TYPES OF COMPLETE INFINITELY SHEETED PLANES

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Abstract. We will answer negatively to the question whether the completeness of infinitely sheeted covering surfaces of the extended complex plane have anything to do with their types being parabolic or hyperbolic. This will be accomplished by giving a one parameter family $\{W[\alpha] : \alpha \in \mathcal{A}\}$ of complete infinitely sheeted planes $W[\alpha]$ depending on the parameter set $\mathcal{A}$ of sequences $\alpha = (a_n)_{n \geq 1}$ of real numbers $0 < a_n \leq 1/2$ $(n \geq 1)$ such that $W[\alpha]$ is parabolic for 'small' α's and hyperbolic for 'large' α's.

§1. Introduction

The purpose of this paper is to show the existence of a one parameter family $(W[\alpha])_{\alpha \in \mathcal{A}}$ of complete infinitely sheeted planes $W[\alpha]$ such that $W[\alpha] \in O_G$ for 'small' $\alpha \in \mathcal{A}$ and $W[\alpha] \notin O_G$ for 'large' $\alpha \in \mathcal{A}$.

Recall that a covering (Riemann) surface X, or more precisely, (X, Y, f) of Y is a triple of two Riemann surfaces X and Y and an analytic mapping f of X to Y. It is said to be *complete* (cf. [2]) if every $a \in Y$ has a closed parametric disc K around a such that each component of $f^{-1}(K)$ is compact. When (X, Y, f) is smooth, i.e. there is no branch point in X, the completeness of (X, Y, f) is equivalent to the regularity of (X, Y, f), where (X, Y, f) is *regular* if, for any arc γ on Y and any point $\tilde{a} \in X$ lying over the initial point a of γ, there always exists a continuation $\tilde{\gamma}$ on X along γ starting from $\tilde{a}$, i.e. there is an arc $\tilde{\gamma}$ on X with its initial point $\tilde{a}$ such that $f(\tilde{\gamma}) = \gamma$ (cf. [2]).

Let P be the plane, or more precisely, P be the finite complex plane $\mathbf{C} : |z| < \infty$ or the infinite (i.e. extended) complex plane $\widehat{\mathbf{C}} : |z| \leq +\infty$ (i.e. $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, the Riemann sphere). If the covering surface (X, P, π) of P satisfies the following two conditions, then X, or (X, P, π), is referred to as a *multisheeted plane*: the cardinal number $\operatorname{card} \pi^{-1}(a)$ is a constant

$\mu_X \in \mathbf{N} \cup \{\aleph_0\}$ for every $a \in P$, where $\mathbf{N}$ is the set of positive integers and $\aleph_0 = \operatorname{card} \mathbf{N}$; for any cover continuation (X', P, π') of (X, P, π), the closed set $X' \setminus X$ is of logarithmic capacity zero measured on X', where a cover continuation (X', P, π') of (X, P, π) is a covering surface of P such that X is a subsurface of X' such that $\pi'|X = \pi$. If the number, which we call the *sheet number* of X, $\mu_X \in \mathbf{N}$ ($\mu_X = \aleph_0$, resp.), then (X, P, π) is said to be a *finitely* (*infinitely*, resp.) *sheeted plane*.

What we call our *type problem* is to judge whether $X \in O_G$ or not for a given multisheeted plane (X, P, π) (cf. [7], [11]), where O_G is the class of parabolic (i.e. not hyperbolic) Riemann surfaces characterized by the nonexistence of Green function on them (cf. e.g. [8], [9]). If (X, P, π) is finitely sheeted plane, then we can easily see that $X \in O_G$ along with the parabolicity or the ellipticity of P. In view of this, we will consider only infinitely sheeted planes in our type problem. We have been having a feeling that complete infinitely sheeted planes are apt to become parabolic. In fact, we have shown the existence of a family $\mathcal{W}$ of infinitely sheeted planes $(X, \mathbf{C}, \pi)$ which is an equivalence class of infinitely sheeted planes with respect to the similarity, where $(X_1, \mathbf{C}, \pi_1)$ and $(X_2, \mathbf{C}, \pi_2)$ are similar to each other if there is a topological mapping T of X_1 onto X_2 with the property that $p \in X_1$ is a branch point of multiplicity m if and only if $T(p) \in X_2$ is a branch point of multiplicity m and in this case $\pi_1(p) = \pi_2(T(p))$, such that $X \in O_G$ for a $(X, \mathbf{C}, \pi) \in \mathcal{W}$ if and only if $(X, \mathbf{C}, \pi)$ is complete (cf. [5] and [6]). Nevertheless, the *main purpose* of this paper is to maintain that the parabolicity and the completeness of infinitely sheeted planes have absolutely nothing to do with each other. This will be shown by constructing a one sequential parameter family $(W[\alpha])_{\alpha \in \mathcal{A}}$ of *complete* infinitely sheeted planes $(W[\alpha], \widehat{\mathbf{C}}, \pi)$ ($\alpha \in \mathcal{A}$) such that $W[\alpha] \in O_G$ if α is "small" and $W[\alpha] \notin O_G$ if α is "large", where $\mathcal{A}$ is the family of sequences $\alpha = (a_n)_{n \in \mathbf{N}}$ with $0 < a_n \leq 1/2$. The one sequential parameter family $(W[\alpha] \setminus \pi^{-1}(\infty))_{\alpha \in \mathcal{A}}$ of $(W[\alpha] \setminus \pi^{-1}(\infty), \mathbf{C}, \pi)$ ($\alpha \in \mathcal{A}$) plays the same role as that of $(W[\alpha])_{\alpha \in \mathcal{A}}$.

