S. Moroianu
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HOMOLOGY AND RESIDUES OF ADIABATIC PSEUDODIFFERENTIAL OPERATORS

SERGIU MOROIANU

Abstract. We compute the Hochschild homology groups of the adiabatic algebra $\Psi_a(X)$, a deformation of the algebra of pseudodifferential operators $\Psi(X)$ when X is the total space of a fibration of closed manifolds. We deduce the existence and uniqueness of traces on $\Psi_a(X)$ and some of its ideals and quotients, in the spirit of the noncommutative residue of Wodzicki and Guillemin. We introduce certain higher homological versions of the residue trace. When the base of the fibration is S^1, these functionals are related to the η function of Atiyah-Patodi-Singer.

§1. Introduction

The interplay between analytic properties of manifolds and algebras associated to them is a fruitful topic of current research. Indeed, appropriate algebraic constructions lead to various analytical objects like de Rham cohomology [7], the index of elliptic operators, [17], [11], the residue trace [17] and the η-invariant [18]. One may even speculate that analysis on manifolds could in principle be reduced to "formal" analysis in the spirit of [22].

In the present paper we study algebraic properties of the algebra $\Psi_a(X)$ of adiabatic pseudodifferential operators associated to a fibration $X \to M$ of closed manifolds, in particular we compute its (topological) Hochschild homology groups. Mazzeo and Melrose [13] have used this algebra in their study of adiabatic families of Riemannian metrics, i.e., time-dependent families of metrics on the total space X such that the base directions blow up at $t = 0$:

$$g_t = dv^2 + \frac{dh^2}{t^2}.$$

The algebra $\Psi_a(X)$ can also be obtained from the methods of [24]. In particular, if X has a spin structure and $M = S^1$, it turns out that the

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family D_t of Dirac operators corresponding to the metric g_t is an adiabatic family of differential operators. In [20], [21] we use the adiabatic algebra and some results of this paper to study adiabatic limits of the eta functions of any elliptic adiabatic family of first order differential operators, extending the holonomy formula of Witten [27] and Bismut-Freed [3], whose proof only holds for Dirac operators.

The definition of the algebra $\Psi_a(X)$ is lengthy (see Section 2) so let us try to give in this introduction the simple idea behind it. Let t be a parameter in $\mathbb{R}_+ =: [0, \infty)$, x local coordinates on the base M, y local coordinates on the fiber F. Intuitively, the algebra $\Psi_a(X)$ associated to the fibration $X \to M$ is made of pseudodifferential operators of the form $a(t, x, y, tD_x, D_y)$, where a is a classical symbol in the last two variables, smooth in the parameters t, x, y for $t \geq 0$. For fixed $t > 0$ we recover the usual algebra of classical pseudodifferential operators on X. The adiabatic family of Dirac operators discussed above is of this form, and one hopes (as it is indeed the case) that one can reasonably invert it inside the adiabatic algebra. The main point is that D_0 is no longer elliptic, hence classical elliptic analysis fails. Therefore, the adiabatic algebra gives a better understanding of families of (pseudo)differential operators like D_t in the degenerate limit $t \to 0$.

For reasons explained below, we adjoin the inverse of t to the algebra, thus admitting Laurent expansions in t at $t = 0$. We will denote by $\Psi_a^{i,j}(X)$ the space of adiabatic operators of order i and vanishing at least to order $-j$ at $t = 0$. The second index will be omitted when it is not restricted or it is clear from the context. Thus for instance, $\Psi_a^{-\infty}(X)$ denotes the ideal of those adiabatic operators which are smoothing for each $t > 0$.

To give an outline of this work, recall first the description by Melrose and Nistor [17] of the Fredholm index map as the boundary map in Hochschild homology arising from the short exact sequence

$$0 \to \Psi^{-\infty}(X) \to \Psi(X) \to S(T^*X) \to 0,$$

where $\Psi(X)$ is the algebra of classical pseudodifferential operators on X, and $S(X)$ is defined by this sequence. More precisely, $HH(\Psi^{-\infty}(X))$ is concentrated in dimension 0 and isomorphic to $\mathbb{C}$ through the trace map. Given an elliptic operator A, they constructed a class in $HH_1(S(X))$ which maps through the boundary map to the index of A.

We consider the analogous short exact sequence in the adiabatic setting

$$0 \to \Psi_a^{-\infty}(X) \to \Psi_a(X) \to A_\sigma(X) \to 0. \tag{1}$$

In Section 4 we compute the Hochschild homology of the total symbol algebra $A_\sigma := \Psi_a(X)/\Psi_a^{-\infty}(X)$, by using the spectral sequence argument from [5]. The principal symbols of adiabatic operators live naturally on a modified cotangent bundle ${}^a T^*X \to X \times \mathbb{R}_+$. There is a natural map ϕ_a from $T^*X \times \mathbb{R}_+$ to ${}^a T^*X$ which is an isomorphism for $t > 0$ but fails to be an isomorphism at $t = 0$. To compute the first differential d_1 in the spectral sequence, we replace the symplectic duality operator $*$ of Brylinski [4] by a (non-canonical) duality $*_a$ on differential forms on ${}^a T^*X$, obtained from $\phi_a^{-1*} * \phi_a^*$ by means of a connection ∇ in $X \to M$. A completely non-obvious result is the convergence of the spectral sequence.

THEOREM 1.1. *The homology groups of the total symbol algebra $A_\sigma(X)$ are*

$$HH_k(A_\sigma(X)) \cong \Big(H^{2N-k}(S^*X \times S^1) \oplus H^{2N+1-k}(S^*X \times S^1)\Big) \otimes C^\infty(\mathbb{R}_+)[t^{-1}].$$

This result is similar to that of Brylinski and Getzler [5]. Note that the theorem would fail if we did not adjoin t^{-1}, since the operator $*_a$ involves negative powers of t. In particular, the residue trace for positive t extends to $A_\sigma(X)$ as a trace functional singular in t at $t = 0$.

A remarkable difference from the case of the algebra $\Psi(X)$ is that now the "smoothing" ideal $\Psi_a^{-\infty}(X)$ has homology in dimensions from 0 to $n+1$. In dimension 0, the homology of $\Psi_a^{-\infty}(X)$ is 1-dimensional as a $C^\infty(\mathbb{R}_+)[t^{-1}]$-module. In higher dimensions it is concentrated near $t = 0$, in other words it is inherited from the quotient $I_\partial(X) := \Psi_a^{-\infty}(X)/\Psi_a^{-\infty,-\infty}(X)$. This quotient is a $\mathcal{L}$-vector space, where $\mathcal{L}$ is the field $\mathbb{C}[[t, t^{-1}]$ of formal Laurent series. The homology of $I_\partial(X)$ is computed in Sections 5 and 6 by using the spectral sequence ${}^\partial E$ with respect to the filtration by the powers of t.

THEOREM 1.2. *The homology of $I_\partial(X) = \Psi_a^{-\infty}(X)/\Psi_a^{-\infty,-\infty}(X)$ is*

$$HH_k(I_\partial(X)) \cong \Big(H^{n-k}(M, \mathcal{O}M) \oplus H^{n-k+1}(M, \mathcal{O}M)\Big) \otimes \mathcal{L}$$

where $\mathcal{O}M$ is the orientation bundle of M. Moreover,

$$HH_k(\Psi_a^{-\infty}(X)) \cong \begin{cases} HH_k(I_\partial(X)) & \text{if } k \geq 2, \\ H^{n-1}(M, \mathcal{O}M) \otimes \mathcal{L} \oplus C^\infty(\mathbb{R}_+)[t^{-1}] & \text{if } k = 1, \\ C^\infty(\mathbb{R}_+)[t^{-1}] & \text{if } k = 0, \end{cases}$$

and the map induced in homology by the quotient map $\Psi_a^{-\infty}(X) \to I_\partial(X)$ *is the obvious surjection.*

The first isomorphism is realized by the map $*_a\chi_a$, where χ_a is a noncommutative analog of the Hochschild-Kostant-Rosenberg map χ defined on Hochschild chains by:

$$c_0 \otimes \ldots \otimes c_k \overset{\chi_a}{\mapsto} \mathrm{Tr_V}(\hat{c}_0 \nabla^t \hat{c}_1 \wedge \ldots \wedge \nabla^t \hat{c}_k). \tag{2}$$

The right-hand side will be explained in Section 6; for the moment, let us only say that $\mathrm{Tr_V}$ is the fiberwise trace applied to a family of suspended operators in the sense of Melrose [16], ∇^t is a "covariant derivative" which differentiates suspended operators in horizontal directions, and the output is a $(2n-k)$-form on T^*M whose coefficients are Schwartz along the fibers. We deduce the existence of a unique continuous trace on $I_\partial(X)$ (up to multiplication by $\mathcal{L}$), which moreover extends the usual operator trace on $\Psi_a^{-\infty}(X)$ for $t > 0$.

