

## ON HYPERBOLICITY OF BALANCED DOMAINS

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**Abstract.** We compare the hyperbolicity with respect to the Lempert function with the other hyperbolicities in the class of pseudoconvex balanced domains.

### §1. Introduction and main results

In complex analysis, various notions of hyperbolicity of a given domain are investigated by many authors. For example, it is known that any domain  $G$  in  $\mathbb{C}^n$  that is hyperbolic with respect to the Kobayashi pseudodistance  $k_G$  (shortly *k-hyperbolic*) is automatically *Brody hyperbolic*, which means that it does not contain non-trivial entire curve. Analogous to the *k-hyperbolicity*, we can define the notion of hyperbolicity with respect to the Lempert function  $\tilde{k}_G$  (shortly  *$\tilde{k}$ -hyperbolic*) in a given domain  $G$  in  $\mathbb{C}^n$ ; in fact, the latter term seems to be a simpler notion than *k-hyperbolicity* and it first appeared in Zwonek's work [8].

Now we consider the following families of domains in all  $\mathbb{C}^n$ 's:

$\mathfrak{G}_K$  := the family of all *k-hyperbolic* domains.

$\mathfrak{G}_L$  := the family of all  *$\tilde{k}$ -hyperbolic* domains.

$\mathfrak{G}_B$  := the family of all Brody hyperbolic domains.

Since the family  $\underline{d} := (d_G)_{G:\text{domain}}$  ( $d = k$  or  $\tilde{k}$ ) satisfies the so-called decreasing property, it is clear that

$$(1) \quad \mathfrak{G}_K \subset \mathfrak{G}_L \subset \mathfrak{G}_B.$$

A natural question is whether or not the converses of both inclusions in (1) are true, that is, whether the following implications are true for any domain in  $\mathbb{C}^n$ :

$$\text{Brody hyperbolic} \xrightarrow{\text{(I)}} \tilde{k}\text{-hyperbolic} \xrightarrow{\text{(II)}} k\text{-hyperbolic}.$$

In [8], as a positive answer, Zwonek showed that

$$(2) \quad \mathfrak{G}_K \cap \mathcal{PR} = \mathfrak{G}_L \cap \mathcal{PR} = \mathfrak{G}_B \cap \mathcal{PR},$$

where  $\mathcal{PR}$  is the family of all pseudoconvex Reinhardt domains.

On the other hand, the relationship between both terms ‘ $k$ - and Brody hyperbolic’ are well-known. Namely, there is an example of a domain belonging to  $\mathfrak{G}_B \setminus \mathfrak{G}_K$ . For instance, such a non-pseudoconvex balanced domain  $G_{ET}$  is given by Eisenman and Taylor (e.g. see [4, p. 104]); another one that is a pseudoconvex Hartogs domain  $G_B$ , but not balanced, is obtained by Barth [2] (cf. see Section 3). These counterexamples imply that at least one of (I) or (II) does not hold in general. In fact, it recently turned out that  $\mathfrak{G}_K \subsetneq \mathfrak{G}_L \subsetneq \mathfrak{G}_B$  (see [6]). We would like to point out that a large part of such counterexamples was found in the class of Hartogs type domains including  $G_{ET}$  and  $G_B$ . More explicitly, let  $\mathcal{H}$  be the family of all domains in the form of  $\{(z, w) \in G \times \mathbb{C}^m : h(w) \exp(u(z)) < 1\}$ , where  $G \subset \mathbb{C}^n$  is a domain,  $u$  (resp.  $h$ ) is upper semicontinuous on  $G$  (resp. on  $\mathbb{C}^m$ ) such that  $u \not\equiv -\infty$ ,  $h \geq 0$  with  $h \not\equiv 0$ , and  $h(\lambda w) = |\lambda|h(w)$ ,  $\lambda \in \mathbb{C}$ ,  $w \in \mathbb{C}^m$ . Then we have

$$(3) \quad \mathfrak{G}_K \cap \mathcal{H} \subsetneq \mathfrak{G}_L \cap \mathcal{H} \subsetneq \mathfrak{G}_B \cap \mathcal{H}.$$

In particular,  $G_{ET}$  and  $G_B$  belong to  $(\mathfrak{G}_B \cap \mathcal{H}) \setminus \mathfrak{G}_L$ .

The geometrical symmetry of the Hartogs type domains is very weak in comparison to the one of pseudoconvex Reinhardt domains. Therefore we are interested in the class of balanced domains which have stronger symmetry. Observe that

$$(4) \quad \mathfrak{G}_K \cap \mathcal{D} \subset \mathfrak{G}_L \cap \mathcal{D} \subsetneq \mathfrak{G}_B \cap \mathcal{D},$$

where  $\mathcal{D}$  is the family of all balanced domains in all  $\mathbb{C}^n$ 's.

At this point, a natural question, motivated by our discussion above, is whether a similar phenomenon as in (2) happens in the class of all pseudoconvex balanced domains, that is, whether both implications (I) and (II) hold for any pseudoconvex balanced domain.