§2. Construction of a one parameter family

Consider the class $\mathcal{A}$ of sequences $\alpha = (a_n)_{n \in \mathbf{N}}$ of real numbers a_n with $0 < a_n \leq 1/2$ ($n \in \mathbf{N}$). We view $\mathcal{A} = \{\alpha\}$ the sequential parameters family. We will construct a complete infinitely sheeted plane $(W[\alpha], \widehat{\mathbf{C}}, \pi)$ for each $\alpha \in \mathcal{A}$ so that we obtain a one sequential parameter family $(W[\alpha])_{\alpha \in \mathcal{A}}$ of complete infinitely sheeted planes $W[\alpha]$.

Let $J = [0, 1/2]$ be the slit in $\widehat{\mathbf{C}}$. We denote by S the surface obtained from two copies of $\widehat{\mathbf{C}} \setminus J$ by pasting $\widehat{\mathbf{C}} \setminus J$ and $\widehat{\mathbf{C}} \setminus J$ crosswise along J. The closure of one of $\widehat{\mathbf{C}} \setminus J$ in S will be called the upper sheet of S and denoted by S^+. The other closure of $\widehat{\mathbf{C}} \setminus J$ left in S is then referred to as the lower sheet of S and denoted by S^-. The surface S is conformally the sphere $\widehat{\mathbf{C}}$ but presently represented as a two-sheeted plane.

In addition to $\mathcal{A}$ and S we use the index set $\Lambda = \{-2, -1, 1, 2\}$ as another ingredient in our construction of $W[\alpha]$ ($\alpha \in \mathcal{A}$). An *admissible n-tuple* $i_1 i_2 \cdots i_n$ of elements in Λ is an ordered n-tuple of $i_1, i_2, \ldots, i_n$ in Λ such that $i_k \neq -i_{k-1}$ ($k = 2, \ldots, n$). There are $4 = 4 \cdot 3^{1-1}$ admissible 1-tuples i_1 in Λ, $4 \cdot 3 = 4 \cdot 3^{2-1}$ admissible 2-tuples $i_1 i_2$, $4 \cdot 3^2 = 4 \cdot 3^{3-1}$ admissible 3-tuples $i_1 i_2 i_3$, and so on, $\ldots$, and in general $4 \cdot 3^{n-1}$ admissible n-tuples $i_1 \cdots i_n$.

First consider 4 slits $I_{0i} = [|i|, |i| + 1/2]$ ($i \in \Lambda$) in S by the following rule: $I_{0i} \subset S^+$ for $i > 0$ and $I_{0i} \subset S^-$ for $i < 0$ so that I_{01} and I_{02} are in S^+ and $I_{0,-1}$ and $I_{0,-2}$ are in S^-. Then we consider the 4-slitted S:

$$\tag{1} F_0 := S \setminus \bigcup_{i \in \Lambda} I_{0i}.$$

We now fix an $\alpha = (a_n)_{n \in \mathbf{N}} \in \mathcal{A}$ so that $0 < a_n \leq 1/2$ ($n \in \mathbf{N}$). Let $I_{ni} := [|i|, |i| + a_n]$ and

$$\tag{2} F_{i_1 \cdots i_n} := S \setminus \left(I_{n-1, i_n} \cup \left(\bigcup_{i \in \Lambda \setminus i_n} I_{ni} \right) \right)$$

for each $i_1 \cdots i_n$ of $4 \cdot 3^{n-1}$ admissible n-tuples in Λ ($n \in \mathbf{N}$). We then form an infinitely sheeted planes $W[\alpha]$, which we express symbolically as follows:

$$\tag{3} W[\alpha] := F_0 + \sum_{i_1} F_{i_1} + \sum_{i_1 i_2} F_{i_1 i_2} + \cdots + \sum_{i_1 \cdots i_n} F_{i_1 \cdots i_n} + \cdots,$$

where $i_1 \cdots i_n$ are admissible n-tuples ($n \in \mathbf{N}$). Here we add a few words to mention the exact procedure in the construction in (3). As the *first step*, we perform the following: each F_{i_1} of the four F_{i_1} ($i_1 \in \Lambda$) is pasted to F_0 crosswise along $I_{0,-i_1}$ in F_0 and I_{0i_1} in F_{i_1}. From the second step on, and in general as the nth *step*, we perform the following: each $F_{i_1 \cdots i_{n-1} i_n}$ of three $F_{i_1 \cdots i_{n-1} i_n}$ ($i_n \in \Lambda \setminus \{-i_{n-1}\}$) is pasted to each $F_{i_1 \cdots i_{n-1}}$ of $4 \cdot 3^{n-2}$ $F_{i_1 \cdots i_{n-1}}$ for every admissible $(n-1)$-tuple $i_1 \cdots i_{n-1}$ in Λ crosswise along $I_{n-1, -i_n}$ in $F_{i_1 \cdots i_{n-1}}$ and I_{n-1, i_n} in $F_{i_1 \cdots i_{n-1} i_n}$.

From the construction it is clear that each $p \in W[\alpha]$ lies above a point in $\widehat{\mathbf{C}}$, which we denote by $\pi(p)$. Then $p \mapsto \pi(p)$ is an analytic mapping of $W[\alpha]$ to $\widehat{\mathbf{C}}$. Now it is easy to see that $W[\alpha]$, or more precisely $(W[\alpha], \widehat{\mathbf{C}}, \pi)$, is a *complete infinitely sheeted plane*. Moreover $W[\alpha]$ is *planar*. Concerning the one parameter family $(W[\alpha])_{\alpha \in \mathcal{A}}$ constructed above, we have the following result.