The ideal $\Psi_a^{-\infty}$ is H-unital in the sense of [28]. This technical condition is actually fulfilled by all the ideals arising in this paper and we will not mention it further; a detailed proof of this fact is given in [19]. The interested reader should also consult [2] for more on this subject. Thus, there exists a long exact sequence of Hochschild homology groups induced from the short exact sequence (1) and in particular of boundary maps

$$\delta : HH_k(A_\sigma(X)) \to HH_{k-1}(\Psi_a^{-\infty}(X)).$$

Section 7 contains the second type of results of this paper, based on the homological computations from the previous sections. We first describe the boundary maps δ in terms of our identifications of Hochschild homology.

THEOREM 1.3. *The composition*

$$HH_k(A_\sigma(X)) \xrightarrow{\delta} HH_{k-1}(\Psi_a^{-\infty}(X)) \to HH_{k-1}(I_\partial(X))$$

is given in terms of the isomorphisms from Theorems 1.1 and 1.2 by integration along the fibers of $S^*X \to M$ *in the first factor, and by the Laurent expansion map* $C^\infty(\mathbb{R}_+)[t^{-1}] \to \mathcal{L}$ *in the second factor.*

To prove this, we introduce certain higher analogs of the residue trace, with values in $\Lambda^*(M, \mathcal{O}M)$. Namely, let $c = c_0 \otimes \ldots \otimes c_k \in C_k(\Psi_a(X))$ and define

$$R(c) = \mathrm{Res}_{z=0} \int_{{}^aT^*M_{|t=0}/M} *_a\chi_a(Q^z c) \tag{3}$$

where Q is a positive adiabatic operator of order 1. We prove that R is the first component (in the filtration by the order of vanishing at $t = 0$) of some Hochschild cochain. This means that while we restrict the order of vanishing, we allow arbitrary operator orders in c, as in the zeta-function definition of the residue trace.

THEOREM 1.4. *The map*

$$R : C_k(\Psi_a(X)) \to \Lambda^{n-k}(M, \mathcal{O}M)$$

descends to $C_k(\Psi_a(X)/\Psi_a^{-\infty}(X))$ and has the following properties:

(1) *If $c \in C_k(\Psi_a(X)/\Psi_a^{-\infty}(X))$ is Hochschild exact, then $R(c)$ is de Rham exact.*

(2) *If $c \in C_k(\Psi_a(X)/\Psi_a^{-\infty}(X))$ is Hochschild closed, then $R(c)$ is de Rham closed.*

Therefore R descends to Hochschild homology, with values in the de Rham cohomology of the base M. For comparison, the only such explicit cochain known in $\Psi(X)$ is the residue trace, in dimension 0. If $M = \{*\}$ then $\Psi_a(X) \cong C^\infty(\mathbb{R}_+, \Psi(X))$, and our residues are only defined in dimensions 0 and 1. In that case both of them are variants of the residue trace.

Finally, in Section 8 we examine the action of our functionals on the Melrose-Nistor cycle $\mathrm{tr}(A \otimes B)$, where $A \in \Psi_a(X)$ is elliptic, B is a parametrix of A and tr is the generalized trace functional [12]. We recover essentially the residue of the eta function of a self-adjoint pseudodifferential operator on a closed manifold (this residue is known to vanish by the results of [1] and [8]). Thus the residues R encode highly non-trivial analytic information, and will hopefully find other interesting applications.

NOTATION. All symbols in this paper are classical (i.e., one-step polyhomogeneous) of possibly complex order. We denote by $S_{\mathrm{cl}}(V)$ the classical symbols, by $\mathcal{S}(V)$ the Schwartz (rapidly vanishing) functions, and by $S(V) := S_{\mathrm{cl}}(V)/\mathcal{S}(V)$ the formal symbols on the total space of a vector bundle $V \to X$. The superscript i in a filtered algebra denotes objects in filtration order i, while $[i]$ denotes homogeneous degree i in the associated graded algebra. For instance,

$$S^z(V) \cong \prod_{j \in \mathbb{N}} S^{[z-j]}(V),$$

the space of formal sums of homogeneous functions of homogeneity bounded above on the complement of the zero-section.

The field $\mathbb{C}[[t, t^{-1}]$ of formal Laurent series is denoted by $\mathcal{L}$.

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§2. Overview of the adiabatic algebra

2.1. The adiabatic algebra

The construction of the adiabatic algebra is based on the general formalism from [15]. It is a particular case of an algebra constructed in [13], and can also be defined as in [24]. For simplicity we describe it in the scalar case, noting that it extends easily to operators between sections of vector bundles.

Let $X^{m+n} \xrightarrow{\pi} M^n$ be a fibration of compact manifolds. The adiabatic limit of this geometric data means exploding the base while keeping the fibers fixed, so that the points in the base become disconnected, at least in a first approximation. From this point of view, the underlying space of the adiabatic limit as $t \to 0$ is the product $X \times \mathbb{R}_+$. The central idea in Melrose's approach is that the limiting value $t = 0$, where the interesting phenomena occur, is of the same nature as the positive values of t, provided one works in the appropriate "adiabatic category".

Denote by TX/M the vertical sub-bundle of TX. A smooth family $v : \mathbb{R}_+ \to \Gamma(X, TX)$ of vector fields is called *adiabatic* if $v(0) \in \Gamma(X, TX/M)$. Thus, an adiabatic vector field is a family of vector fields on X which in the limit does not "see" the base. Mazzeo and Melrose [13] remarked that the adiabatic vector fields are the sections of a vector bundle over $X \times \mathbb{R}_+$, called ${}^{a}TX$. If $X = F \times M \to M$ is a trivial fibration, then ${}^{a}TX$ is canonically isomorphic to $TX \times \mathbb{R}_+$. In general, the vector bundles $TX \times \mathbb{R}_+$ and ${}^{a}TX$ are *non-canonically* isomorphic as bundles over $X \times \mathbb{R}_+$. When $X = M$ we denote by ${}^{a}TM$ the adiabatic tangent bundle of the identity fibration.

Definition 2.1. Let $\phi^a : {}^{a}TX \to TX \times \mathbb{R}_+$ be the tautological map of vector bundles which transforms a section of ${}^{a}TX$ into itself, viewed as a time-dependent section of TX.

Denote by $\phi_a : T^*X \times \mathbb{R}_+ \to {}^aT^*X$ the dual map.

Following [15], [13] we construct an algebra of pseudodifferential operators in which the adiabatic vector fields are the differential operators of order exactly 1. This algebra is a space of conormal distributions on a radial blow-up of $X \times X \times \mathbb{R}_+$. More precisely, let $X \times_M X$ be the fiber diagonal in $X \times X$. Let S^+ be the positive half-sphere of the normal bundle $N(X \times_M X \times \{0\})$ inside $X^2 \times \mathbb{R}_+$.

Definition 2.2. Let

$$X_a^2 = [X^2 \times \mathbb{R}_+; X \times_M X \times \{0\}].$$

As a set, this is $(X^2 \times \mathbb{R}_+ \setminus X \times_M X \times \{0\}) \cup S^+$. The *front face* of the blow-up, denoted ff, is just S^+. The set X_a^2 has a natural C^∞ structure, defined by gluing the two parts along the normal geodesic flow of some Riemannian metric. As manifold with corners, it has one boundary component of codimension 1 if $n = 0$, three if $n = 1$ and two otherwise.

There exists a natural blow-down map $\beta : X_a^2 \to X^2 \times \mathbb{R}_+$. Let Δ be the diagonal in X^2. We define Δ_a, the *lifted diagonal*, to be the closure in X_a^2 of $\beta^{-1}(\Delta \times (0, \infty))$. Note that Δ_a is transversal to ff. Let $2X_a^2$ be the double of X_a^2 across the front face. By definition, a distribution on X_a^2, conormal to the lifted diagonal, is *smooth to the front face* if it is the restriction of a distribution on $2X_a^2$, conormal to $2\Delta_a$.

Let $\Omega_a \to X_a^2$ be the pull-back via $\pi_R \circ \beta$ of the density bundle $\Omega = \Omega^{m+n}({}^aT^*X)$, where $\pi_R : X^2 \to X$ is the projection on the second factor.

The set of adiabatic differential operators (defined as compositions of adiabatic vector fields and multiplication operators) corresponds via Schwartz's kernel theorem to the space of Ω_a-valued distributions on X_a^2 which are supported on the lifted diagonal and extendible across the front face. In light of this fact, let $\Psi_a^{i,0}(X)$ be the space of those Ω_a-valued distributions on X_a^2 which are classical (i.e., 1-step poly-homogeneous) conormal to Δ_a of order i, vanish rapidly to the boundary faces of X_a^2 other than ff, and are extendible across ff. This space forms a module over $C^\infty(\mathbb{R}_+)$. For $j \in \mathbb{C}, i \in \mathbb{R}$ define further $\Psi_a^{i,j}(X) := t^{-j}\Psi_a^{i,0}(X)$.

Definition 2.3. The adiabatic algebra of $X \to M$ is defined by

$$\Psi_a^{\mathbb{Z},\mathbb{Z}}(X) := \bigcup_{i,j \in \mathbb{Z}} \Psi_a^{i,j}(X).$$

The fiber of the natural projection $X_a^2 \to \mathbb{R}_+$ for $t > 0$ is just X^2, hence transverse to Δ_a and therefore the restriction of adiabatic operators to such a fiber is well-defined. Clearly, such a restriction gives the Schwartz kernel of a classical pseudodifferential operator on X. Thus for all $t > 0$ we get a map from $\Psi_a^{\mathbb{Z},\mathbb{Z}}(X)$ into $\Psi^{\mathbb{Z}}(X)$.

