_{k \in \mathbb{Z}_+} 2^{ks} \big|\widetilde{D}_k f\big| \Big)(y)$$

for all $y \in B$.

Let

$$g(y) = CM^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(y).$$

By Lemma 2.2 and the $L^p(X)$-boundedness of M for $p \in (1,\infty)$, we obtain

$$\begin{aligned} \|g\|_{L^p(X)} &\le C\Big\|M^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)\Big\|_{L^p(X)} \\ &\le C\Big\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big\|_{L^p(X)} \\ &= C\|f\|_{F^s_{p,\infty}(X)}. \end{aligned} \tag{2.28}$$

The estimates (2.24), (2.27) and (2.28) show that $f \in B^s_p(X)$ and

$$\|f\|_{B^s_p(X)} \le C\|f\|_{F^s_{p,\infty}(X)}.$$

Now suppose $f \in B^s_p(X)$. By Definition 2.6, for any given $\epsilon > 0$, there exists a function $g \in L^p(X)$ such that (2.23) holds for any ball $B \subset X$ and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B^s_p(X)} + \epsilon. \tag{2.29}$$

Let B_1 be the same as in the proof of Theorem 2.1. For $k \in \mathbb{N}$, by (2.7) and Definition 2.6, we have

$$\begin{aligned} |D_k f(x)| &= \left|\int_X D_k(x,y) f(y)\, d\mu(y)\right| \\ &= \left|\int_X D_k(x,y)\left[f(y) - \fint_{B_1} f(z)\, d\mu(z)\right] d\mu(y)\right| \\ &\le C2^{-ks}\int_X |D_k(x,y)|\,|g(y)|\, d\mu(y) \\ &\le C2^{-ks} Mg(x). \end{aligned} \tag{2.30}$$

The estimates (2.14) and (2.30) tell us that

$$\sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f(x)| \le C[Mf(x) + Mg(x)]. \tag{2.31}$$

Finally the estimates (2.31) and (2.29) and the $L^p(X)$-boundedness of M yield that

$$\begin{aligned}\|f\|_{F^s_{p,\infty}(X)} &= \left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f|\right\|_{L^p(X)}\\ &\le C\big\{\|Mf\|_{L^p(X)} + \|Mg\|_{L^p(X)}\big\}\\ &\le C\big\{\|f\|_{L^p(X)} + \|g\|_{L^p(X)}\big\}\\ &\le C\big[\|f\|_{B^s_p(X)} + \epsilon\big].\end{aligned}$$

Letting $\epsilon \to 0$, we obtain that $B^s_p(X) \subset F^s_{p,\infty}(X)$ and

$$\|f\|_{F^s_{p,\infty}(X)} \le C\|f\|_{B^s_p(X)}.$$

This finishes the proof of Theorem 2.3. □

We now introduce some other space of functions and we will prove that these spaces under some restrictions are just the Triebel-Lizorkin spaces.

Definition 2.7. Let $1 < p \le \infty$ and $s > 0$. The space $A^s_p(X)$ is the set of functions $f \in L^p(X)$ satisfying that there exist some $q \in [1, p)$ and a non-negative function $g \in L^p(X)$ such that

$$\fint_B \left| f(x) - \fint_B f(y)\, d\mu(y)\right| d\mu(x) \le r(B)^s \left(\fint_B g^q(x)\, d\mu(x)\right)^{1/q} \tag{2.32}$$

for every ball $B \subset X$. Moreover, if $f \in A^s_p(X)$, we define its norm by

$$\|f\|_{A^s_p(X)} = \|f\|_{L^p(X)} + \inf_{\{g\}} \|g\|_{L^p(X)},$$

where the infimum is taken over all functions g satisfying (2.32).

The following theorem will indicate the relations among these spaces, the Triebel-Lizorkin spaces and the spaces $B^s_p(X)$ under some restrictions.