THEOREM. *The complete infinitely sheeted plane $W[\alpha]$ determined by the sequence $\alpha = (a_n)_{n \in \mathbf{N}}$ with $0 < a_n \leq 1/2$ $(n \in \mathbf{N})$ is hyperbolic, i.e. $W[\alpha] \notin O_G$, if α is large in the sense that*

$$(4) \qquad \inf_{n \in \mathbf{N}} a_n > 0;$$

and $W[\alpha]$ is parabolic, i.e. $W[\alpha] \in O_G$, if α is small in the sense that

$$(5) \qquad \sum_{n \in \mathbf{N}} \frac{1}{3^n} \log \frac{1}{a_n} = +\infty.$$

The proof of the second part of our theorem above will be given in the next Section 3 and the first part in Section 4. The proof of the second part is easy and very short. Since the proof given in Section 4 is considerably long, the section is divided into 3 Subsections 4.1–4.3.

§3. Proof of Theorem: parabolicity

We now start the proof of our theorem stated in Section 2. In this present short Section 3 we prove that if $\alpha = (a_n)_{n \in \mathbf{N}} \in \mathcal{A}$ is so small as to satisfy (5), then $W[\alpha] \in O_G$. For simplicity we set $W := W[\alpha]$. Let W_n $(n \geq 0)$ be regular subregions of W given by

$$(6) \quad W_0 := F_0, \quad W_n := F_0 + \sum_{i_1} F_{i_1} + \sum_{i_1 i_2} F_{i_1 i_2} + \cdots + \sum_{i_1 \cdots i_n} F_{i_1 \cdots i_n} \ (n \in \mathbf{N}),$$

where $i_1 \cdots i_k$ $(1 \leq k \leq n)$ are admissible k-tuples $(1 \leq k \leq n)$. Then $(W_n)_{n \geq 0}$ forms a regular exhaustion of W, which is called the *standard exhaustion* of W in this present paper. For each $n \in \mathbf{N}$ let w_n be the continuous function on $\overline{W}_n \setminus W_{n-1}$ such that w_n is harmonic on $W_n \setminus \overline{W}_{n-1}$ with boundary values $w_n|\partial W_n = 1$ and $w_n|\partial W_{n-1} = 0$. The function w_n is referred to as the harmonic measure of ∂W_n on $W_n \setminus \overline{W}_{n-1}$. The Dirichlet integral $D_{W_n \setminus \overline{W}_{n-1}}(w_n)$ of w_n over $W_n \setminus \overline{W}_{n-1}$ is, by definition,

$$D_{W_n \setminus \overline{W}_{n-1}}(w_n) := \int_{W_n \setminus \overline{W}_{n-1}} |\nabla w_n(z)|^2 \, dxdy \quad (z = x + \sqrt{-1}y).$$

Then the modulus μ_n of the configuration $W_n \setminus \overline{W}_{n-1}$ is given by

$$\mu_n := 2\pi/D_{W_n\setminus\overline{W}_{n-1}}(w_n).$$

By virtue of (5) we will derive

(7)
$$\sum_{n\in\mathbf{N}} \mu_n = +\infty.$$

Before proceeding to the proof of the above (7), we pause here to make the following simple observation. We denote by A_a for $0 < a \leq 1/4$ the annulus bounded by the ellipse centered at the origin 0 with major axis 1 and minor axis $\sqrt{1-4a^2}$ and the segment $[-a, a]$, by B_a the annulus bounded by the ellipse centered at 0 with major axis $1/a$ and minor axis $\sqrt{1-4a^2}/a$ and the segment $[-1, 1]$, and finally by C_a the ring domain bounded by circles $|z| = 2a/(1 + \sqrt{1-4a^2})$ and $|z| = 1$. By the conformal mapping $w = (z+1/z)/2$, C_a is mapped onto B_a. By the conformal mapping $w = az$, B_a is mapped onto A_a. In view of $\operatorname{mod} A_a = \operatorname{mod} B_a = \operatorname{mod} C_a$, where e.g. $\operatorname{mod} A_a$ means the modulus of A_a, and

$$\operatorname{mod} C_a = \log\Big[\big(1 + \sqrt{1-4a^2}\big)/2a\Big],$$

we see that

(8)
$$\operatorname{mod} A_a = \log\Big[\big(1 + \sqrt{1-4a^2}\big)/2a\Big] = 2\pi/D_{A_n}(w),$$

where w is the harmonic measure of the segment $[-a, a]$ on A_a. From the above (8) it can be derived that

(9)
$$\frac{1}{2}\log\frac{1}{a} \leq \operatorname{mod} A_a \leq \log\frac{1}{a}.$$

We now turn to the proof of (7). Observe that $W_n \setminus \overline{W}_{n-1}$ consists of $4 \cdot 3^{n-1}$ components $F_{i_1\cdots i_n}$ for all admissible n-tuples $i_1\cdots i_n$. Since the modulus $\mu_{i_1\cdots i_n}$ of $F_{i_1\cdots i_n}$ is given by $2\pi/D_{F_{i_1\cdots i_n}}(w_n)$, where w_n is the harmonic measure of ∂W_n on $W_n \setminus \overline{W}_{n-1}$, and

$$D_{W_n\setminus\overline{W}_{n-1}}(w_n) = \sum_{i_1\cdots i_n} D_{F_{i_1\cdots i_n}}(w_n),$$

we conclude that

(10)
$$1/\mu_n = \sum_{i_1\cdots i_n} 1/\mu_{i_1\cdots i_n}.$$

Let $A := A_{a_{n-1}/2} + (|i_n| + a_{n-1}/2)$, where $a_0 = 1/2$, and w be the harmonic measure of the segment part of ∂A on A. Since $A \subset F_{i_1 \cdots i_n}$, the Dirichlet principle implies