For this let us first recall some results. Azukawa [1] modified the domain  $G_B$  by Barth and found, using an idea of Sadullaev [7], an example of an unbounded pseudoconvex balanced domain  $D_A \subset \mathbb{C}^2$  belonging to  $\mathfrak{G}_B \setminus \mathfrak{G}_K$  (see Section 3). Afterwards Kodama [4] obtained that a balanced domain

in  $\mathbb{C}^n$  is  $k$ -hyperbolic iff it is bounded. These results imply that at least one of (I) or (II) does not hold in general, even in the class of pseudoconvex balanced domains.

We can ask whether or not the domain  $D_A \subset \mathbb{C}^2$  is  $\tilde{k}$ -hyperbolic. I regret that I do not know what the complete answer will be. However, our first aim is to give a partial answer to this question. Namely,

PROPOSITION 1. *Let  $D := D_A \subset \mathbb{C}^2$ . For any  $z, w \in D \setminus (\{0\} \times \mathbb{C} \setminus \{0\})$  with  $z \neq w$  one has*

$$\tilde{k}_D(z, w) > 0.$$

In order to prove this assertion in case  $z_1 = w_1 \neq 0$ , we will modify an idea of Azukawa that he used in [1] to prove the Brody hyperbolicity of  $D = D_A$ . However it remains still an open problem to estimate the value of  $\tilde{k}_D(z, w)$  provided that  $z, w \in D \cap (\{0\} \times \mathbb{C} \setminus \{0\})$  with  $z \neq w$ .

To conclude, we can also ask whether the implication (I) holds for any pseudoconvex balanced domain in  $\mathbb{C}^n$ . In contrast to (2), the answer for  $n \geq 3$  is, in general, ‘No’. More explicitly, we have the following result:

THEOREM 2. *For any  $n \geq 3$  there is a pseudoconvex balanced domain  $D = D_h$  in  $\mathbb{C}^n$  with  $h^{-1}(0) = \{0\}$  which is Brody hyperbolic but not  $\tilde{k}$ -hyperbolic.*

In conclusion, we have that

$$(5) \quad \mathfrak{G}_K \cap \mathcal{PD} \subset \mathfrak{G}_L \cap \mathcal{PD} \subsetneq \mathfrak{G}_B \cap \mathcal{PD}$$

where  $\mathcal{PD}$  is the family of all pseudoconvex balanced domains in all  $\mathbb{C}^n$ 's, although it is not known whether the implications (I) and (II) are true for any pseudoconvex balanced domain in  $\mathbb{C}^n$ ,  $n = 2$  resp.  $n \geq 2$ .

**Acknowledgements.** The author would like to express his hearty gratitude to my supervisor Professor P. Pflug and Professor M. Jarnicki for their helpful suggestions and fruitful discussions.

## §2. Basic notions, properties, and an example

By  $\|\cdot\| := \|\cdot\|_n$  we denote the Euclidean norm on  $\mathbb{C}^n$ ,  $|\cdot| := \|\cdot\|_1$ , and by  $\mathbb{B}_n(z, r)$  the Euclidean open ball with center  $z \in \mathbb{C}^n$  and radius  $r > 0$ .

Let  $E := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$  and let  $G \subset \mathbb{C}^n$  be a domain. By  $\mathcal{O}(E, G)$  we denote the family of all holomorphic mappings from  $E$  into  $G$ . Put

$$\tilde{k}_G(z, w) := \inf\{p(\lambda, \zeta) : \exists \varphi \in \mathcal{O}(E, G), \varphi(\lambda) = z, \varphi(\zeta) = w\}$$

where  $p(\lambda, \zeta) := \tanh^{-1}\left(\left|\frac{\lambda - \zeta}{1 - \lambda\bar{\zeta}}\right|\right)$  is the Poincaré distance on  $E$ , and set

$$k_G := \text{the largest pseudodistance not exceeding } \tilde{k}_G.$$

The function  $\tilde{k}_G$  (resp.  $k_G$ ) is called the *Lempert function* (resp. *Kobayashi pseudodistance*) on  $G$ . It is known that the family  $\underline{d}$  ( $d = k$  or  $\tilde{k}$ ) has the *decreasing property*, i.e. for any domain  $\Omega \subset \mathbb{C}^m$  and any holomorphic mapping  $f : G \rightarrow \Omega$ ,

$$d_\Omega(f(z), f(w)) \leq d_G(z, w), \quad z, w \in G.$$

We say that a domain  $G$  is *d-hyperbolic* ( $d = k$  or  $\tilde{k}$ ) if  $d_G(z, w) > 0$  whenever  $z \neq w$ . Clearly, any  $k$ -hyperbolic domain is also  $\tilde{k}$ -hyperbolic. Note that any bounded domain is  $k$ -hyperbolic but its converse does not hold in general. For example, any domain  $G \subset \mathbb{C} \setminus \{0, 1\}$  is  $k$ -hyperbolic.

A domain  $D \subset \mathbb{C}^n$  is called *balanced* if  $\lambda z \in D$  for any  $\lambda \in \bar{E}$  and  $z \in D$ . For a balanced domain  $D \subset \mathbb{C}^n$  there exists a unique upper semicontinuous function  $h = h_D$  on  $\mathbb{C}^n$  such that  $h$  is absolutely homogeneous on  $\mathbb{C}^n$ , i.e.  $h(\lambda z) = |\lambda|h(z)$ ,  $\lambda \in \mathbb{C}$ ,  $z \in \mathbb{C}^n$ , and  $D = \{z \in \mathbb{C}^n : h(z) < 1\} =: D_h$ . The function  $h = h_D$  is called the *Minkowski function* of  $D$ .