Theorem 2.4. *Let $1 < p < \infty$ and $0 < s < \theta$. Then $A^s_p(X) = F^s_{p,\infty}(X) = B^s_p(X)$ with equivalent norms.*

Proof. Let $f \in A^s_p(X)$. For any given $\epsilon > 0$, there is a non-negative function $g \in L^p(X)$ and some $q \in (0, p)$ satisfying (2.32) and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{A^s_p(X)} + \epsilon. \tag{2.33}$$

Let $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 2.4. For $k \in \mathbb{N}$, let $B_1 = B(x, 2C_1 2^{-k})$. By (2.7), we then have

$$
\begin{aligned}
|D_k f(x)| &= \left|\int_X D_k(x,y) f(y)\, d\mu(y)\right| \\
&= \left|\int_X D_k(x,y)\left[f(y) - \fint_{B_1} f(z)\, d\mu(z)\right] d\mu(y)\right| \\
&\le C \fint_{B_1}\left|f(y) - \fint_{B_1} f(z)\, d\mu(z)\right| d\mu(y) \\
&\le C2^{-ks}\left(\fint_{B_1} g^q(y)\, d\mu(y)\right)^{1/q} \\
&\le C2^{-ks}\{M(g^q)(x)\}^{1/q}.
\end{aligned} \tag{2.34}
$$

Thus, the estimates (2.14), (2.34) and (2.33), the fact $q < p$ and the $L^{p/q}(X)$-boundedness of M tell us that

$$
\begin{aligned}
\|f\|_{F^s_{p,\infty}(X)} &= \left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f|\right\|_{L^p(X)} \\
&\le C\|Mf\|_{L^p(X)} + C\left\|\{M(g^q)\}^{1/q}\right\|_{L^p(X)} \\
&\le C\|f\|_{L^p(X)} + C\|g\|_{L^p(X)} \\
&\le C\left\{\|f\|_{A^s_p(X)} + \epsilon\right\}.
\end{aligned}
$$

Letting $\epsilon \to 0$, we obtain that $A^s_p(X) \subset F^s_{p,\infty}(X)$ and

$$
\|f\|_{F^s_{p,\infty}(X)} \le C\|f\|_{A^s_p(X)}. \tag{2.35}
$$

On another hand, let $f \in B^s_p(X)$. Then, by Definition 2.6, there is a function $g \in L^p(X)$ such that (2.23) holds and

$$
\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B^s_p(X)} + \epsilon. \tag{2.36}
$$

Then the estimate (2.23) and the Hölder inequality tell us that for any ball $B \subset X$ and any $q \in [1, p)$,

$$
\begin{aligned}
\fint_B\left|f(x) - \fint_B f(y)\, d\mu(y)\right| d\mu(x) &\le r(B)^s \fint_B g(x)\, d\mu(x) \\
&\le r(B)^s\left\{\fint_B g^q(x)\, d\mu(x)\right\}^{1/q}.
\end{aligned}
$$

Thus, $f \in A_p^s(X)$ and, by (2.36),

$$\|f\|_{A_p^s(X)} \le \|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B_p^s(X)} + \epsilon.$$

Letting $\epsilon \to 0$, we finally obtain that $B_p^s(X) \subset A_p^s(X)$ and

$$\|f\|_{A_p^s(X)} \le \|f\|_{B_p^s(X)}. \tag{2.37}$$

Then (2.35) and (2.37) together with Theorem 2.3 imply Theorem 2.4 and we finish this proof. ∎

§3. Sobolev spaces

The main purpose of this section is to give several new characterizations of the Hajłasz-Sobolev spaces of order 1 on spaces of homogeneous type. We will first show that the space $A_p^1(X)$ is the same space as the space $B_p^1(X)$ by using some ideas from [21]. In fact, Theorem 3.1 below is similar to Theorem 1 in [21]. However, we do not suppose that the space of homogeneous type, X, has the segment property as in [21].