$$D_A(w) = D_{F_{i_1 \cdots i_n}}(w) \geq D_{F_{i_1 \cdots i_n}}(w_n),$$

where w is extended to $F_{i_1 \cdots i_n}$ from A by setting $w = 0$ on $F_{i_1 \cdots i_n} \setminus A$. Thus $\mathrm{mod}\, A_{a_{n-1}/2} = \mathrm{mod}\, A \leq \mu_{i_1 \cdots i_n}$. Using (10) we see that

$$1/\mu_n \leq \sum_{i_1 \cdots i_n} 1 \Big/ \Big(\frac{1}{2} \log \frac{1}{a_{n-1}}\Big) = 4 \cdot 3^{n-1} \Big/ \Big(\frac{1}{2} \log \frac{1}{a_{n-1}}\Big)$$

or $\mu_n \geq 8^{-1} 3^{-(n-1)} \log(1/a_{n-1})$. Therefore (5) implies

$$\sum_{n \in \mathbf{N}} \mu_n \geq 8^{-1} \sum_{n \in \mathbf{N}} 3^{-(n-1)} \log(1/a_{n-1}) = +\infty,$$

which is the relation (7) to be proved.

By the Sario-Noshiro modular criterion for the parabolicity of W (cf. e.g. [8], [9], etc.), the existence of the regular exhaustion $(W_n)_{n \in \mathbf{N}}$, which we call the standard exhaustion of W in this paper, of W with (7) concludes that $W \in O_G$. ∎

§4. Proof of Theorem: hyperbolicity

Under the assumption that the sequence $\alpha = (a_n)_{n \in \mathbf{N}}$ is large in the sense that $c := \inf_{n \in \mathbf{N}} a_n > 0$, we will show that $W[\alpha]$ is hyperbolic, i.e. $W[\alpha] \notin O_G$. As the largest member in $\mathcal{A}$ will be referred to the sequence $\sigma := (s_n)_{n \in \mathbf{N}}$ such that $s_n = 1/2$ $(n \in \mathbf{N})$ so that

$$\sigma = (1/2, 1/2, \ldots, 1/2, \ldots).$$

The *plan* of our proof for $W[\alpha] \notin O_G$ goes as follows. First we will show in Subsection 4.1 the existence of a quasiconformal mapping T of $W[\sigma]$ onto $W[\alpha]$. By the quasiconformal invariance of the parabolicity (i.e. if there is a quasiconformal mapping of a Riemann surface R_1 onto another R_2, then $R_1 \in O_G$ if and only if $R_2 \in O_G$ (cf. e.g. [9])), we can conclude our objective $W[\alpha] \notin O_G$ if we show that $W[\sigma] \notin O_G$. Then, as the second step, we will derive in Subsections 4.2 and 4.3 a contradiction from the erroneous assumption $W[\sigma] \in O_G$.

4.1. Construction of a quasiconformal mapping

In this first Subsection 4.1, we will construct a quasiconformal mapping T of $W[\sigma]$ onto $W[\alpha]$. We consider, as we did in Section 3, the standard exhaustions for covering surfaces constructed in Section 2. Recall that the *standard exhaustion* $(W_n[\alpha])_{n \geq 0}$ of $W[\alpha]$ is given by

$$W_0[\alpha] := F_0, \quad W_n[\alpha] := F_0 + \sum_{i_1} F_{i_1} + \cdots + \sum_{i_1 \cdots i_n} F_{i_1 \cdots i_n} \ (n \geq 1),$$

where $F_{i_1 \cdots i_k}$ will also be denoted by $F_{i_1 \cdots i_k}[\alpha]$ indicating the dependence on α. Take a *standard neighborhood* $U_n[\alpha]$ of each $\partial W_n[\alpha]$ ($n \in \mathbf{N}$) characterized by the following three conditions: a) each component of $U_n[\alpha]$ is a piecewise smooth annulus containing only one component of $\partial W_n[\alpha]$, where a region is piecewise smooth if the region is relatively compact and its relative boundary consists of a finite number of mutually disjoint piecewise smooth Jordan curves; b) any two different components of $\bigcup_{n \geq 0} U_n[\alpha]$ are disjoint in their closures; c) $W[\alpha] \setminus \bigcup_{n \geq 0} \overline{U_n[\alpha]}$ is an open set independent of α so that, for example,

$$W[\alpha] \setminus \bigcup_{n \geq 0} \overline{U_n[\alpha]} = W[\sigma] \setminus \bigcup_{n \geq 0} \overline{U_n[\sigma]}.$$

Let $\gamma[\alpha]$ be a Jordan curve which is a component of $\partial W_n[\alpha]$ and let $V[\alpha]$ be the annulus containing $\gamma[\alpha]$ which is a component of $U_n[\alpha]$.

The construction of T goes as follows. First let the mapping T be defined as the identity between $W[\sigma] \setminus \bigcup_{n \geq 0} U_n[\sigma]$ and $W[\alpha] \setminus \bigcup_{n \geq 0} U_n[\alpha]$. The construction will be over if we define a K-quasiconformal mapping of $V[\sigma]$ onto $V[\alpha]$ for every $V[\alpha]$ and its corresponding $V[\sigma]$ such that $T : \partial V[\sigma] \mapsto \partial V[\alpha]$ is the identity and $K \leq K_0$, where K_0 is a constant independent of the choice of $V[\alpha]$ (and $V[\sigma]$).