The algebra structure on $\Psi_a^{\mathbb{Z},\mathbb{Z}}(X)$ is given by the following theorem.

THEOREM 2.4. *The space $\Psi_a^{\mathbb{Z},\mathbb{Z}}(X)$ has a $C^\infty(\mathbb{R}_+)[t^{-1}]$-algebra structure which induces the usual algebra structure on $\Psi^{\mathbb{Z}}(X)$ for every fixed $t > 0$.*

Proof. The composition rule is defined by constructing a triple blow-up space X_a^3, such that there exist three maps π_L, π_C, π_R to X_a^2 which are b-submersions [15]. This space is defined as the iterated blow-up

$$X_a^3 := [X^3 \times \mathbb{R}_+; \Delta_3 \times \{0\}, D_{1,2} \times \{0\}, D_{1,3} \times \{0\}, D_{2,3} \times \{0\}]$$

where Δ_3 is the triple diagonal, and $D_{a,b}$ is the fiber-diagonal in X^3 in the components $a, b \in \{1, 2, 3\}$. Using the properties of conormal distributions, set

$$A \circ B := (\pi_C)_*((\pi_R^* A)(\pi_L^* B)). \tag{4}$$

Associativity follows from the action of $\Psi_a^{\mathbb{Z},\mathbb{Z}}(X)$ on $C^\infty(X \times \mathbb{R}_+)$. Alternately, it also follows from the associativity of multiplication on $\Psi^{\mathbb{Z}}(X)$.

For $\mathcal{E}, \mathcal{F}$ vector bundles over X, define $\Psi_a^{i,j}(X, \mathcal{E}, \mathcal{F})$ by considering distributional kernels as above with values in $\Omega_a \otimes \mathcal{E} \boxtimes \mathcal{F}$. This is again an algebra if $\mathcal{E} = \mathcal{F}$. By Morita equivalence, for homology purposes it is enough to consider scalar operators.

2.2. Ideals, quotients and filtrations

The algebra $\Psi_a^{\mathbb{Z},\mathbb{Z}}(X)$ has two increasing filtrations. The first superscript specifies the order of singularity at the lifted diagonal. The ideal $\Psi_a^{-\infty,\mathbb{Z}}(X)$ is called the *smoothing ideal* by abuse of notation; an operator in this ideal has smooth Schwartz kernel and is actually smoothing for fixed $t > 0$ as an operator on X. while the quotient $A_\sigma(X) := \Psi_a^{\mathbb{Z},\mathbb{Z}}(X)/\Psi_a^{-\infty,\mathbb{Z}}(X)$ is called the *adiabatic symbol algebra.*

The second superscript denotes the negative of the order of vanishing at $t = 0$, so both filtrations are increasing. Let $A_\partial(X) := \Psi_a^{\mathbb{Z},\mathbb{Z}}(X)/\Psi_a^{\mathbb{Z},-\infty}(X)$ be the *boundary algebra.*

We will use the notation F_i for the first filtration, and T_j for the second.

Let $I_\partial(X) := \Psi_a^{-\infty,\mathbb{Z}}(X)/\Psi_a^{-\infty,-\infty}(X)$ be the *boundary ideal.* Let $A_{\partial,\sigma}(X) := \Psi_a^{\mathbb{Z},\mathbb{Z}}(X)/(\Psi_a^{-\infty,\mathbb{Z}}(X) + \Psi_a^{\mathbb{Z},-\infty}(X))$ be the *boundary symbol algebra.* The following diagram summarizes the different ideals and quotients:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Psi_a^{-\infty,-\infty}(X) \arrow[r] \arrow[d] & \Psi_a^{\mathbb{Z},-\infty}(X) \arrow[r] \arrow[d] & I_\sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Psi_a^{-\infty,\mathbb{Z}}(X) \arrow[r] \arrow[d] & \Psi_a(X) \arrow[r] \arrow[d] & A_\sigma(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_\partial(X) \arrow[r] \arrow[d] & A_\partial(X) \arrow[r] \arrow[d] & A_{\partial,\sigma}(X) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{5}$$

The horizontal and vertical sequences are exact.

2.3. The description of the product

PROPOSITION 2.5. *The adiabatic structure map ϕ_a induces a canonical splitting*

$$ {}^aTX_{|t=0} \cong TX/M \oplus \pi^{*a}TM_{|t=0}.$$

Proof. There exists a tautological inclusion map $TX/M \times \mathbb{R}_+ \to {}^aTX$, which views a time-dependent family of vertical vector fields as an adiabatic family. Hence TX/M is a subspace of ${}^aTX_{|t=0}$. The null-space of $\phi^a_{|t=0}$, π^*TM and $\pi^{*a}TM_{|t=0}$ are canonically isomorphic, and form a complement of TX/M in ${}^aTX_{|t=0}$.

Hence, ${}^aT^*X_{|t=0}$ is canonically isomorphic to $(TX/M)^* \oplus \pi^{*a}T^*M_{|t=0}$.

From the definition of the blow-up, it follows immediately that the normal bundle to the lifted diagonal in X_a^2 is canonically isomorphic to ${}^aTX$. In other words, X_a^2 resolves the degeneracy of adiabatic vector fields.

PROPOSITION 2.6. *The interior of* ff *is naturally diffeomorphic to the lift of ${}^aTM$ to $X \times_M X$.*

Proof. By definition, $\mathrm{ff}^\circ \to X \times_M X$ is a fiber bundle with contractible fibers. Define the 0-section as the class of the tangent vector ∂_t on X_a^2. Take an adiabatic vector field v which lives in ${}^aTM$ at $t=0$ (see Proposition 2.5) and lift it via $\pi_R \circ \beta$ to X_a^2. The restriction of the lift $\tilde{v}$ to the front face is tangent to the fibers of ff and depends only on the restriction of v at $t=0$. Integrating such vector fields gives the desired identification.

Thus ff° has a natural vector bundle structure over $X \times_M X$.

The product structure on $\Psi_a(X)$ is given by (4), which is rather inexplicit. We have more explicit descriptions of the product on $A_\sigma(X)$, $I_\partial(X)$ and $\Psi_a^{-\infty,-\infty}(X)$. First, $\Psi_a^{-\infty,-\infty}(X)$ is isomorphic to the space of rapidly vanishing families of smoothing operators on X endowed with the standard product.

Let $A_\sigma^{[i],\mathbb{Z}} := A_\sigma^{i,\mathbb{Z}}/A_\sigma^{i-1,\mathbb{Z}}$ be the filtration quotients of $A_\sigma(X)$. As in the standard case, the associated graded algebra $A_\sigma^{[\mathbb{Z}],\mathbb{Z}}$ is isomorphic to the algebra of homogeneous symbols on the adiabatic cotangent bundle ${}^aT^*X$.

Let us fix an *adiabatic quantization*, i.e., a diffeomorphism from an open subset of the radial compactification $\overline{{}^aTX}$ to a subset of X_a^2, which

(1) extends the diffeomorphism from the zero section to the lifted diagonal;

(2) induces the identity map between the normal bundle to the zero section and the normal bundle to Δ_a (they are both canonically isomorphic to ${}^aTX$);

(3) extends the natural diffeomorphism from $\pi^*TM_{|t=0}$ to $\mathrm{ff}^\circ_{\mathrm{ff}\cap\Delta_a}$;

(4) induces the identity map on the normal bundle to these two spaces (they are both canonically isomorphic to TX/M).

This choice allows us to find a vector space isomorphism called *total symbol map*

$$A_\sigma(X) \to S({}^aT^*X)[t^{-1}] \tag{6}$$

in the following way: take an adiabatic operator, cut it off near $\mathrm{ff} \cup \pi^*TM_{|t=0}$, pull it back via the quantization and apply Fourier transform in the fibers of ${}^aTX$. Only the leading component of the resulting symbol is independent of the choices made.

Let us define the *adiabatic Poisson bracket* $\{\ ,\ \}_t$ as the push-forward of the Poisson bracket on T^*X under the canonical map ϕ_a.

Proposition 2.7. *For any choice of adiabatic quantization, the isomorphism* (6) *induces the canonical isomorphism of graded algebras*

$$A_\sigma^{[\mathbb{Z}],\mathbb{Z}}(X) \cong S^{[\mathbb{Z}]}({}^aT^*X)[t^{-1}].$$

Moreover, the product induced on $S({}^aT^*X)[t^{-1}]$ *by the map* (6) *is a smooth t-dependent* star product, *i.e., it is given by a series of bi-differential operators*

$$* = \sum_{i\geq 0} P_i(t) \tag{7}$$

such that $P_0(t)(a,b) = ab$, $P_1(t)(a,b) - P_1(t)(b,a) = \{a,b\}_t$, *and* P_i *is of homogeneity* $-i$ *in the cotangent fibers.*

Proof. The first claim follows directly from the construction of the total symbol map, which starts with the principal symbol. The product structure on $\Psi_a(X)$ is local in t since $C^\infty(\mathbb{R}_+)[t^{-1}]$ is central. Therefore, for any $t > 0$ the statement follows from the corresponding result for $\Psi^{\mathbb{Z}}(X)$. Since for every i, the product map $\Psi_a^{i,i} \otimes \Psi_a^{i,i} \to \Psi_a^{2i,2i}$ is continuous, it follows that $P_i(t)$ extends down to $t = 0$ as a smooth differential operator, and that the identities hold down to $t = 0$.