Now let us recall some basic properties that will be needed in the sequel. By  $PSH(G)$  we denote the family of all plurisubharmonic functions on a domain  $G \subset \mathbb{C}^n$ . For a balanced domain  $D = D_h \subset \mathbb{C}^n$  it is known that:

- $D \subset\subset \mathbb{C}^n \iff \exists C > 0: h(z) \geq C\|z\|, z \in \mathbb{C}^n$ .
- $h \in PSH(\mathbb{C}^n) \implies \tilde{k}_D(0, z) = p(0, h(z)), z \in D$ .
- $D$  is pseudoconvex  $\iff \log h \in PSH(\mathbb{C}^n) \iff h \in PSH(\mathbb{C}^n)$ .
- $D$  is Brody hyperbolic  $\implies h^{-1}(0) = \{0\}$ .
- (due to Siciak) If  $n = 2$  and  $D$  is pseudoconvex, then

$$D \text{ is Brody hyperbolic } \iff h^{-1}(0) = \{0\}.$$

We refer to e.g. [3] for more information; in particular, see Theorem 7.1.3 in [3] for a proof of the Siciak's result.

Before we deal with Azukawa's example, we would like to observe a general situation, as follows:

EXAMPLE 3. Fix  $n > 1$ . Let  $g : \mathbb{C}^{n-1} \rightarrow [0, \infty)$  be upper semicontinuous such that  $\ell_g := \lim_{\|z'\| \rightarrow \infty} g(z')/\|z'\|$  exists and is finite. Define  $h : \mathbb{C}^{n-1} \times \mathbb{C} \rightarrow [0, \infty)$  by

$$h(z) = h(z', z_n) = h_g(z', z_n) := \begin{cases} |z_n|g\left(\frac{z'}{z_n}\right) & (z_n \neq 0), \\ \ell_g\|z'\| & (z_n = 0), \end{cases}$$

where  $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ . Clearly,  $h$  is absolutely homogeneous, upper semicontinuous on  $\mathbb{C}^n$ . Now we will consider the balanced domain  $D = D_h \subset \mathbb{C}^n$  and the following condition:

$$(6) \quad \exists C > 0 : h(z', 1) \geq C\|z'\|, \quad z' \in \mathbb{C}^{n-1}.$$

Put  $\pi_1(z) := z'$  and  $\pi_2(z) := z_n$ . It is easy to show that if  $D$  is bounded in  $\mathbb{C}^n$ , then there is a  $C > 0$  such that  $g(z') > C\|z'\|$  for any  $z' \in \mathbb{C}^{n-1}$ ; in particular,  $g^{-1}(0) = \emptyset$  and  $h$  satisfies (6) with  $h^{-1}(0) = \{0\}$ . Conversely, if  $h$  satisfies (6) and  $h^{-1}(0) = \{0\}$ , then  $\pi_1(D)$  is bounded in  $\mathbb{C}^{n-1}$ , so  $\pi_2(D) \neq \mathbb{C}$  iff  $D$  is bounded in  $\mathbb{C}^n$ .

Moreover, if  $h$  satisfies (6) and  $h \in PSH(\mathbb{C}^n)$  with  $h^{-1}(0) = \{0\}$ , and if  $\{0'\} \times \mathbb{C} \not\subset D$ , then  $D$  is Brody hyperbolic. For this suppose the contrary. Then there is a mapping  $\varphi := (f, g) \in \mathcal{O}(\mathbb{C}, D)$ ,  $\varphi \neq \text{constant}$ , where  $f \in \mathcal{O}(\mathbb{C}, \pi_1(D))$  and  $g \in \mathcal{O}(\mathbb{C}, \pi_2(D))$ , and also the map  $f$  must be a constant, i.e.,  $f \equiv a'$  for some  $a' \in \pi_1(D)$ . But since  $\varphi$  is not a constant, one has  $\pi_2(D) = \mathbb{C}$  and thus Little Picard Theorem yields that  $g(\mathbb{C}) \supset \mathbb{C} \setminus \{\lambda_0\}$  for some  $\lambda_0 \in \mathbb{C}$ , and because of  $h \in PSH(\mathbb{C}^n)$ , it follows from the removable singularity theorem and the Liouville type theorem for subharmonic functions that  $h(a', \cdot) \equiv \text{constant} < 1$  on  $\mathbb{C}$ ; moreover,  $a' \neq 0$  by our assumption. On the other hand, one has  $h(\zeta a', \zeta \lambda) < 1$ ,  $\zeta \in \bar{E}$ ,  $\lambda \in \mathbb{C}$ , so  $h(\zeta a', \lambda) < 1$ ,  $\zeta \in \bar{E} \setminus \{0\}$ ,  $\lambda \in \mathbb{C}$ . For each  $\lambda \in \mathbb{C}$ , the function  $u_\lambda : \zeta \mapsto h(\zeta a', \lambda)$  is subharmonic on  $E$ . By the submean value property, we have  $u_\lambda(0) \leq 1$  for any  $\lambda \in \mathbb{C}$ , and hence, by the maximum principle, we get that  $u_\lambda(0) < 1$  for any  $\lambda \in \mathbb{C}$ , which contradicts the assumption that  $\{0'\} \times \mathbb{C} \not\subset D$ .