Suppose $\gamma[\alpha]$ comes from the slit $I_{n,i_{n+1}} = I_{n,i_{n+1}}[\alpha] = [|i_{n+1}|, |i_{n+1}| + a_n]$ in $F_{i_1 \cdots i_n i_{n+1}} = F_{i_1 \cdots i_n i_{n+1}}[\alpha]$ and the slit $I_{n,-i_{n+1}} = I_{n,-i_{n+1}}[\alpha]$ in $F_{i_1 \cdots i_n} = F_{i_1 \cdots i_n}[\alpha]$, which are pasted crosswise and give rise to the analytic Jordan curve $\gamma[\alpha]$. Let $\gamma[\sigma]$ be the corresponding one to $\gamma[\alpha]$ in $W[\sigma]$ so that it comes from $I_{n+1,i_{n+1}}[\sigma] = [|i_{n+1}|, |i_{n+1}| + s_n]$ in $F_{i_1 \cdots i_n i_{n+1}}[\sigma]$ and $I_{n,-i_{n+1}}[\sigma] = [|-i_{n+1}|, |-i_{n+1}| + s_n]$ in $F_{i_1 \cdots i_n}[\sigma]$, which are pasted crosswise and give rise to $\gamma[\sigma]$.

We denote by $Z(d)$ the slitted square

$$Z(d) := Q \setminus J(d),$$

where $Q = (0,1) \times (0,1)$ is the unit square and $J(d) = [1/4, 1/4+d] + \sqrt{-1}/2$ is the line segment of the length $0 < d \le 1/2$ and set

$$Z_i(d) := Z(d) - \sqrt{-1}/2 + |i| - 1/4$$

for $i \in \Lambda$. Then $Z_{i_{n+1}}(a_n)$ is viewed as being contained in $F_{i_1 \cdots i_n i_{n+1}}[\alpha]$ and the slit of $Z_{i_{n+1}}(a_n)$ is identical with $I_{n,i_{n+1}}[\alpha]$. Similarly $Z_{-i_{n+1}}(a_n)$ is viewed as being contained in $F_{i_1 \cdots i_n}[\alpha]$ and the slit of $Z_{-i_{n+1}}(a_n)$ is identical with $I_{n,-i_{n+1}}[\alpha]$. Thus as $V[\alpha]$ we take the annulus obtained by pasting $Z_{i_{n+1}}(a_n)$ to $Z_{-i_{n+1}}(a_n)$ crosswise along $I_{n,i_{n+1}}$ and $I_{n,-i_{n+1}}$. If we can construct a quasiconformal mapping T of $Z_{i_{n+1}}(s_n) \cup I_{n,i_{n+1}}[\sigma]$ to $Z_{i_{n+1}}(a_n) \cup I_{n,i_{n+1}}[\alpha]$ such that T is identity on their boundaries and

$$T(I_{n,i_{n+1}}[\sigma]) = I_{n,i_{n+1}}[\alpha],$$

then it can be continued symmetrically to the mapping T of $V[\sigma]$ to $V[\alpha]$ with the dilatation less than a fixed constant K_0 independent of $\gamma[\sigma]$ and $\gamma[\alpha]$.

Based upon the observation above we, hence, only have to show the existence of a quaiconformal mapping w from the square $Q = (0,1) \times (0,1)$ to itself fixing ∂Q such that $w(I(1/2)) = I(a_n)$, where

$$I(1/2) := [1/4, 1/4 + 1/2] + \sqrt{-1}/2, \quad I(a_n) := [1/4, 1/4 + a_n] + \sqrt{-1}/2$$

and the dilatation K is bounded by a constant K_0 depending only on c. For simplicity we set $a_n =: d$ in this proof so that $I(d) := I(a_n)$ and $0 < c \le d \le 1/2$.

Consider a homeomorphic mapping w of Q onto itself given by

$$w(z) := u(z) + \sqrt{-1}y \quad (z = x + \sqrt{-1}y),$$

where $u(z) := x$ for $0 \le x \le 1/4$, $2(\xi(y) - 1/4)(x - 1/4) + 1/4$ for $1/4 \le x \le 3/4$, and $4(1 - \xi(y))(x - 1) + 1$ for $3/4 \le x \le 1$, where

$$\xi(y) := (1 - 2d)|y - 1/2| + (1/4 + d),$$

which maps $[0,1]$ to $[1/4 + d, 3/4]$. Then $w_x = u_x$ and $w_y = u_y + \sqrt{-1}$ and thus $w_z = (w_x - \sqrt{-1}w_y)/2 = (u_x + 1 - \sqrt{-1}u_y)/2$ and $w_{\bar{z}} = (w_x + \sqrt{-1}w_y)/2 = (u_x - 1 + \sqrt{-1}u_y)/2$ so that the candidate of complex dilatation

$$\mu(z) := w_{\bar{z}}(z)/w_z(z)$$

satisfies $|\mu|^2 = (P - p)/(P + p)$, where $P := (u_x^2 + u_y^2 + 1)/2$ and $p := u_x$. Since $P \leq P_0 := 11/2$ and $p \geq p_0 := 2d$, we have

$$|\mu|^2 = \frac{P - p}{P + p} \leq \frac{P_0 - p_0}{P_0 + p_0} = \left(\frac{K - 1}{K + 1}\right)^2$$

or equivalently

$$|\mu| \leq \frac{K - 1}{K + 1},$$

where

$$K := \frac{11}{4d} + \sqrt{\frac{121}{16d^2} - 1} \leq \frac{11}{4c} + \sqrt{\frac{121}{16c^2} - 1} =: K_0$$

and

$$K \geq \frac{11}{4 \cdot (1/2)} + \sqrt{\frac{121}{16 \cdot (1/2)^2} - 1} = \frac{11}{2} + \sqrt{\frac{1}{4} + 29} > 1.$$

Hence μ defined above is a measurable function on $\overline{Q}$ such that $|\mu| \leq (K - 1)/(K + 1)$ on $\overline{Q}$ so that μ is eligible to be a complex dilatation on $\overline{Q}$. Clearly $w : \overline{Q} \mapsto \overline{Q}$ is a homeomorphism, the identity on ∂Q, ACL on $\overline{Q}$ and $w(I(1/2)) = I(d)$, and satisfies the Beltrami equation

$$w_{\overline{z}}(z) = \mu(z)w_z(z)$$

almost everywhere on $\overline{Q}$, i.e. w is a generalized solution of the above Beltrami equation on $\overline{Q}$. Therefore w is a required K-quasiconformal mapping of $\overline{Q}$ to $\overline{Q}$ (cf. e.g. [3]) fixing ∂Q pointwise with $K \leq K_0$, a fixed constant depending only upon c.