THEOREM 3.1. *Let $1 \le q < \infty$, f be a locally integrable function on X for which there is a non-negative function $g \in L^q(X)$ such that the Poincaré inequality*

$$\fint_B \left| f(x) - \fint_B f(z)\,d\mu(z) \right| d\mu(x) \le Cr(B) \left(\fint_B g^q(x)\,d\mu(x) \right)^{1/q} \tag{3.1}$$

holds for every ball $B \subset X$. Then for μ-a. e. $x \in B$,

$$\left| f(x) - \fint_B f(z)\,d\mu(z) \right| \le Cr(B) M(g^q)(x)^{1/q},$$

where C is independent of x and B.

Proof. Let $x \in B$ be a Lebesgue point of $\left| f(y) - \fint_B f(z)\,d\mu(z) \right|$ and $g(y)$. Let $B_0 = B$ and $B_j = B(x, 2^{-j} r(B))$ for $j \in \mathbb{N}$. Then

$$\begin{aligned}
\left| f(x) - \fint_B f(z)\,d\mu(z) \right| &= \lim_{j\to\infty} \fint_{B_j} \left| f(y) - \fint_B f(z)\,d\mu(z) \right| d\mu(y) \\
&\le \limsup_{j\to\infty} \fint_{B_j} \left| f(y) - \fint_{B_j} f(z)\,d\mu(z) \right| d\mu(y) \\
&\quad + \limsup_{j\to\infty} \left| \fint_{B_j} f(z)\,d\mu(z) - \fint_B f(z)\,d\mu(z) \right| \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, by (3.1), we have

$$I_1 \le C \limsup_{j\to\infty} r(B_j)\left(\fint_{B_j} g^q(y)\,d\mu(y)\right)^{1/q} = 0\cdot g(x) = 0.$$

We further control I_2 by

$$\begin{aligned}
I_2 &\le \limsup_{j\to\infty} \sum_{l=0}^{j-1}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right| \\
&\le \sum_{l=1}^{\infty}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right| \\
&\quad + \left|\fint_{B_1} f(z)\,d\mu(z) - \fint_{B} f(z)\,d\mu(z)\right| \\
&= I_2^1 + I_2^2.
\end{aligned}$$

For I_2^1, the facts that $B_{l+1}\subset B_l$ and $\mu(B_{l+1})\sim\mu(B_l)$ for $l\in\mathbb{N}$ and (3.1) imply that

$$\begin{aligned}
I_2^1 &\le C\sum_{l=1}^{\infty}\fint_{B_l}\left|f(y) - \fint_{B_l} f(z)\,d\mu(z)\right| d\mu(y) \\
&\le C\sum_{l=1}^{\infty} 2^{-l} r(B)\left[\fint_{B_l} g^q(y)\,d\mu(y)\right]^{1/q} \\
&\le Cr(B)M(g^q)(x)^{1/q}.
\end{aligned}$$

We estimate I_2^2. Let $\bar{B} = B(x, 2Ar(B))$. Then $B\cup B_1\subset\bar{B}$ and

$$\mu(\bar{B})\sim\mu(B)\sim\mu(B_1).$$

Therefore,

$$\begin{aligned}
I_2^2 &\le \left|\fint_{B_1} f(z)\,d\mu(z) - \fint_{\bar{B}} f(z)\,d\mu(z)\right| + \left|\fint_{\bar{B}} f(z)\,d\mu(z) - \fint_{B} f(z)\,d\mu(z)\right| \\
&\le \fint_{B_1}\left|f(y) - \fint_{\bar{B}} f(z)\,d\mu(z)\right| d\mu(y) + \fint_{B}\left|f(y) - \fint_{\bar{B}} f(z)\,d\mu(z)\right| d\mu(y) \\
&\le C\fint_{\bar{B}}\left|f(y) - \fint_{\bar{B}} f(z)\,d\mu(z)\right| d\mu(y) \\
&\le Cr(B)M(g^q)(x)^{1/q}.
\end{aligned}$$

This proves Theorem 3.1. ∎

Using Theorem 3.1, we can establish the relation between the space $A_p^1(X)$ and $B_p^1(X)$.