### §3. Azukawa's example

We now start by defining the balanced domain  $D_A \subset \mathbb{C}^2$  mentioned above.

Define a function  $u : \mathbb{C} \rightarrow [-\infty, \infty)$  by

$$u(\lambda) := \max \left\{ \log |\lambda|, \sum_{k=2}^{\infty} \frac{1}{k^2} \log \left| \lambda - \frac{1}{k} \right| \right\}, \quad \lambda \in \mathbb{C}.$$

Note that the Hartogs domain  $G_B$  mentioned in Section 1 is defined by  $\{z \in E \times \mathbb{C} : |z_2| \exp(u(z_1)) < 1\}$ . Put  $g := \exp \circ u$ . Observe that

$$|\lambda| \leq \exp(u(\lambda)) \leq \max\{|\lambda|, (|\lambda| + 1)^{\left(\frac{\pi^2}{6} - 1\right)}\}, \quad \lambda \in \mathbb{C},$$

so  $\ell_g = 1$ . Thus we get an absolutely homogeneous function  $h = h_g \in PSH(\mathbb{C}^2)$  with  $h^{-1}(0) = \{0\}$  and  $D_A := D_h \subset \mathbb{C}^2$  is just the balanced domain constructed by Azukawa in [1].

From now on we deal only with the balanced domain  $D := D_A$ .

Observe that  $h(\lambda, 1) \geq |\lambda|$ ,  $\lambda \in \mathbb{C}$  and  $\{0\} \times \mathbb{C} \not\subset D$ , so  $D$  is Brody hyperbolic; compare it with Siciak's result and the proof of Lemma 6.3 in [1].

Now we will prove the proposition.

*Proof of Proposition 1.* Note that  $\{z_1 \in \mathbb{C} : z \in D\}$  is bounded in  $\mathbb{C}$ ; in fact,  $\{z_1 \in \mathbb{C} : z \in D\} = E$ . By the decreasing property of  $\tilde{k}$ ,

$$(7) \quad \tilde{k}_D(z, w) > 0, \quad z, w \in D, z_1 \neq w_1.$$

But since  $D$  is not bounded in  $\mathbb{C}^2$ , one has  $\{z_2 \in \mathbb{C} : z \in D\} = \mathbb{C}$ . On the other hand, because  $\tilde{k}_D(0, z) = p(0, h(z))$ ,  $z \in D$ , it is clear that

$$(8) \quad \tilde{k}_D(0, z) > 0, \quad z \in D \text{ with } z \neq 0.$$

To finish the proof of Proposition 1, it remains to verify that

$$(9) \quad \tilde{k}_D(z, w) > 0, \quad z, w \in D, z \neq w, z_1 = w_1 \neq 0.$$

In order to verify (9) suppose that there are two points  $(a, z_2), (a, w_2) \in D \cap (E \setminus \{0\} \times \mathbb{C})$  such that  $\tilde{k}_D((a, z_2), (a, w_2)) = 0$ . Then there are sequences  $(s_j)_{j \geq 1} \subset \mathbb{R}$  and  $(\varphi_j)_{j \geq 1} \subset \mathcal{O}(s_j E, D)$ ,  $\varphi_j := (f_j, g_j)$ , such that

$$f_j, g_j \in \mathcal{O}(s_j E, \mathbb{C}), \quad \varphi_j(0) = (a, z_2), \varphi_j(1) = (a, w_2), \\ 1 < s_j < s_{j+1} \xrightarrow{j \rightarrow \infty} \infty.$$

Fix  $0 < \epsilon \ll \frac{1}{2} \min\{|a|, 1 - |a|\}$ . Put  $D_a := D \cap (\mathbb{B}_1(a, \epsilon) \times \mathbb{C})$ . Note that  $D \subset E \times \mathbb{C}$ . In virtue of Montel's Theorem, the family  $\mathcal{F} := \{f_j : j \geq 1\}$  is normally convergent in  $\mathcal{O}(\mathbb{C}, \bar{E})$ , i.e., there exists a sequence  $(f_{j_\nu})_{\nu \geq 1} \subset \mathcal{F}$  which converges uniformly on compact subsets to a map  $F \in \mathcal{O}(\mathbb{C}, \bar{E})$ . In particular,  $F(0) = a$ , so Liouville's theorem implies that  $F \equiv \text{constant} = a$ . Put  $K(j) := s_j \bar{E}$ ,  $j \geq 1$ . Then every  $K(j)$  is a compact subset of  $\mathbb{C}$ , hence for every  $j \geq 1$  there is a  $\nu_{K(j)} \in \mathbb{N}$  such that  $j_{\nu_{K(j)}} > j$  and  $f_{j_\nu}(s_j E) \subset f_{j_\nu}(K(j)) \subset \mathbb{B}_1(a, \epsilon)$ ,  $\nu \geq \nu_{K(j)}$ . Say  $\tilde{\varphi}_j := (\tilde{f}_j, \tilde{g}_j) := \varphi_{j_{\nu_{K(j)}}$  for  $j \geq 1$ . Then one has

$$\tilde{f}_j, \tilde{g}_j \in \mathcal{O}(s_j E, \mathbb{C}), \quad \tilde{\varphi}_j(0) = (a, z_2), \quad \tilde{\varphi}_j(1) = (a, w_2).$$

In this point of view, by taking a subsequence and renumbering if necessary, without loss of generality we may assume that  $f_j(s_j E) \subset \mathbb{B}_1(a, \epsilon)$  for any  $j \geq 1$ .