4.2. Exhausting regions with short boundaries

We are in the position to prove $W[\sigma] \notin O_G$. We will do this by contradiction so that we now make the erroneous assumption that $W := W[\sigma] \in O_G$. Then there exists an Evans-Selberg potential h on W with its pole at $a \in W$ (cf. [4], see also [9] and [10]): h is a harmonic function on $W \setminus a$ such that h has the negative logarithmic pole at $a \in W$, i.e. $z \mapsto h(z) + \log(1/|z|)$ is harmonic at a for the local parameter z at a with $z(a) = 0$, and $h(z) \to +\infty$ as z tends to the Alexandroff point of W. We consider the polar coordinate $re^{\sqrt{-1}\theta}$ on W constructed from h, where $r = e^h$ and θ is the multivalued conjugate harmonic function of h on $W \setminus a$ so that $d\theta = *dh$ on $W \setminus a$. Using a branch θ of θ we can use $re^{\sqrt{-1}\theta}$ as

local parameters at each point of W except for an isolated set of points with $|\nabla h| = 0$.

For each $0 < t < +\infty$ we set

$$R(t) := \{z \in W : h(z) < \log t\} = \{z \in W : r = r(z) < t\}$$

so that $a \in R(t)$ since $r(a) = 0$. Clearly $R(t)$ is a relatively compact region and its relative boundary $\Gamma(t) := \partial R(t)$ consists of a finite number of Jordan curves which may have common points at where $*dh = 0$. Except for these points which are isolated in W the above Jordan curves are analytic. Hence except for a countable number of $0 < t < +\infty$, $R(t)$ is a regular region and $\Gamma(t)$ consists of a finite number of mutually disjoint analytic Jordan curves. Clearly

$$\int_{\Gamma(t)} d\theta = 2\pi.$$

In the covering surface $(W, \widehat{\mathbf{C}}, \pi)$ we consider a (pseudo-)metric on W induced from the spherical metric on $\widehat{\mathbf{C}}$ by the projection π. The area of a set $G \subset W$ is denoted by $|G|$ and the length of a curve $\gamma \subset W$ is also denoted by $|\gamma|$ in the above induced metric. Put $A(t) := |R(t)|$, i.e. the area of $R(t)$. By denoting $\pi'(z) = d\pi(z)/dz$, we have

$$A(t) = \iint_{R(t)} \left(\frac{|\pi'(z)|}{1 + |\pi(z)|^2} \right)^2 r\, dr d\theta$$

$$= \int_0^t \left(\int_{\Gamma(r)} \left(\frac{|\pi'(z)|}{1 + |\pi(z)|^2} \right)^2 r\, d\theta \right) dr \quad (z = re^{\sqrt{-1}\theta}).$$

Clearly $t \mapsto A(t)$ is continuous and increasing, and

$$A(\infty) := \iint_W \left(\frac{|\pi'(z)|}{1 + |\pi(z)|^2} \right)^2 r\, dr d\theta$$

$$= \mu_W \iint_{F_0} \left(\frac{|\pi'(z)|}{1 + |\pi(z)|^2} \right)^2 r\, dr d\theta = +\infty.$$

Hence we conclude that

(11) $$\lim_{t \uparrow \infty} A(t) = A(\infty) = +\infty.$$

Put $L(t) := |\Gamma(t)|$, the length of $\Gamma(t) = \partial R(t)$, so that

$$L(t) = \int_{\Gamma(t)} \frac{|\pi'(z)|}{1 + |\pi(z)|^2} t\, d\theta \quad (z = re^{\sqrt{-1}\theta}),$$

which is also continuous in t. Denoting by $A'(t) = dA(t)/dt$, we have

$$A'(t) = \int_{\Gamma(t)} \left(\frac{|\pi'(z)|}{1 + |\pi(z)|^2} \right)^2 t\, d\theta.$$

By the Schwarz inequality we see that

$$L(t)^2 = \left(\int_{\Gamma(t)} \frac{|\pi'(z)|}{1 + |\pi(z)|^2} t\, d\theta \right)^2$$
$$\leq \left(\int_{\Gamma(t)} \left(\frac{|\pi'(z)|}{1 + |\pi(z)|^2} \right)^2 t\, d\theta \right) \cdot \left(\int_{\Gamma(t)} t\, d\theta \right) = 2\pi t A'(t)$$

and thus we have obtained the inequality

(12) $$L(t)^2 \leq 2\pi t A'(t).$$

Choose and then fix an arbitrary $1/2 < \tau < 1$. We consider

$$\Xi(s) := \{ t \in [s, \infty) : L(t) \geq A(t)^\tau \},$$

which is closed in $[s, \infty)$. Choose an $r_0 > 0$ such that $R(r_0)$ is a regular region containing $\overline{F}_0$ so that $\partial R(r_0)$ consists of at least four components. Suppose that a sequence $0 < r_0 < r_1 < \cdots < r_{n-1}$ for an $n \in \mathbf{N}$, $n > 1$, has been chosen such that $L(r_k) < A(r_k)^\tau$ and $R(r_k)$ is regular for $1 \leq k \leq n-1$. Take an $s_n \in (\max\{n, r_{n-1}\}, \infty)$. Considering the logarithmic length

$$\mathcal{L}(X) := \int_X \frac{dt}{t}$$

for measurable subsets $X \subset (0, \infty)$, we observe that

$$\mathcal{L}([s_n, \infty)) = \int_{s_n}^\infty \frac{dt}{t} = [\log t]_{s_n}^\infty = +\infty.$$