THEOREM 3.2. *Let $1 < p \le \infty$. Then $A_p^1(X) = B_p^1(X)$ with equivalent norms.*

Proof. Let $f \in A_p^1(X)$. For any given $\epsilon > 0$, Definition 2.7 then tells us that there exist some $q \in [1, p)$ and a non-negative function $g \in L^p(X)$ such that (2.32) holds for any ball $B \subset X$, and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{A_p^1(X)} + \epsilon.$$

Therefore, Theorem 3.1 yields that

$$\left| f(x) - \fint_B f(z)\, d\mu(z) \right| \le C_5 r(B) M(g^q)(x)^{1/q}$$

for a. e. $x \in B$ and all $B \subset X$. Let $h = C_5 M(g^q)(x)^{1/q}$. Then (2.23) holds with g replaced by h, and

$$\begin{aligned} \|f\|_{L^p(X)} + \|h\|_{L^p(X)} &\le \|f\|_{L^p(X)} + C_5 \big\| M(g^q)^{1/q} \big\|_{L^p(X)} \\ &\le C \big\{ \|f\|_{L^p(X)} + \|g\|_{L^p(X)} \big\} \\ &\le C \big\{ \|f\|_{A_p^1(X)} + \epsilon \big\}, \end{aligned}$$

where we used the $L^{p/q}(X)$-boundedness of M. Let $\epsilon \to 0$. We therefore obtain that $A_p^1(X) \subset B_p^1(X)$ and

$$\|f\|_{B_p^1(X)} \le C \|f\|_{A_p^1(X)}.$$

Conversely, by an argument similar to the proof of Theorem 2.4, it is easy to show that $B_p^1(X) \subset A_p^1(X)$ and

$$\|f\|_{A_p^1(X)} \le C \|f\|_{B_p^1(X)},$$

which finishes the proof of Theorem 3.2. $\square$

We now recall the definition of the Sobolev spaces of Hajłasz in [9]; see also [10], [18].

Definition 3.1. Let $1 < p \le \infty$. The Sobolev space $W^1_p(X)$ is defined by

$$\begin{aligned} W^1_p(X) = \{u \in L^p(X) &: \text{there is a set } E \subset X,\ \mu(E) = 0, \\ &\text{and a function } g \ge 0,\ g \in L^p(X) \text{ such that} \\ &|u(x) - u(y)| \le \rho(x,y)[g(x) + g(y)] \text{ for all } x,\ y \in X \setminus E\}, \end{aligned}$$

where g is called a generalized gradient of u. Moreover, we define

$$\|u\|_{W^1_p(X)} = \|u\|_{L^p(X)} + \inf_{\{g\}} \|g\|_{L^p(X)},$$

where the infimum is taken over all generalized gradients of the function u in the definition of $W^1_p(X)$.

The following theorem indicates the relations among the Hajłasz-Sobolev space, the space $A^1_p(X)$ and the space $B^1_p(X)$.

Theorem 3.3. *Let $1 < p \le \infty$. Then $W^1_p(X) = A^1_p(X) = B^1_p(X)$ with equivalent norms.*

Proof. Let $1 < p \le \infty$ and $f \in W^1_p(X)$. For any given $\epsilon > 0$, then there is a non-negative function $g \in L^p(X)$ such that

$$|f(x) - f(y)| \le \rho(x,y)[g(x) + g(y)]$$

for a. e. x, $y \in X$, and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{W^1_p(X)} + \epsilon.$$