Choose a decreasing sequence  $(r_j)_{j \geq 2} \subset \mathbb{R}$  such that

$$\lim_{j \rightarrow \infty} r_j = 0, \quad r_j + r_{j+1} < \frac{1}{j(j+1)} \quad (j \geq 2), \quad \alpha := \sum_{k=2}^{\infty} \frac{\log r_k}{k^2} > -\infty.$$

Put  $\Omega_0 := \mathbb{B}_1(a, \epsilon) \times \mathbb{B}_1(0, e^{-\alpha})$ . For each  $j \geq 2$  we define

$$\Omega_j := \bigcup_{|a-x| < \epsilon} (\{x\} \times S_j^x),$$

where  $S_j^x := \{\zeta \in \mathbb{C} : |\frac{x}{\zeta} - \frac{1}{j}| < r_j\}$ . Obviously, every  $\Omega_j \subset \mathbb{C}^2$  is open and  $\Omega_j \cap \Omega_k = \emptyset$  for  $j \neq k$ . Moreover, we have

$$D_a \subset \Omega_0 \cup \left( \bigcup_{j=2}^{\infty} \Omega_j \right).$$

For this, it is enough to check that

$$(D_a \cap (\mathbb{C} \times \mathbb{C}^*)) \setminus \left( \bigcup_{j \geq 2} \Omega_j \right) \subset \mathbb{B}_1(a, \epsilon) \times \mathbb{B}_1(0, e^{-\alpha}).$$

More explicitly, let  $(x, \lambda) \in (D_a \cap (\mathbb{C} \times \mathbb{C})) \setminus \left( \bigcup_{j \geq 2} \Omega_j \right)$  with  $\lambda \neq 0$ . Then  $(x, \lambda) \in D$ ,  $x \in \mathbb{B}_1(a, \epsilon)$ , and  $\lambda \notin \left( \bigcup_{j \geq 2} S_j^x \right)$ , that is,  $|\frac{x}{\lambda} - \frac{1}{j}| \geq r_j$  for any

$j \geq 2$ . Moreover, one has

$$\begin{aligned} 1 > |\lambda| \exp\left(u\left(\frac{x}{\lambda}\right)\right) &\geq |\lambda| \exp\left(\sum_{k=2}^{\infty} \frac{1}{k^2} \log\left|\frac{x}{\lambda} - \frac{1}{k}\right|\right) \\ &\geq |\lambda| \exp\left(\sum_{k=2}^{\infty} \frac{1}{k^2} \log r_k\right), \end{aligned}$$

which implies that  $|\lambda| < \exp(-\alpha)$ .

Observe that

$$S_j^x \subset \left\{ \zeta \in \mathbb{C} : \frac{|x|}{\frac{1}{j} + r_j} < |\zeta| < \frac{|x|}{\frac{1}{j} - r_j} \right\}, \quad j \geq 2, x \in \mathbb{B}_1(a, \epsilon).$$

Hence, there is a number  $j_0 \geq 0$  with  $j_0 \neq 1$  such that  $\Omega_0 \cap \Omega_j \neq \emptyset$  ( $j \leq j_0$ ),  $\Omega_0 \cap \Omega_j = \emptyset$  ( $j > j_0$ ), and  $\{\zeta \in \mathbb{C} : \exists x \in \mathbb{C}, (x, \zeta) \in \tilde{\Omega}_0\} \subset \subset \mathbb{C}$ , where  $\tilde{\Omega}_0 := \Omega_0 \cup \left(\bigcup_{j=2}^{j_0} \Omega_j\right)$ . Moreover,

$$\tilde{\Omega}_0 \cap \left(\bigcup_{j>j_0} \Omega_j\right) = \emptyset, \quad D_a \subset \tilde{\Omega}_0 \cup \left(\bigcup_{j>j_0} \Omega_j\right).$$

On the other hand, since  $\varphi_j(s_j E)$  is connected in  $D_a$  for any  $j \geq 1$ , we obtain that

$$\text{either } \{(a, z_2), (a, w_2)\} \subset \bigcup_{j \geq 1} \varphi_j(s_j E) \subset \tilde{\Omega}_0$$

$$\text{or } \{(a, z_2), (a, w_2)\} \subset \bigcup_{j \geq 1} \varphi_j(s_j E) \subset \Omega_{j_a} \quad \text{for some } j_a \in \mathbb{N}, j_a > j_0.$$