Using $1/t \leq 2\pi A'(t)/L(t)^2$ for $t \in [s_n, \infty)$ as a consequence of (12) and $2\pi A'(t)/L(t)^2 \leq 2\pi A'(t)/A(t)^{2\tau}$ for $t \in \Xi(s_n)$, we deduce

$$\mathcal{L}(\Xi(s_n)) = \int_{\Xi(s_n)} \frac{dt}{t} \leq 2\pi \int_{\Xi(s_n)} \frac{A'(t)}{A(t)^{2\tau}}\, dt$$
$$\leq 2\pi \int_{s_n}^\infty \frac{A'(t)}{A(t)^{2\tau}}\, dt = 2\pi \int_{A(s_n)}^{A(\infty)} \frac{dA}{A^{2\tau}}$$
$$= 2\pi \left[\frac{1}{1 - 2\tau} A^{-2\tau+1} \right]_{A(s_n)}^\infty = \frac{2\pi}{2\tau - 1} A(s_n)^{-2\tau+1} < \infty.$$

Hence $\mathcal{L}([s_n,\infty) \setminus \Xi(s_n)) = \infty - (2\pi/(2\tau-1))A(s_n)^{-2\tau+1} = +\infty$. This shows that the set $[s_n,\infty) \setminus \Xi(s_n) \neq \emptyset$ is open and a fortiori we can find an $r_n \in (s_n,\infty)$ with $r_n \notin \Xi(s_n)$, i.e. $L(r_n) < A(r_n)^\tau$, such that $R(r_n)$ is regular. Since $r_n > s_n > \max\{n, r_{n-1}\}$, we have

$$0 < r_0 < r_1 < r_2 < \cdots < r_{n-1} < r_n \uparrow \infty.$$

Using this sequence we finally define

$$(13) \qquad R_n := R(r_n), \quad \Gamma_n = \Gamma(r_n) = \partial R(r_n) \quad (n \geq 0).$$

Then $(R_n)_{n\geq 0}$ is a regular exhaustion of W and

$$|\Gamma_n|/|R_n| = L(r_n)/A(r_n) < A(r_n)^\tau/A(r_n) = A(r_n)^{\tau-1} \downarrow 0 \quad (n \uparrow \infty).$$

Thus the boundaries Γ_n of the exhausting regions R_n of W are relatively short in the sense that

$$(14) \qquad \lim_{n\to\infty} \frac{|\Gamma_n|}{|R_n|} = 0.$$

4.3. Exhausting regions with long boundaries

As a consequence of $W := W[\sigma] \in O_G$, we have seen in the preceding Subsection 4.2 the existence of a regular exhaustion $(R_n)_{n\geq 0}$ of W given in (13) that the boundaries $\Gamma_n = \partial R_n$ of R_n are so short as to satisfy (14). We will see from a different view point that the same exhaustion $(R_n)_{n\geq 0}$ satisfies

$$(15) \qquad \liminf_{n\to\infty} \frac{|\Gamma_n|}{|R_n|} > 0$$

as a consequence of $W \in O_G$, which contradicts (14). Thus we should not have assumed that $W \in O_G$ and the proof of $W := W[\sigma] \notin O_G$ will be over. The proof of (15) goes as follows.

Besides the main covering surface $(W, \widehat{\mathbf{C}}, \pi)$ we have two other covering surfaces (W, F, π_1) and $(F, \widehat{\mathbf{C}}, \pi_2)$ involved in $(W, \widehat{\mathbf{C}}, \pi)$ with the relation $\pi = \pi_2 \circ \pi_1$, where F is the closed surface of genus 2 obtained from F_0 by pasting crosswise along I_{01} and $I_{0,-1}$ and also along I_{02} and $I_{0,-2}$. The metric on F is the one induced from the spherical metric on $\widehat{\mathbf{C}}$ by π_2 and that on W is the one induced from that on F by π_1, which is identical with the one induced from the spherical metric on $\widehat{\mathbf{C}}$ by π already considered in

the preceding Subsection 4.2. The area (length, resp.) of a set G (an arc γ, resp.) is denoted as before by the same notation $|G|$ ($|\gamma|$, resp.) on either one of three surfaces $(F, \widehat{\mathbf{C}}, \pi_2)$, (W, F, π_1), and $(W, \widehat{\mathbf{C}}, \pi)$.

We denote by $\Phi(W)$ ($\Phi(F)$, resp.) the set of closed curves φ on W (F, resp.) which are not of homotop null. We maintain