Thus, for any ball $B \subset X$ and a. e. $x \in B$,

$$\begin{aligned} \left| f(x) - \fint_B f(y)\, d\mu(y) \right| &\le \fint_B |f(x) - f(y)|\, d\mu(y) \\ &\le r(B) \fint_B [g(x) + g(y)]\, d\mu(y) \\ &\le r(B)[g(x) + Mg(x)] \\ &\le 2r(B)Mg(x), \end{aligned}$$

where we used the estimate (2.25). Thus, (2.23) holds with g replaced by $2Mg$, and

$$\begin{aligned}\|f\|_{B^1_p(X)} &\le \|f\|_{L^p(X)} + 2\|Mg\|_{L^p(X)}\\ &\le \|f\|_{L^p(X)} + 2C\|g\|_{L^p(X)}\\ &\le C\big[\|f\|_{W^1_p(X)} + \epsilon\big],\end{aligned}$$

where we used the $L^p(X)$-boundedness of M. Letting $\epsilon \to 0$, we then obtain that $W^1_p(X) \subset B^1_p(X)$ and

$$\|f\|_{B^1_p(X)} \le C\|f\|_{W^1_p(X)}.$$

Let $f \in B^1_p(X)$. Then, for any given $\epsilon > 0$, there is a non-negative function $g \in L^p(X)$ such that for any ball $B \subset X$ and a. e. $x \in B$,

$$\left|f(x) - \fint_B f(y)\,d\mu(y)\right| \le r(B)g(x)$$

and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B^1_p(X)} + \epsilon.$$

From this, it follows that for a. e. x, $y \in B$,

$$|f(x) - f(y)| \le r(B)[g(x) + g(y)].$$

By suitably choosing $r(B)$, we finally obtain that for a. e. x, $y \in X$,

$$|f(x) - f(y)| \le \rho(x,y)[g(x) + g(y)].$$

Thus, $f \in W^1_p(X)$ and

$$\|f\|_{W^1_p(X)} \le \|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B^1_p(X)} + \epsilon.$$

Letting $\epsilon \to 0$, we obtain that $B^1_p(X) \subset W^1_p(X)$ and

$$\|f\|_{W^1_p(X)} \le \|f\|_{B^1_p(X)}.$$

Thus, $W^1_p(X) = B^1_p(X)$ with equivalent norms. Furthermore, this fact and Theorem 3.2 imply Theorem 3.3. This finishes the proof of Theorem 3.3. $\square$

Finally, the following theorem gives another new characterization of the Sobolev space $W^1_p(X)$.

THEOREM 3.4. *Let $1 \le u < p \le \infty$. Then $f \in W^1_p(X)$ if and only if $f \in L^p(X)$ and $f^1_{u,C_4} \in L^p(X)$. Moreover,*

$$\|f\|_{W^1_p(X)} \sim \|f\|_{L^p(X)} + \left\|f^1_{u,C_4}\right\|_{L^p(X)}.$$

Proof. Let $f \in W^1_p(X)$. Then Theorem 3.3 tells us that $f \in B^1_p(X)$. By Definition 2.6, for any given $\epsilon > 0$, there is a function $g \in L^p(X)$ such that (2.23) holds for all $B \subset X$ and a. e. $x \in B$, and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B^1_p(X)} + \epsilon.$$

Thus,

$$\begin{aligned} f^1_{u,C_4}(x) &= \sup_{0<t<C_4} t^{-1}\left[\fint_{B(x,t)}\left|f(y) - \fint_{B(x,t)} f(z)\,d\mu(z)\right|^u d\mu(y)\right]^{1/u} \\ &\le \sup_{0<t<C_4}\left[\fint_{B(x,t)} g^u(y)\,d\mu(y)\right]^{1/u} \\ &\le M(g^u)(x)^{1/u}. \end{aligned}$$

Therefore, the $L^{p/u}(X)$-boundedness of M implies that

$$\begin{aligned} \|f\|_{L^p(X)} + \left\|f^1_{u,C_4}\right\|_{L^p(X)} &\le \|f\|_{L^p(X)} + \left\|M(g^u)^{1/u}\right\|_{L^p(X)} \\ &\le \|f\|_{L^p(X)} + C\|g\|_{L^p(X)} \\ &\le C\big(\|f\|_{B^1_p(X)} + \epsilon\big). \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\|f\|_{L^p(X)} + \left\|f^1_{u,C_4}\right\|_{L^p(X)} \le C\|f\|_{B^1_p(X)} \le C\|f\|_{W^1_p(X)}.$$