Consider now the case where $X = M$. Adiabatic limits in this setting are called *semi-classical limits.* Let $\mathcal{L} = \mathbb{C}[[t]][t^{-1}]$ denote the field of formal Laurent series in the variable t. An adiabatic quantization map gives now vector space isomorphisms

$$A_\partial(M) \cong S_{\mathrm{cl}}({}^aTM_{|t=0})\hat{\otimes}\mathcal{L} \qquad I_\partial(M) \cong \mathcal{S}({}^aTM_{|t=0})\hat{\otimes}\mathcal{L} \tag{8}$$

(the hat means that we allow infinite sums of tensors, provided that they are finite, or equivalently convergent, in each degree). The first isomorphism from (8) is compatible with (6) in the following sense: for an operator $A \in \Psi_a(M)$, we get a formal symbols with Laurent coefficients over $\mathbb{R}_+$ from (6), and a Laurent series of classical symbols from (8); these two objects must map to the same formal series in $S({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}$. One may ask how this compatibility translates in terms of the product. The answer is given below.

Proposition 2.8. *Let* $M \to M$ *be the identity fibration. The product induced on* $S_{\mathrm{cl}}({}^aTM_{|t=0})\hat{\otimes}\mathcal{L}$ *by* (8) *is given by the same formula as the product* (7) *on formal symbols, specialized to the case* $X = M$.

Proof. In the semi-classical limit algebra, unlike the general adiabatic case, it is not hard to see that the product on $A_\partial(M)$ is local, hence given by a star-product. The induced product on formal symbols is by definition (7); moreover the bi-differential operators appearing in the product have polynomial coefficients, thus we can recover them from their action on formal symbols.

In general, the ideal $I_\partial(X)$ is a module over $I_\partial(M)$, generated by the algebra of smoothing operators in the fibers of $X \to M$.

A Taylor coefficient at $t = 0$ of an element in $\Psi_a(X)$ becomes, after Fourier transform in the fibers of ${}^aTM_{|t=0}$, a classical symbol on ${}^aT^*M_{|t=0}$ taking values in the space of classical pseudodifferential operators on the fibers of X. Note however that adiabatic operators have joint symbolic behavior in the fibers and in the base. Then the product on Laurent series at $t = 0$ is a combination of a star-product on ${}^aT^*M$ and of the usual convolution product in the fibers.

Let $\Psi_a^{\mathbb{Z},[0]}(X)$ be the *vertical algebra* of $\Psi_a(X)$,

$$\Psi_a^{\mathbb{Z},[0]}(X) := \Psi_a^{\mathbb{Z},0}(X)/\Psi_a^{\mathbb{Z},1}(X).$$

The algebra $\Psi_a^{\mathbb{Z},[0]}(X)$ is a fibered version of the suspended algebra [16], [18]. Let $\Psi_a^{-\infty,[0]}(X) := I_\partial^{\mathbb{Z},0}(X)/I_\partial^{\mathbb{Z},1}(X) \hookrightarrow \Psi_a^{\mathbb{Z},[0]}(X)$ be the *vertical smoothing ideal* associated to $I_\partial(X)$. An element in $\Psi_a^{-\infty,[0]}(X)$ is the restriction to the front face of the Schwartz kernel of a non-singular smoothing adiabatic operator. On ff, the density bundle Ω_a decomposes as $\Omega_R \otimes \Omega({}^aTX)$, where Ω_R is the bundle of densities in the second F factor. We identify such a restriction with its Fourier transform in the fibers of ${}^aTM_{|t=0}$. Hence by Fourier transform,

$$\Psi_a^{-\infty,[0]}(X) \xrightarrow{\cong} \mathcal{S}(X \times_M X \times_M {}^aT^*X, \Omega_R).$$

Note that the isomorphisms (8) induce, by passage to the associated graded objects, canonical isomorphisms

$$\begin{aligned}
\Psi_a^{\mathbb{Z},[0]}(M) &\cong S_{\mathrm{cl}}({}^aT^*M),\\
\Psi_a^{-\infty,[0]}(M) &\cong \mathcal{S}({}^aT^*M),\\
\Psi_a^{[\mathbb{Z}],[0]}(M) &\cong S^{[\mathbb{Z}]}({}^aT^*M).
\end{aligned}$$

§3. Hochschild homology and derivations

3.1. Hochschild Homology

Let A be an unital algebra over $\mathbb{C}$. Let $C_i(A) = A^{\otimes i+1}$ be the space of *Hochschild chains*. For $j = 0, \ldots, i$, define $b_j : C_i(A) \to C_{i-1}(A)$ by

$$a_0 \otimes \ldots \otimes a_j \otimes a_{j+1} \otimes \ldots \otimes a_i \stackrel{b_j}{\mapsto} a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_i$$
$$a_0 \otimes \ldots \otimes a_i \stackrel{b_i}{\mapsto} a_i a_0 \otimes \ldots \otimes a_{i-1}.$$

Define the *Hochschild boundary map* by $b = \sum_{j=0}^{i}(-1)^j b_j$. The homology of the complex (C_*, b) is the *Hochschild homology* of the algebra A (relative to A).

If A is not unital, let $\bar{A}$ be the augmented algebra $A \oplus \mathbb{C}$. We define the chain spaces of A by $C_k(A) = \ker(\bar{A}^{\otimes k+1} \to \mathbb{C}^{\otimes k+1})$. This definition, applied to an unital algebra, gives different chain spaces from the ones above. Nevertheless, there are canonical chain maps which induce isomorphisms on homology.

Benameur-Nistor [2] defined the Hochschild chains for a large class of algebras with topology. We use these chain spaces for the adiabatic algebra. Like the usual algebra of pseudodifferential operators, this is not a topological algebra, in the sense that the product is not jointly continuous. This is however only a minor problem. For every $j_1, j_2, k_1, k_2 \in \mathbb{C} \cup \{-\infty\}$, the multiplication map

$$\Psi_a^{j_1,k_1}(X) \otimes \Psi_a^{j_2,k_2}(X) \to \Psi_a^{j_1+j_2,k_1+k_2}(X)$$

is continuous with respect to natural Fréchet topologies.

The two filtrations by the order, respectively by the negative of the vanishing order at $t = 0$, are compatible with multiplication in $\Psi_a(X)$. They induce filtrations on the Hochschild chain spaces in the following way: a pure tensor $a_0 \otimes \ldots \otimes a_k$ is said to belong to $C_k^{i,j;l}$ if $a_0 \in \Psi_a^{i_0,j_0}(X), \ldots, a_k \in \Psi_a^{i_k,j_k}(X)$, $i_0 + \ldots + i_k = i$, $j_0 + \ldots + j_k = j$ and $i_\alpha < l$, $j_\alpha < l$ for $\alpha = 1, \ldots, k$. Then $C_k^{i,j;l}$ is defined as the closure of the linear span of pure tensors with respect to the projective tensor product topology in $C_k^{i,j;l}$, and we define

$$C_k := \bigcup_{i,j,l \in \mathbb{Z}} C_k^{i,j;l}.$$

See [2] for more comments on this issue.

The boundary map b is compatible with these two filtrations, hence we get filtrations on the Hochschild complexes of $\Psi_a(X)$, $\Psi_a^{-\infty}(X)$, $A_\sigma(X)$, etc. We will denote the associated spectral sequences by ${}^\sigma E$, respectively ${}^\partial E$, or simply by E when no confusion can arise.

Note that although $A_\partial(X)$ is an $\mathcal{L}$–vector space, our definition of Hochschild homology involves only tensors over $\mathbb{C}$.

Let us recall the Hochschild-Kostant-Rosenberg map [10]. Let A be a commutative $\mathbb{C}$-algebra. Define

$$\chi : C_k(A) \to \Omega^k_{A/\mathbb{C}} \qquad a_0 \otimes \ldots \otimes a_k \mapsto a_0 da_1 \wedge \ldots \wedge da_k. \tag{9}$$

Proposition 3.1. *Let Y be a compact manifold, possibly with boundary. The map χ induces isomorphisms*

$$HH_k(C^\infty(Y)) \cong \Lambda^k(Y) \qquad HH_k(\dot{C}^\infty(Y)) \cong \dot{\Lambda}^k(Y),$$

where the dot denotes rapid vanishing to the boundary of Y.

This result is known as the *Hochschild-Kostant-Rosenberg* theorem, and it was proved in [7] for the algebra $C^\infty(Y)$ when Y is a closed manifold. The other cases follow from this one, by considering for instance the double of a manifold with boundary, and the H-unital ideals of rapidly vanishing functions.