Therefore, we can choose a number  $R > 0$  so large that  $g_j(s_j E) \subset \mathbb{B}_1(0, R)$  for  $j \geq 1$ . So, using Montel's theorem and Liouville's theorem, we can get that  $z_2 = w_2$ , as desired. The proposition is proved.  $\square$

#### §4. Proof of Theorem 2

In order to prove our theorem we will first show the following statement:

LEMMA 4. *Let  $a \in \mathbb{C} \setminus \{0\}$ ,  $b, c \in \mathbb{C}$  with  $b \neq c$ . For  $M_1 > 0$  and  $M_2 > |a|$ , there exists a Brody hyperbolic, pseudoconvex, balanced domain  $D$  in  $\mathbb{C}^3$  such that  $D \subset (M_1 E) \times (M_2 E) \times \mathbb{C}$  and  $\tilde{k}_D((0, a, b), (0, a, c)) = 0$ . In particular,  $D$  is not  $k$ -hyperbolic.*



*Proof.* Fix an increasing sequence  $(r_j)_{j \geq 1} \subset \mathbb{R}$  such that  $1 < r_j \rightarrow \infty$  as  $j \rightarrow \infty$  and  $\lim_{j \rightarrow \infty} \log(\log(r_j^2 + r_j))/\log j \in \mathbb{R}$ . For  $j \geq 1$  take a number  $s_j$  so that  $r_j(r_j + 1) < 1/s_j < 2r_j(r_j + 1)$ . Moreover, we define a sequence  $(\varphi_j)_{j \geq 1} \subset \mathcal{O}(r_j E, \mathbb{C}^3)$  of mappings  $\varphi_j =: (\varphi_j^1, \varphi_j^2, \varphi_j^3)$  by

$$\varphi_j^1(\lambda) := s_j \lambda(\lambda - 1), \quad \varphi_j^2(\lambda) := a, \quad \varphi_j^3(\lambda) := (c - b)\lambda + b, \quad \lambda \in r_j E$$

and set

$$Q_j(z) := z_1 z_2 - \frac{a s_j}{(c - b)^2} \left( z_3 - \frac{b}{a} z_2 \right) \left( z_3 - \frac{c}{a} z_2 \right), \quad z = (z_1, z_2, z_3) \in \mathbb{C}^3.$$

For each  $j \geq 1$ , put  $\epsilon_j := 2^{-j-1}$ ,  $t_j := \sqrt{j}/s_j$ , and  $\eta_j := t_j s_j$ . It is easy to see that

$$\sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{\eta_j} > -\infty, \quad \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} > -\infty.$$

Next, we define a function  $h : \mathbb{C}^n \rightarrow \mathbb{R}_{\geq 0}$  by

$$h(z) := \max \left\{ \frac{|z_1|}{M_1}, \frac{|z_2|}{M_2}, h_0(z) \right\}, \quad z = (z_1, z_2, z_3) \in \mathbb{C}^3,$$

where

$$h_0(z) := \prod_{j=1}^{\infty} \left( \frac{|Q_j(z)|}{\eta_j} \right)^{\epsilon_j} = \exp \left( \sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(z)|}{\eta_j} \right), \quad z \in \mathbb{C}^3.$$

Now we are going to show that the domain  $D = D_h := \{z \in \mathbb{C}^3 : h(z) < 1\}$  satisfies the required conditions that we claimed in Theorem 2.

1°.  $h_0$  is absolutely homogeneous on  $\mathbb{C}^3$ , so is  $h$ ; moreover,  $h^{-1}(0) = \{0\}$ .

*Subproof.* Let  $z \in \mathbb{C}^3$ ,  $\lambda \in \mathbb{C}$ . Clearly,  $|Q_j(\lambda z)| = |\lambda|^2 |Q_j(z)|$ ,  $j \geq 1$ , and also  $h_0(\lambda z) = |\lambda| h_0(z)$ . Hence,  $h_0$  is absolutely homogeneous, so is  $h$ .

Note that  $h^{-1}(0) \subset \{0\} \times \{0\} \times \mathbb{C}$ . Since

$$\sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(0, 0, \lambda)|}{\eta_j} = \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} + \frac{1}{2} \log \frac{|a||\lambda|^2}{|c - b|^2},$$

we have that  $h_0(0, 0, \lambda) = 0$  iff  $\lambda = 0$ , so  $h^{-1}(0) = \{0\}$ .

2°.  $h_0$  is plurisubharmonic in  $\mathbb{C}^3$ , so is  $h$ ; in particular,  $D$  is pseudoconvex.

Recall that the plurisubharmonicity is a local property and the limit of a decreasing sequence of plurisubharmonic functions is also plurisubharmonic.