Let now $f \in L^p(X)$ and $f^1_{u,C_4} \in L^p(X)$. Let x be a Lebesgue point of f. For any fixed $0 < t < C_4$, consider the Lebesgue point of the function

$$f(v) - \fint_{B(x,t)} f(z)\,d\mu(z).$$

Moreover, let $B_0 = B(x,t)$ and $B_j = B(y, 2^{-j}t/A)$ for $j \in \mathbb{N}$. Then, for

a. e. $y \in B(x, t/2A)$, we have

$$\begin{aligned}\left|f(y) - \fint_{B(x,t)} f(z)\,d\mu(z)\right| &= \lim_{j\to\infty} \fint_{B_j} \left|f(v) - \fint_{B(x,t)} f(z)\,d\mu(z)\right| d\mu(v)\\ &\le \limsup_{j\to\infty} \fint_{B_j} \left|f(v) - \fint_{B_j} f(z)\,d\mu(z)\right| d\mu(v)\\ &\quad + \limsup_{j\to\infty}\left|\fint_{B_j} f(z)\,d\mu(z) - \fint_{B_0} f(z)\,d\mu(z)\right|\\ &= J_1 + J_2.\end{aligned}$$

The Hölder inequality tells us that

$$\begin{aligned}J_1 &\le \limsup_{j\to\infty}\left\{\fint_{B_j}\left|f(v) - \fint_{B_j} f(z)\,d\mu(z)\right|^u d\mu(v)\right\}^{1/u}\\ &\le \limsup_{j\to\infty} 2^{-j} t f^1_{u,C_4}(y)\\ &= 0.\end{aligned}$$

We now estimate J_2. The Hölder inequality and the fact that $\mu(B_{l+1}) \sim \mu(B_l)$ for $l \in \mathbb{N}$ yield that

$$\begin{aligned}J_2 &\le \limsup_{j\to\infty} \sum_{l=0}^{j-1}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right|\\ &\le \sum_{l=1}^{\infty}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right|\\ &\quad + \left|\fint_{B_1} f(z)\,d\mu(z) - \fint_{B_0} f(z)\,d\mu(z)\right|\\ &\le \sum_{l=1}^{\infty}\left\{\fint_{B_{l+1}}\left|f(v) - \fint_{B_l} f(z)\,d\mu(z)\right|^u d\mu(v)\right\}^{1/u}\\ &\quad + C\left\{\fint_{B_0}\left|f(v) - \fint_{B_0} f(z)\,d\mu(z)\right| d\mu(v)\right\}^{1/u}\\ &\le C\sum_{l=1}^{\infty} 2^{-l} t f^1_{u,1}(y) + Ctf^1_{u,1}(x)\\ &\le Ct\big[f^1_{u,1}(y) + f^1_{u,1}(x)\big].\end{aligned}$$

Thus, for a. e. $y \in B(x, t/2A)$,

$$|f(y) - f(x)| \le Ct\big[f^1_{u,1}(y) + f^1_{u,1}(x)\big].$$

By suitably choosing t, we obtain that for a. e. x, $y \in X$,

$$|f(y) - f(x)| \le C_6\rho(x, y)\big[f^1_{u,1}(y) + f^1_{u,1}(x)\big].$$

Therefore, letting $g(x) = C_6 f^1_{u,1}(x)$, we then verify that $f \in W^1_p(X)$ and

$$\begin{aligned}\|f\|_{W^1_p(X)} &\le \|f\|_{L^p(X)} + \|g\|_{L^p(X)}\\ &\le \|f\|_{L^p(X)} + C_6\|f^1_{u,1}\|_{L^p(X)}\\ &\le C\big[\|f\|_{L^p(X)} + \|f^1_{u,C_4}\|_{L^p(X)}\big].\end{aligned}$$

This proves Theorem 3.4. ∎

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