3.2. The action of derivations on Hochschild Homology

Let d be a derivation on an algebra A. We define the inner product, respectively the Lie derivative with respect to d on $HH(A)$ by the following chain maps [12]:

Definition 3.2.

$$a_0 \otimes a_1 \otimes \ldots \otimes a_k \overset{e_d}{\mapsto} (-1)^{k+1} d(a_k) a_0 \otimes \ldots \otimes a_{k-1};$$
$$a_0 \otimes a_1 \otimes \ldots \otimes a_k \overset{L_d}{\mapsto} \sum_i a_0 \otimes \ldots \otimes d(a_i) \otimes \ldots \otimes a_k.$$

If d is an inner derivation, then e_d and L_d both vanish on $HH(A)$ [12].

Proposition 3.3.

(1) $[L_{d_1}, e_{d_2}] = e_{[d_1,d_2]}$.

(2) $e_{d_1}e_{d_2} = -e_{d_2}e_{d_1}$.

Proof. The first statement is true at chain level [25]. For the second, we note that $d_1 \otimes d_2 = -d_2 \otimes d_1$ in the graded commutative ring $HH^*(A, A)$. The claim follows from the action of this ring on $HH_*(A)$.

If d is continuous, then e_d and L_d extend to the topological chain spaces and to topological Hochschild homology.

3.3. The derivative with respect to t

If we restrict adiabatic operators to $t > 0$, we remark a canonical derivation preserving the filtrations, namely $D := t\frac{d}{dt}$. In the blow-up picture, it is given by the Lie derivative in the direction of the lift of the vector field $t\frac{d}{dt}$ to the interior of X_a^2 through the blow-down map. This vector field vanishes on the fiber–diagonal at $t = 0$, hence its lift extends to ff as a (non-degenerate) vector field, tangent to the fibers of ff°. In light of Proposition 2.5, we can identify it with the radial vector field in $\pi^{*a}TM$. We can therefore extend the action D to the whole algebra $\Psi_a(X)$; by continuity, it will stay a derivation.

Let us choose a connection in $X \to M$. It induces an extension of the splitting ${}^aT^*X_{|t=0} \cong (TX/M)^* \oplus \pi^{*a}T^*M_{|t=0}$ to ${}^aT^*X$ over $\mathbb{R}_+$, and defines a splitting $T^*X \cong (TX/M)^* \oplus \pi^*T^*M$. Recall that ${}^aT^*M$ and $T^*M \times \mathbb{R}_+$ are canonically isomorphic. Then the map ϕ_a (Definition 2.1) takes the form

$$\phi_a = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \tag{10}$$

Let $\mathcal{R}_M$ be the radial vector field in π^*T^*M. We can lift $\mathcal{R}_M$ to ${}^aT^*X$ by using the connection.

Proposition 3.4. *On $A_\sigma^{[\mathbb{Z}],\mathbb{Z}}$ and $A_\partial^{\mathbb{Z},[\mathbb{Z}]}$, the derivation D takes the following form:*

$$D = t\frac{d}{dt} + \mathcal{R}_M. \tag{11}$$

Proof. From the above expression of ϕ_a,

$$(\phi_a)_*\left(\frac{d}{dt}\right) = \frac{d}{dt} + t^{-1}\mathcal{R}_M. \tag{12}$$

In any quantization independent of t for $t > t_0 > 0$, the action of $t\frac{d}{dt}$ on symbols is just the action of the vector field $t\frac{d}{dt}$ on functions. The blow-down map β induces the isomorphism $\phi_a : T^*X \times (0, \infty) \to {}^aT^*X_{|t>0}$. The relation between principal symbols is simply pull-back via ϕ_a, hence the induced derivation on $A_\sigma^{[\mathbb{Z}],\mathbb{Z}}$ for $t > 0$ is $\phi_{a*}(t\frac{d}{dt}) = t\frac{d}{dt} + \mathcal{R}_M$. By continuity, this is valid down to $t = 0$.

We have implicitly proved that the partial Fourier transform in the horizontal fibers of the operator $L_{\beta^*(t\frac{d}{dt})}$ equals $L_{t\frac{d}{dt}+\mathcal{R}_M}$. This implies the result for $A_\partial^{\mathbb{Z},[\mathbb{Z}]}$.

We note that (11) is a sum of two well-defined terms on the front face, while in the interior they depend on the connection. Their sum is of course independent of this choice.

Remark that D preserves all ideals like $I_\partial(X), \Psi_a^{-\infty}(X)$, etc. As in Definition 3.2, we get an action e_D on the Hochschild homology of these algebras. Moreover, D preserves the two algebra filtrations, hence it induces maps of spectral sequences for the Hochschild complexes.

3.4. The conjugation by $\log Q$

Let $Q \in \Psi_a^{1,0}(X)$ be a positive elliptic adiabatic operator of order 1 acting on the sections of a vector bundle over X. Then the complex powers Q^z, $z \in \mathbb{C}$, are adiabatic operators of complex order z. Indeed, by the result of Seeley [26], for any fixed $t > 0$ the complex powers of the restriction Q_t are classical pseudodifferential operators. Using the method of Bucicovschi [6] extending a proof by Guillemin [9], one can construct the complex powers of the images of Q in $A_\partial(X)$, respectively in $A_\sigma(X)$. Moreover, these families are unique. Patching them together gives the family $\{Q^z\}_{z\in\mathbb{C}}$.

We choose Q with $\sigma_1(Q) = r$ where $r : {}^aT^*X \to \mathbb{R}$ is the length function of some non-degenerate metric on ${}^aTX$.

Let $D_z(a) := (Q^{-z}aQ^z - a)/z$, and let D_Q be the following derivation on $\Psi_a(X)$:

$$D_Q(a) := \lim_{z\to 0} D_z(a)_{|z=0} = \frac{d}{dz}\left(Q^{-z}aQ^z\right)_{|z=0}. \tag{13}$$

PROPOSITION 3.5. *$[D_Q, D]$ is an inner derivation.*

Proof. Follows easily from the fact that

$$\frac{d}{dz}(DQ^z)_{|z=0} \in \Psi_a(X).$$

$\square$

LEMMA 3.6. *D_Q decreases the order by 1. If A is an operator of order j, then*

$$\sigma_{j-1}(D_Q A) = \frac{1}{ir}\{r, \sigma_j(A)\}_t,$$

where $\{,\}_t$ is the adiabatic Poisson bracket.

Proof. The symbol of Q^z is r^z. The result follows from Proposition 2.7.

§4. The homology of the algebra of adiabatic symbols

In this section we compute $HH_*(A_\sigma(X))$ with the spectral sequence ${}^\sigma E$. Let $N = m + n = \dim X$. Recall that all operators in this paper have classical (i.e., 1-step, polyhomogeneous) symbols.

4.1. The E_2 term

Recall that the subscript $[i]$ denotes homogeneity i in radial directions. We first compute the E_1 term.

PROPOSITION 4.1. *The first term in ${}^\sigma E(C_*(A_\sigma(X)))$ is given by*

$$E_1^{i,k}(A_\sigma(X)) \simeq \Lambda_{[i]}^{i+k}({}^aT^*X \setminus \{0\})[t^{-1}].$$

Proof. From Proposition 2.7, the associated graded algebra $GA_\sigma(X) \cong S({}^aT^*X)[t^{-1}]$ is the graded commutative algebra of formal symbols on ${}^aT^*X$, with t^{-1} adjoined. This implies that $E_1^{i,k}(A_\sigma(X))$ is the component of homogeneity i of $HH_{i+k}(S({}^aT^*X)[t^{-1}]))$. By Proposition 3.1,

$$HH_k(S({}^aT^*X)[t^{-1}]) = \Lambda_S^k({}^aT^*X)[t^{-1}].$$

The isomorphism is given by the map χ.

Consider the following inclusions of algebras

$$\Psi_a^{\mathbb{Z},-\infty}(X) \hookrightarrow C^\infty(\mathbb{R}_+, \Psi^{\mathbb{Z}}(X) \tag{14}$$

$$\Psi_a^{\mathbb{Z},-\infty}(X) \hookrightarrow \Psi_a(X). \tag{15}$$

The second map is induced by the blow-down map β. Let us examine the induced map on the spectral sequences of the symbol algebras. Note that all the terms of these spectral sequences are $HH_*(C^\infty(\mathbb{R}_+)[t^{-1}])$–modules, hence local in t. As in Proposition 4.1, we see that

$$E_1^{i,k}(A_\sigma^{\mathbb{Z},-\infty}(X)) \cong \dot\Lambda_{[i]}^{i+k}({}^aT^*X \setminus \{0\})[t^{-1}];$$
$$E_1^{i,k}(C^\infty(\mathbb{R}_+, S(X))) \cong \Lambda_{[i]}^{i+k}(\mathbb{R}_+ \times T^*X \setminus \{0\})[t^{-1}]$$

where the dot denotes rapid vanishing at $t = 0$.