*Subproof.* Fix  $\alpha > 0$ . Then for any  $z \in (\alpha E)^3$

$$|Q_j(z)| \leq \alpha^2 + s_j \underbrace{\frac{|a|\alpha^2}{|c-b|^2} \left( 1 + \frac{|b+c|}{|a|} + \frac{|bc|}{|a|^2} \right)}_{=: M_\alpha = M_\alpha(a,b,c) > 0}, \quad j \geq 1.$$

Since  $\lim_{j \rightarrow \infty} \eta_j = \infty$  and  $0 < s_j < 1$ , there is a number  $j_\alpha \in \mathbb{N}$  such that

$$\frac{|Q_j(z)|}{\eta_j} \leq \frac{1 + M_\alpha s_j}{\eta_j} \leq \frac{1 + M_\alpha}{\eta_j} < 1, \quad z \in \mathbb{B}_3(0, \alpha), \quad j \geq j_\alpha.$$

This implies that  $h_0 \in PSH(\mathbb{B}_3(0, \alpha))$ . Since  $\alpha > 0$  is arbitrary, one has  $h_0 \in PSH(\mathbb{C}^3)$  and also  $h \in PSH(\mathbb{C}^3)$ ; moreover,  $D$  is pseudoconvex.

3°.  $\tilde{k}_D((0, a, b), (0, a, c)) = 0$ , and so  $D$  is not  $\tilde{k}$ -hyperbolic; in particular,  $D$  is unbounded and not  $k$ -hyperbolic.

*Subproof.* It is easy to check that  $(Q_j \circ \varphi_j)(\lambda) = 0, \lambda \in r_j E, j \geq 1$ . This implies that  $\varphi_j(r_j E) \subset D$ , i.e.  $\varphi_j \in \mathcal{O}(r_j E, D)$  for any  $j \geq 1$ . In particular,  $\varphi_j(0) = (0, a, b)$  and  $\varphi_j(1) = (0, a, c)$ .

4°.  $D$  is Brody hyperbolic.

*Subproof.* Let  $f := (f_1, f_2, f_3) \in \mathcal{O}(\mathbb{C}, D)$ , where  $f_j \in \mathcal{O}(\mathbb{C}), j = 1, 2, 3$ . Since  $D \subset (M_1 E) \times (M_2 E) \times \mathbb{C}$ , it follows from Liouville's theorem that  $f_1 = \text{constant} =: \zeta_1$  and  $f_2 = \text{constant} =: \zeta_2$ . Suppose that  $f$  is not a constant, i.e.  $f_3$  is not a constant. By the Little Picard Theorem, there is an  $\lambda_0 \in \mathbb{C}$  with  $f_3(\mathbb{C}) \supset \mathbb{C} \setminus \{\lambda_0\}$ , so  $h(\zeta_1, \zeta_2, \cdot) < 1$  on  $\mathbb{C} \setminus \{\lambda_0\}$ . Then  $h(\zeta_1, \zeta_2, \cdot) < 1$  on  $\mathbb{C}$ ; in particular,  $h_0(\zeta_1, \zeta_2, \cdot) < 1$  on  $\mathbb{C}$ . Thus, by the Liouville type theorem for subharmonic functions, we conclude that  $h_0(\zeta_1, \zeta_2, \cdot) \equiv \text{constant}$  on  $\mathbb{C}$ . Observe that

$$h_0(\zeta_1, \zeta_2, \lambda) = 0$$

for any  $\lambda \in \mathbb{C}$  such that  $Q_j(\zeta_1, \zeta_2, \lambda) = 0$  for some  $j \geq 1$ . Therefore, in order to get a contradiction, it is enough to verify that for every  $(\zeta_1, \zeta_2) \in (M_1 E) \times (M_2 E)$  there is an  $\lambda = \lambda_{(\zeta_1, \zeta_2)} \in \mathbb{C}$  such that

$$\log h_0(\zeta_1, \zeta_2, \lambda) > -\infty.$$

To show this fix a point  $(\zeta_1, \zeta_2) \in (M_1 E) \times (M_2 E)$ . Now we shall discuss the following four cases.

(i) *Case  $\zeta_2 = 0$* : Obviously,

$$\log h_0(\zeta_1, 0, \lambda) = \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} + \frac{1}{2} \log \frac{|a||\lambda|^2}{|c-b|^2} > -\infty, \quad \lambda \neq 0.$$

(ii) *Case  $\zeta_1 = 0$  and  $\zeta_2 \neq 0$* : Take a point  $\lambda \in \mathbb{C} \setminus \{b\zeta_2/a, c\zeta_2/a\}$ . Then it is easy to check that

$$\log h_0(0, \zeta_2, \lambda) = \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} + \frac{1}{2} \log \frac{|a| |(\lambda - \frac{b}{a}\zeta_2)(\lambda - \frac{c}{a}\zeta_2)|}{|c-b|^2} > -\infty.$$

(iii) *Case  $\zeta_1\zeta_2 \neq 0$  and  $\frac{1}{s_j} \neq \frac{bc\zeta_2}{a(c-b)^2\zeta_1}$  ( $j \geq 1$ )*: Observe that

$$|Q_j(\zeta_1, \zeta_2, 0)| = \left| \zeta_1\zeta_2 - \frac{bc\zeta_2^2 s_j}{a(c-b)^2} \right| \geq |\zeta_1\zeta_2| - s_j \frac{|bc\zeta_2^2|}{|a||c-b|^2}, \quad j \geq 1.$$