(16) $$a(W) := \inf_{\varphi \in \Phi(W)} |\varphi| > 0.$$

For a proof observe first that (W, F, π_1) is smooth and complete (so that regular) and therefore the monodromy theorem assures that $\pi_1(\varphi) \in \Phi(F)$ if and only if $\varphi \in \Phi(W) : \pi_1(\Phi(W) = \Phi(F)$. Since the metric on W may be viewed as induced from one on F by π_1, we see that $|\varphi| \geq |\pi_1(\varphi)|$. Hence

$$a(W) = \inf_{\varphi \in \Phi(W)} |\varphi| \geq \inf_{\varphi \in \Phi(W)} |\pi_1(\varphi)| = \inf_{\varphi \in \pi_1(\Phi(W))} |\varphi| = \inf_{\varphi \in \Phi(F)} |\varphi|,$$

by which we only have to prove

(17) $$a(F) := \inf_{\varphi \in \Phi(F)} |\varphi| > 0$$

to conclude (16), i.e. $a(W) \geq a(F) > 0$. Fix an arbitrary $p \in F$ and denote by $\Phi_p(F)$ the set of $\varphi \in \Phi(F)$ passing through p and consider

$$f(p) := \inf_{\varphi \in \Phi_p(F)} |\varphi|.$$

Then $a(F) = \inf_{p \in F} f(p)$. We maintain that $f(p) > 0$. Contrary to the assertion assume that $f(p) = 0$ for a $p \in F$. Choose an open parametric disc B centered at p and let ε be the distance between p and $F \setminus B$ measured by the induced spherical metric on F. Then $f(p) = 0$ implies that there is a $\varphi_0 \in \Phi_p(F)$ such that $|\varphi_0| < \varepsilon$ so that $\varphi_0 \subset B$, which is however impossible since B is simply connected and φ_0 is not homotop null. Otherwise every $\varphi \in \Phi_p(F)$ is not contained in B, i.e. $\varphi \cap (F \setminus B) \neq \emptyset$, and a fortiori $|\varphi| \geq \varepsilon$, which leads to $f(p) = \inf_{\varphi \in \Phi_p(F)} |\varphi| \geq \varepsilon$, contradicting the present erroneous assumption $f(p) = 0$. Thus we have seen that $f(p) > 0$ for every $p \in F$. Next choose an arbitrary pair (p, q) of points in F and an arc $\gamma \subset F$ connecting p and q. For any $\varphi \in \Phi_q(F)$, we have $\gamma \varphi \gamma^{-1} \in \Phi_p(F)$ and $|\gamma \varphi \gamma^{-1}| \leq |\varphi| + 2|\gamma|$. Hence $f(q) \leq f(p) + 2d_F(p, q)$, which implies the (Lipschitz) continuity of f on F, i.e. $|f(p) - f(q)| \leq 2d_F(p, q)$ for every pair (p, q) of points in F, where $d_F(p, q)$ is the induced spherical distance on F. Thus $f \in C(F)$ and $f > 0$ on F implies that $\inf_F f > 0$, which proves (17).

Concerning the particular exhaustion $(R_n)_{n \geq 0}$ introduced in the preceding Subsection 4.2, we suppose the relative boundary $\partial R_n =: \Gamma_n$ consists of a finite number ν_n of components Γ_{ni} $(1 \leq i \leq \nu_n): \Gamma_n = \bigcup_{1 \leq i \leq \nu_n} \Gamma_{ni}$. In view of the choice of $R_0 = R(r_0)$ we have $\nu_n \geq 4$ $(n \in \mathbf{N})$. Observe that Γ_{ni} is not homologue null. Otherwise Γ_{ni} is the boundary of a relatively closed region not containing a, which we denote by (Γ_{ni}). Since $h = r_n$ on $\partial(\Gamma_{ni}) = \Gamma_{ni}$, the maximum principle yields $h \equiv r_n$ on (Γ_{ni}) and thus h must be a constant r_n on W, which is clearly a contradiction. Since the homology group $\mathcal{H}(W)$ of W is isomorphic to the factor group $\mathcal{F}(\mathcal{W})/\mathcal{F}^{(\infty)}(\mathcal{W})$ of the homotopy group $\mathcal{F}(\mathcal{W})$ of W by its commutator subgroup $\mathcal{F}^{(\infty)}(\mathcal{W})$, being homotop null implies being homologue null for any closed curve on W and thus we see that $\Gamma_{ni} \in \Phi(W)$ $(1 \leq i \leq \nu_n)$ so that $|\Gamma_n| = \sum_{1 \leq i \leq \nu_n} \Gamma_{ni} \geq \sum_{1 \leq i \leq \nu_n} a(W)$, i.e.

$$(18) \qquad\qquad |\Gamma_n| \geq \nu_n a(W) \quad (n \in \mathbf{N}).$$

We denote by $\rho(R_n)$ ($\rho(F)$, resp.) the Euler characteristic of R_n (F, resp.) multiplied by -1. As a relatively compact subsurface of the *planar* surface W, R_n is a planar surface bounded by ν_n Jordan curves Γ_{ni} and a fortiori $\rho(R_n) = \nu_n - 2$. Since the genus $g(F)$ of F is 2, i.e. $g(F) = 2$, we see that $\rho(F) = 2g(F) - 2 = 2$. Recall the main theorem of the Ahlfors theory of covering surfaces (cf. [1], see also e.g. [7], [10]) applied to the covering surface (R_n, F, π_1):

$$(19) \qquad\qquad \max\{\rho(R_n), 0\} \geq \rho(F) \frac{|R_n|}{|F|} - h_F |\Gamma_n|,$$

where h_F is a *positive* number depending only upon F and independent of R_n. The inequality (18) implies $|\Gamma_n|/a(W) \geq \nu_n$ and the relation $\rho(R_n) = \nu_n - 2$ yields $\nu_n > \rho(R_n)$. Hemce $|\Gamma_n|/a(W) > \rho(R_n)$ holds. These with $\rho(F) = 2$ transform (19) into the following form:

$$|\Gamma_n|/a(W) > 2|R_n|/|F| - h_F |\Gamma_n|.$$

Rewriting the above inequality with $b := 2a(W)/|F|(a(W)h_F + 1) > 0$, we obtain the inequality $|\Gamma_n|/|R_n| > b > 0$ for every $n \in \mathbf{N}$ and thus we can finally deduce (15): $\liminf_{n \to \infty} |\Gamma_n|/|R_n| \geq b > 0$. $\quad\square$

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