For $t > 0$, the canonical map $\phi_a : T^*X \times \mathbb{R}_+ \to {}^aT^*X$ is an isomorphism. Then the map induced by (14), respectively (15) on E_0 and E_1 terms is the inclusion, respectively the pull-back by ϕ_a. The point of this argument is that we already know the differential d_1 for the algebra of families of symbols, by the results of [5]. Denote the differential in the spectral sequence of $A_\sigma(X)$ by a superscript a. For $t > 0$ we get

$$d_1^a = (\phi_a^*)^{-1} d_1 \phi_a^*. \tag{16}$$

We will extend (16) by continuity down to $t = 0$. Choose a connection in $X \to M$ as in Section 3.3. Let $d_v = d - dt \otimes \mathcal{L}_{\frac{d}{dt}}$ be the de Rham differential in the vertical directions in a product decomposition of ${}^aT^*X$. Let $*$ be the symplectic duality operator on $\Lambda(T^*X)$ introduced by Brylinski [4]. Let $*_\phi$ be the conjugation of $*$ by ϕ_a:

$$*_\phi = \phi_a^{*-1} * \phi_a^*.$$

Proposition 4.2. *For $t > 0$, the differential d_1^a can be written (up to sign) in terms of the product structure on ${}^aT^*X$ as*

$$d_1^a = *_\phi(d_v - t^{-1} dt \otimes \mathcal{L}_{\mathcal{R}_M}) *_\phi . \tag{17}$$

Proof. The differential d_1 for the algebra $S(X) = \Psi^{\mathbb{Z}}(X)/\Psi^{-\infty}(X)$ was computed in [4], [5]. Up to sign, which is irrelevant for homology, it equals $*d*$, where d is the de Rham differential on $T^*X \setminus 0$. Hence, in the spectral sequence of the families algebra $C^\infty(\mathbb{R}_+, S(X))$, we get $d_1 = *(d - dt \otimes \frac{\partial}{\partial t})*$. From (16) and (12), it follows that

$$\begin{aligned} d_1^a &= *_\phi(\phi_a^{*-1}(d - dt \otimes \mathcal{L}_{\frac{d}{dt}})\phi_a^*) *_\phi \\ &= *_\phi(d - dt \otimes \mathcal{L}_{\phi_{a*}(\frac{d}{dt})}) *_\phi \\ &= *_\phi(d - dt \otimes (\frac{\partial}{\partial t} + t^{-1}\mathcal{L}_{\mathcal{R}_M})) *_\phi . \end{aligned}$$

(we have used the invariance by pull-back of d).

Let ${}^v\Lambda^k_{[i]}$ denote the space $\imath_{\frac{d}{dt}}\Lambda^k_{[i]}({}^a T^*X\setminus\{0\})[t^{-1}]$ of forms that do not contain dt.

Proposition 4.3. *The involution $*_\phi$ extends to $\Lambda^*({}^a T^*X)[t^{-1}]$. Its restriction to ${}^v\Lambda^{i+j}_{[i]}$ acts as follows:*

$$
{}^v\Lambda^{i+j}_{[i]} \stackrel{*_\phi}{\to} {}^v\Lambda^{2N-i-j}_{[N-j]} \oplus dt\wedge {}^v\Lambda^{2N-i-j-1}_{[N-j]} \oplus dt\wedge {}^v\Lambda^{2N-i-j+1}_{[N-j+1]}. \tag{18}
$$

Proof. Follows from the following observations: First, the operator $*$ commutes with dt and, as noted in [4], it maps $\Lambda^{i+j}_{[i]}(T^*X)$ to $\Lambda^{2N-i-j}_{[N-j]}(T^*X)$. Secondly, ϕ_a^* maps ${}^v\Lambda^k_{[i]}$ onto $(1+t^{-1}dt\otimes\imath_{\mathcal{R}_M})\Lambda^k_{[i]}$, and ϕ^{*-1} maps $\Lambda^k_{[i]}$ onto $(1-t^{-1}dt\otimes\imath_{\mathcal{R}_M})^v\Lambda^k_{[i]}$.

As a corollary, the identity (17) holds by continuity down to $t=0$.

Definition 4.4. Fix a product structure on ${}^a T^*X$. Define the *adiabatic duality operator* $*_a$ on $\Lambda^*({}^a T^*X)$ by

$$
*_a := (1+t^{-1}dt\otimes\imath_{\mathcal{R}_M})*_\phi. \tag{19}
$$

From the definition, $*_a$ is an isomorphism with inverse $*_\phi(1-t^{-1}dt\otimes\imath_{\mathcal{R}_M})$. Also, it is clear that $*_a$ consists of the first and third terms of $*_\phi$ from the decomposition (18).

Proposition 4.5. *The first differential d_a^1 is conjugated to the vertical de Rham differential:*

$$
*_a d_1^a *_a^{-1} = d_v.
$$

Proof. Direct computation, using (17) and the definition (19).

Hence $d_1^a: E_1^{i,j}(A_\sigma(X))\to E_1^{i-1,j}(A_\sigma(X))$ is conjugated via $*_a$ to

$$
{}^v\Lambda^{2N-i-j}_{[N-j]}\oplus dt\otimes {}^v\Lambda^{2N-i-j+1}_{[N-j+1]} \stackrel{d_v}{\to} {}^v\Lambda^{2N-i-j+1}_{[N-j]}\oplus dt\otimes {}^v\Lambda^{2N-i-j+2}_{[N-j+1]} \tag{20}
$$

Let ${}^aS^*X\to\mathbb{R}_+$ be the sphere bundle of ${}^a T^*X$ viewed as a bundle over $\mathbb{R}_+$. Denote by $C^\infty(\mathbb{R}_+, H_v^*({}^aS^*X))$ the space of sections in the (trivial) fiber cohomology bundle.

Theorem 4.6. *At the E_2 level, the spectral sequence with respect to the total operator degree filtration of the Hochschild complex of the formal symbol algebra is isomorphic to a family of de Rham cohomology spaces indexed by $\mathbb{R}_+$:*

$$E_2^{i,j}(A_\sigma(X)) \cong \begin{cases} C^\infty(\mathbb{R}_+, H_v^{N-i}({}^aS^*X \times S^1))[t^{-1}] & \text{if } j = N, \\ C^\infty(\mathbb{R}_+, H_v^{N-i}({}^aS^*X \times S^1))[t^{-1}]dt & \text{if } j = N+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have seen that $E_2^{i,j}(A_\sigma(X))$ is isomorphic to the homology of the complex (20). Since d_v preserves homogeneity and commutes with $dt\wedge$, this complex splits. We claim that

$$H_*(\Lambda^*_{[k]}({}^aT^*X \setminus \{0\})[t^{-1}], d_v) = 0 \quad \text{for } k \neq 0. \tag{21}$$

Indeed, let $\mathcal{R}$ be the radial vector field on ${}^aT^*X$. Then, for $\nu \in \Lambda^*_{[k]}({}^aT^*X \setminus \{0\}[t^{-1}])$, we have $\mathcal{L}_\mathcal{R}\nu = k\nu$. Assume ν is d_v-closed. Since d/dt is defined using a product decomposition of ${}^aT^*X$, we have

$$\left[\mathcal{R}, \frac{d}{dt}\right] = 0. \tag{22}$$

Thus

$$k\nu = \mathcal{L}_\mathcal{R}\nu = d\imath_\mathcal{R}\nu + \imath_\mathcal{R}d\nu = d_v\imath_\mathcal{R}\nu + \imath_\mathcal{R}d_v\nu = d_v(\imath_\mathcal{R}\nu).$$

Hence if $k \neq 0$ it follows that ν is exact.

On the other hand, it is straightforward to check that

$$H_j(\Lambda^*_{[0]}({}^aT^*X \setminus \{0\})[t^{-1}], d_v) \cong C^\infty(\mathbb{R}_+, H_v^j({}^aS^*X \times S^1))[t^{-1}].$$

□

By the homotopy invariance of cohomology, the isomorphisms from Theorem 4.6 are independent of the choices made.

4.2. Action of derivations and degeneracy

The derivation D introduced in (11) acts by e_D on $C_*(A_\sigma(X))$. Since it commutes with b and preserves the filtration F_i, this action descends to the spectral sequence.

Proposition 4.7. *The effect of e_D on ${*_a}E_1(A_\sigma(X))$, and hence on $E_\infty(A_\sigma(X))$, is contraction by the vector field $t\frac{d}{dt}$.*

Proof. The following commutations hold trivially on $E_1(\Psi_{(0,\infty)})$:

$$\chi e_{t\frac{d}{dt}} = \imath_{t\frac{d}{dt}}\chi \qquad\qquad * \,\imath_{t\frac{d}{dt}} = \imath_{t\frac{d}{dt}} * .$$

Since χ is invariant by pull-backs, we can pull it back via the map ϕ_a. It follows that

$$\chi e_D = \imath_{(\phi_a)_*(t\frac{d}{dt})}\chi \qquad\qquad *_\phi \imath_{(\phi_a)_* t\frac{d}{dt}} = \imath_{(\phi_a)_* t\frac{d}{dt}} *_\phi,$$

which implies $*_\phi \chi\, e_D = \imath_{\phi_*(t\frac{d}{dt})} *_\phi \chi$, and so

$$\begin{aligned}(1 + t^{-1}dt \otimes \imath_{\mathcal{R}_M}) *_\phi \chi e_D &= (1 + t^{-1}dt \otimes \imath_{\mathcal{R}_M})(\imath_{t\frac{d}{dt}} + \imath_{\mathcal{R}_M}) *_\phi \chi \\ &= \imath_{t\frac{d}{dt}}(1 + t^{-1}dt \otimes \imath_{\mathcal{R}_M}) *_\phi \chi.\end{aligned} \tag{23}$$

This identity extends by continuity down to $t = 0$.