Since  $\lim_{j \rightarrow \infty} s_j = 0$ , there is a number  $j_0 \in \mathbb{N}$  such that  $|Q_j(\zeta_1, \zeta_2, 0)| \geq \frac{1}{2}|\zeta_1\zeta_2| > 0$  for  $j \geq j_0$ . By our assumption,

$$Q_j(\zeta_1, \zeta_2, 0) = \zeta_2 \left( \zeta_1 - \frac{bc\zeta_2 s_j}{a(c-b)^2} \right) \neq 0, \quad j \geq 1,$$

which implies that

$$C := \sum_{j=1}^{j_0-1} \epsilon_j \log \frac{|Q_j(\zeta_1, \zeta_2, 0)|}{\eta_j} \in \mathbb{R}.$$

Note that  $\sum_{j=1}^{j_0-1} \epsilon_j \log(1/\eta_j) > -\infty$ . Therefore, we have

$$\begin{aligned} \log h_0(\zeta_1, \zeta_2, 0) &\geq \sum_{j=1}^{j_0-1} \epsilon_j \log \frac{|Q_j(\zeta_1, \zeta_2, 0)|}{\eta_j} + \sum_{j=j_0}^{\infty} \epsilon_j \log \frac{|\zeta_1\zeta_2|}{2\eta_j} \\ &= C + \left( \sum_{j=j_0}^{\infty} \epsilon_j \right) \log \frac{|\zeta_1\zeta_2|}{2} + \sum_{j=j_0}^{\infty} \epsilon_j \log \frac{1}{\eta_j} > -\infty. \end{aligned}$$

(iv) *Case  $\zeta_1\zeta_2 \neq 0$  and  $\frac{1}{s_j} = \frac{bc\zeta_2}{a(c-b)^2\zeta_1}$  for some  $j \geq 1$* : Note that, in this case,  $bc \neq 0$ . Take a point  $\lambda_0 \in \mathbb{C} \setminus \{0\}$  such that

$$|\zeta_1\zeta_2| > \frac{|\zeta_2|^2 |(\lambda_0 - b)(\lambda_0 - c)|}{|a||c-b|^2}.$$

Put  $\lambda := \frac{\zeta_2}{a}\lambda_0$ . Since  $0 < s_j < 1$  for  $j \geq 1$ , it is easy to check that  $Q_j(\zeta_1, \zeta_2, \lambda) \neq 0$ ,  $j \geq 1$ . In particular,

$$|Q_j(\zeta_1, \zeta_2, \lambda)| \geq \frac{|\zeta_1 \zeta_2|}{2}, \quad j \geq 1.$$

By the similar argument as in (iii), we may get that  $\log h_0(\zeta_1, \zeta_2, \lambda) > -\infty$ .

Thus we have the desired condition. This completes the proof of Lemma 4. □

To obtain the required example for  $n \geq 3$ , we can consider the balanced domain  $G := D \times E^{n-3} \subset \mathbb{C}^n$ , so the proof of Theorem 2 is now complete.

But, to find such an example in two-dimensional case, we can't use the same method above.

In the above construction, the value of  $\lim_{\lambda \rightarrow 0, \lambda \neq 0} h(z/\lambda, 1)/\|z/\lambda\|$  depends on the choices of  $z \in \mathbb{C}^2$ . More explicitly:

*Remark 5.* Keep the same notations as above. Clearly,  $|Q_j(\zeta, 0, 1)| = |a|s_j/|c - b|^2$  for  $\zeta \in \mathbb{C}$  and  $j \geq 1$ . Then for any  $z_1 \in \mathbb{C} \setminus \{0\}$ , one has

$$\lim_{\substack{\lambda \rightarrow 0 \\ \lambda \neq 0}} \frac{h_0(z_1/\lambda, 0, 1)}{|z_1/\lambda|} = \lim_{\substack{\lambda \rightarrow 0 \\ \lambda \neq 0}} |\lambda| \frac{\exp\left(\sum_{j=1}^{\infty} \epsilon_j \log \frac{|a|}{t_j |b-c|^2}\right)}{|z_1|} = 0,$$

and also  $\ell' := \lim_{\lambda \rightarrow 0, \lambda \neq 0} h(z_1/\lambda, 0, 1)/|z_1/\lambda| = 1/M_1$ . On the other hand, it is easy to check that

$$h_0\left(0, \frac{z_2}{\lambda}, 1\right) = \frac{1}{|\lambda|} \exp\left(\sum_{j=1}^{\infty} \epsilon_j \log \frac{|a| \left|(\lambda - \frac{b}{a}z_2)(\lambda - \frac{c}{a}z_2)\right|}{t_j |c - b|^2}\right),$$

$z_2 \neq 0, \lambda \neq 0,$

and also

$$\lim_{\substack{\lambda \rightarrow 0 \\ \lambda \neq 0}} \frac{h_0(0, z_2/\lambda, 1)}{|z_2/\lambda|} = \frac{\sqrt{|bc|}}{\sqrt{|a||c - b|}} \exp\left(\sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j}\right) =: M_3 \geq 0.$$

Here,  $M_3$  is positive for  $bc \neq 0$ . Therefore,  $\ell'' := \lim_{\lambda \rightarrow 0, \lambda \neq 0} h(0, z_2/\lambda, 1)/|z_2/\lambda| = \max\{1/M_2, M_3\}$ , so we have, in general,  $\ell' \neq \ell''$ .

Thus, the balanced domain  $D$  constructed in above is, in general, not of the type studied in Example 3.

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