Hence the map e_D is injective on $E_2^{*,m+n+1}(A_\sigma(X))$ and vanishes on the rest of $E_2(A_\sigma(X))$. This implies that the spectral sequence ${}^\sigma E(A_\sigma(X))$ degenerates at E_2.

PROPOSITION 4.8. *The effect of L_D on $[a] \in E_\infty(A_\sigma(X))$ is the Lie derivative $\mathcal{L}_{t\frac{d}{dt}}[a]$ applied to the cohomology class $[a]$ representing a at $E_2 = E_\infty$ in the presentation of Theorem 4.6. In particular, if a is a class of homogeneity k in t, then $L_D(a) = ka$.*

Proof. From the definitions, $\chi \circ L_D = \mathcal{L}_{(\phi_a)_*(t\frac{d}{dt})}\chi$, i.e L_D acts as the Lie derivative $\mathcal{L}_{(\phi_a)_*(t\frac{d}{dt})}$ on E_1, i.e., on forms. This Lie derivative commutes with $*_\phi$ and with $(1 + t^{-1}dt \otimes \imath_{\mathcal{R}_M})$. Finally, $\mathcal{L}_{\mathcal{R}_M} = 0$ on cohomology.

Let $t^{-1}dt\wedge$ be the following operation on $C_*(\Psi_a(X))$:

$$a_0 \otimes \ldots \otimes a_n \overset{t^{-1}dt\wedge}{\longmapsto} \sum_{i=0}^{n}(-1)^i t^{-1} a_0 \otimes \ldots \otimes a_i \otimes t. \otimes \ldots \otimes a_n,$$

Since t belongs to the center of $\Psi_a(X)$, $t^{-1}dt\wedge$ is a chain map.

PROPOSITION 4.9. *Set $\alpha := e_D t^{-1}dt\wedge$, $\beta := t^{-1}dt \wedge e_D$. The following identities hold on homology:*

(1) $(t^{-1}dt\wedge)^2 = 0$.

(2) $e_D t^{-1}dt \wedge + t^{-1}dt \wedge e_D = 1$.

(3) $C_*(A_\sigma(X))$ *and* $HH(A_\sigma(X))$ *split as* $im(e_D t^{-1}dt\wedge) \oplus im(t^{-1}dt \wedge e_D)$.

Proof. The first two statements follow by direct computation. From part (4.9), $\alpha\beta = 0$. Then part (4.9) implies that $\alpha = \alpha(\alpha+\beta) = \alpha^2$ and $\beta = (\alpha+\beta)\beta = \beta^2$, so α and β are idempotents. Since $1 = \alpha+\beta = (\alpha+\beta)^2 = \alpha+\beta+\beta\alpha$, we get also $\beta\alpha = 0$. This proves part (4.9).

Let $r^{-1}dr$ be the generator of $H^1(S^1)$ in Theorem 4.6.

PROPOSITION 4.10. *After identification of E_2 with de Rham cohomology, the derivation D_Q acts as an exterior product:*

$$e_{D_Q} = r^{-1}dr \cup .$$

Proof. First, by Propositions 3.5 and 3.3, e_{D_Q} commutes with $t^{-1}dt\wedge$ and with e_D. This implies that e_{D_Q} preserves the spaces of 0 and 1-forms in t. Let $a \in E_2^{i,m+n}$ be a representative for a homology class with no dt. Then, by Theorem 4.6, $e_{D_Q}(a)$ is determined by its part in homogeneity $i-1$.

From Lemma 3.6, it follows that $\chi(\sigma(e_{D_Q}(a))) = \frac{dr}{r} \wedge \chi(a)$, which implies the desired formula after applying the conjugation $(1+t^{-1}dt \otimes \imath_{\mathcal{R}_M})*_\phi$. The case where the class of a is a multiple of dt follows by multiplication with $t^{-1}dt$.

4.3. Convergence of the spectral sequence

This section is unavoidably technical because of the crucial role played by the two filtrations F_i and T_j in the definition of the topological chain spaces for $\Psi_a(X)$.

THEOREM 4.11. *The spectral sequence for $A_\sigma(X)$ is convergent, in the sense that there exists an isomorphism*

$$HH_k(A_\sigma(X)) \cong E_2^{k-N,N} \oplus E_2^{k-N-1,N+1}.$$

compatible with the filtration $\{T_i\}_{i\in\mathbb{Z}}$, where the E_2 spaces are given in Theorem 4.6.

A homology class is said to belong to F_i if it has a representative in F_i. Thus Theorem 4.11 contains what is usually meant by convergence:

$$F_i/F_{i-1}(HH_k(A_\sigma(X))) \cong \begin{cases} E_2^{k-N,N} & \text{if } i = k-N \\ E_2^{k-N-1,N+1} & \text{if } i = k-N-1 \\ 0 & \text{otherwise} \end{cases}$$

but also something *more*: first, the "residual" homology

$$\bigcap_{i\in\mathbb{Z}} F_i(HH_k(A_\sigma(X)))$$

vanishes; secondly, there exists a (natural) splitting of $HH_k(A_\sigma(X))$ as the direct sum of its filtration quotients, as $C^\infty(\mathbb{R}_+)[t^{-1}]$-modules.

One can construct abstractly a formal extension of any element in E_∞ to a closed "chain". The issue is finding asymptotically summable extensions, i.e., bounding from above the T-order in accordance with the definition of the topological chain spaces (by definition, the *T-order* of a chain $a_0 \otimes \ldots \otimes a_k$ with respect to a increasing filtration T_i is at most l if $a_j \in T_l$ for all j). We claim that we can achieve this, and moreover we can find such an extension of the same boundary filtration order as the starting element in E_∞.

The proof uses some results and ideas from [18].

We identify elements in $\Psi_a^{\mathbb{Z},[0]}$ with the Fourier transform of their Schwartz kernel. The algebra of polynomial functions on ${}^aT^*M$ is central in $\Psi_a^{\mathbb{Z},[0]}$. Let $A_\sigma^{\mathbb{Z},[0]}$ be the symbol algebra of $\Psi_a^{\mathbb{Z},[0]}$.

PROPOSITION 4.12. *The spectral sequence ${}^\sigma E(A_\sigma^{\mathbb{Z},[0]})$ degenerates at E_2.*

Proof. Let $B_{(sus)} \subset A_\sigma^{\mathbb{Z},[0]}$ be the ideal consisting of the symbols that vanish rapidly at the vertical sub-bundle of ${}^aT^*X$. Let $F_{(sus)}$ denote the quotient $A_\sigma^{\mathbb{Z},[0]}/B_{(sus)}$. These algebras are filtered by the operator order, thus we get spectral sequences for each of the three Hochschild complexes.

Since the polynomial functions on ${}^aT^*M$ commute with these algebras, we can define the following two types of natural operations on the spectral sequences: contractions by vector fields with polynomial coefficients, and exterior multiplication by forms with polynomial coefficients. These operations define splittings of the homologies and of the spectral sequences, which are preserved by the differentials. We can also consider the action of the

Lie derivative in the direction of $\mathcal{R}_M$, the radial vector field in ${}^{a}T^*M$. This replaces the dilation argument from [18]. The different spectral sequences split as eigenspaces of this action. By computing the eigenvalues, one can prove as in [18, Propositions 6 and 7 and Lemma 7] that the spectral sequences $E(F_{(sus)})$ and $E(B_{(sus)})$ degenerate at E_2. The same argument shows that the long exact sequence in homology is determined at E_2, hence implying the proposition.

The convergence of these sequences follows from the fact that the chain spaces are defined as inverse limits, and the following lemma:

LEMMA 4.13. *Let $a \in C_k(C^\infty(Y) \otimes \mathcal{L})$ be a boundary of T-order l. Then there exists $c \in C_{k+1}(C^\infty(Y) \otimes \mathcal{L})$ of T-order* $\max\{0, l\}$ *such that $a = b(c)$.*

Proof. If Y is 0-dimensional, the statement follows directly. This and the proof of the Eilenberg-Zilber Theorem imply the result for arbitrary Y.

Note that there is no analog of the index map for the suspended algebra ([18], Lemma 8). Namely, the short exact sequence

$$0 \to \Psi_a^{-\infty,[0]} \to \Psi_a^{\mathbb{Z},[0]} \to A_\sigma^{\mathbb{Z},[0]} \to 0$$

induces a long exact sequence in homology (by H-unitality) which actually splits into short exact sequences. See also Proposition 7.15.

Consider now the graded algebra $A_\partial^{\mathbb{Z},[\mathbb{Z}]}(X)$. It is canonically isomorphic to $\Psi_a^{\mathbb{Z},[0]} \otimes \mathcal{L}$. In the statements above, we can replace the algebras by their tensor product with $\mathcal{L}$. A careful analysis of the T-orders shows that the spectral sequences are still convergent.

We have proved

PROPOSITION 4.14. *Let $a \in C_k^{j,i;l}(\Psi_a(X))$ be a chain of order j, such that $a^{j,[i]}$ survives at ${}^{\sigma}E_2^{j,k-j}(\Psi_a^{\mathbb{Z},[\mathbb{Z}]})$. Then there exists $A \in C_k^{j,i;\max\{l,0\}}(\Psi_a(X))$ such that $A - a \in C_k^{j,i-1}(\Psi_a(X))$ and $b(A) \in C_k^{j-1,i-1}(\Psi_a(X))$.*

PROPOSITION 4.15. *If $a \in C_k^{j,i;l}(\Psi_a(X))$ survives at $E_2^{j,k-j}(A_\sigma(X))$ and $k - j \neq N, N+1$ then there exists $x \in C_{k+1}^{j+1,i+1}(\Psi_a(X))$ such that $bx - a \in C_k^{j-1,i}(\Psi_a(X))$ and moreover the T-order of x is at most* $\max\{T\text{−order}(a), 0\}$.

Proof. Let ν be the form corresponding to a in $E_2^{j,k-j}(A_\sigma(X))$, so that $d_v(\nu) = 0$. From (21), it follows that

$$\nu = C^{-1} d_v(\imath_{\mathcal{R}}\nu)$$

where $C = k-j-N-1$ or $k-j-N$, depending on whether ν does or does not contain dt. The assumption on $k-j$ means exactly that $C \neq 0$. The passage between E_1 and E_2 is done via the isomorphism $(1+t^{-1}dt \otimes \imath_{\mathcal{R}_M})*_\phi$ (Proposition 4.5). Then $\imath_{\mathcal{R}}\nu$ corresponds to

$$c = C^{-1} *_\phi (1 - t^{-1}dt \otimes \imath_{\mathcal{R}_M})\imath_{\mathcal{R}}\nu = C^{-1}\mathcal{R}^{\#_\phi} \wedge *_\phi(1 - t^{-1}dt \otimes \imath_{\mathcal{R}_M})\nu,$$

where $\mathcal{R}^{\#_\phi}$ is the 1-form dual to $\mathcal{R}$ under ω_ϕ. We see that this 1-form is in T_1, since $(\phi_a)_*^{-1}\mathcal{R} = \mathcal{R}$, $\mathcal{R}^{\#} = \alpha$, the canonical 1-form, and $(\phi_a)_*\alpha \in T_1$.

Now find a chain x_0 in $C_{k+1}^{j+1,i+1;\max\{l,0\}}$ representing the class c at $E_1(A_\sigma(X))$. By Proposition 4.14, one can find the desired x.

Proof of Theorem 4.11. By applying Proposition 4.15 repeatedly, it follows that $F_{k-N-1}HH_k(A_\sigma(X)) = 0$. We must still prove that the edge map $F_iHH_k(A_\sigma(X)) \to E_2^{i,k-i}$ is surjective. Start with $a \in T_jF_iC_k$ so that $\sigma(a)$ represents a class in $E_2^{i,N}(A_\sigma(X))$, hence assuming that a contains no dt. This means that we can change the sub-principal symbol of a in T_j so that $b(a) \in F_{i-2}$.

From Proposition 4.14, there exists $A \in T_jC_k(\Psi_a(X))$, $\sigma(A) = a$, such that $b(A) \in T_{j-1}$. Observe that we can assume that the sub-principal symbols of A and a agree up to a chain $\mu \in T_jF_{i-1}$ with the property $b(\mu) \in F_{i-2}$. This follows from the proof of Proposition 4.14. Now $b(A)$ represents a form which is the zero element in $E_2^{i-2,N+1}(A_\sigma(X))$. We can assume that this form contains no dt (if not, project onto the no-dt part). As in Proposition 4.15, it follows that there exists $x \in T_jF_{i-1}$ such that $b(A) - b(x_1) \in T_jF_{i-3}$ and $T - \text{order}(x) \leq \max\{ord(a), 0\}$. Repeated applications of Proposition 4.15 will yield an infinite sequence $x_p \in F_{i-p}$ of chains of uniformly bounded orders and of t-degree j. Due to the decreasing operator orders and bounded T-orders, such a sequence is asymptotically summable in the sense of our definition of chain spaces. It follows that $b(A - \sum x_p) = 0$. The case where the class a contains dt is similar.

COROLLARY 4.16. *The group $HH(A_{\partial,\sigma}(X))$ splits into eigenspaces of the L_D action with integer eigenvalues.*

Proof. First, note that the spectral sequence associated to $A_{\partial,\sigma}(X)$ also degenerates at E^2 and is convergent, by the same argument as above. Since α and β commute with b and L_D, the splitting form Proposition 4.9 is inherited by Hochschild homology and is preserved by L_D. Via the edge inclusion, $E_2^{k-N-1,N+1}(A_{\partial,\sigma}(X))$ is a subspace of $HH_k(A_{\partial,\sigma}(X))$. By Proposition 4.7, β acts the identity and α acts as 0 on this subspace, which therefore coincides with $im(\beta)$. Then the inclusion $im(\alpha) \to HH_k(A_{\partial,\sigma}(X))$, followed by the edge surjection $HH_k(A_{\partial,\sigma}(X)) \to E_2^{k-N,N}(A_{\partial,\sigma}(X))$, is an isomorphism which commutes with L_D. Now use Proposition 4.8.

§5. The semi-classical limit algebra

Consider the case where $X = M$, i.e., the fiber of the fibration $X \to M$ is just one point. In this case, the adiabatic algebra is called the *semi-classical limit* algebra. Consider the short exact sequence

$$0 \to I_\partial(M) \overset{\imath}{\hookrightarrow} A_\partial(M) \overset{p}{\to} A_{\partial,\sigma}(M) \to 0. \tag{24}$$

This sequence is compatible with the filtration T_j, hence it induces a short exact sequence of the associated graded algebras

$$0 \to \mathcal{S}({}^aT^*M_{|t=0}) \otimes \mathcal{L} \overset{\imath}{\hookrightarrow} S_{\mathrm{cl}}({}^aT^*M_{|t=0}) \otimes \mathcal{L} \overset{p}{\to} S({}^aT^*M_{|t=0} \setminus \{0\}) \otimes \mathcal{L} \to 0. \tag{25}$$

In this section, ${}^\partial E$ stands for the spectral sequences with respect to the filtration T_j of the Hochschild complexes of $I_\partial(M)$, $A_\partial(M)$ and $A_{\partial,\sigma}(M)$. The following Proposition is entirely similar to Proposition 4.1. We have kept the powers of t and dt in order to keep track of the filtration.

Proposition 5.1. *Let ${}^\partial E$ denote the spectral sequence of the Hochschild complex with respect to the filtration T_i. Then the E_1 terms of the semi-classical algebras are:*

$$\begin{aligned}
{}^\partial E_1^{i,j}(I_\partial(M)) &\cong E_1^{i,j}(\mathcal{S}({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}) \\
&\cong t^{-i}\Lambda_{\mathcal{S}}^{i+j}({}^aT^*M_{|t=0}) \oplus \Lambda_{\mathcal{S}}^{i-j-1}({}^aT^*M_{|t=0})t^{-i-1}dt; \\
{}^\partial E_1^{i,j}(A_\partial(M)) &\cong E_1^{i,j}(S_{\mathrm{cl}}({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}) \\
&\cong t^{-i}\Lambda_{S_{\mathrm{cl}}}^{i+j}({}^aT^*M_{|t=0}) \oplus \Lambda_{S_{\mathrm{cl}}}^{i-j-1}({}^aT^*M_{|t=0})t^{-i-1}dt; \\
{}^\partial E_1^{i,j}(A_{\partial,\sigma}(M)) &\cong E_1^{i,j}(S({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}) \\
&\cong t^{-i}\Lambda_S^{i+j}({}^aT^*M_{|t=0}) \oplus \Lambda_S^{i-j-1}({}^aT^*M_{|t=0})t^{-i-1}dt.
\end{aligned}$$

Proof. The subscripts $\mathcal{S}$, S_{cl} and S stand for forms with Schwartz, classical symbol, respectively homogeneous symbol coefficients The first isomorphism for each space follows from the fact that the induced product on $A_\partial^{\mathbb{Z},[\mathbb{Z}]}(M)$ is the usual commutative product on functions (Proposition 2.8). The second isomorphism for each space is realized by the Hochschild-Kostant-Rosenberg map χ (Definition 9).

Let d_1 be the first differential in the above spectral sequences.

PROPOSITION 5.2. *In the spectral sequences $E(I_\partial(M))$, $E(A_\partial(M))$ and $E(A_{\partial,\sigma}(M))$, the first differential takes the form*

$$d_1 = (*_a)^{-1} d_v *_a \, . \tag{26}$$

Proof. We first prove the assertion for the algebra $A_{\partial,\sigma}(M)$. Let $a \in C_{i+j}^{[s],[i]}(A_{\partial,\sigma}(M))$ be a chain of pure homogeneity s in the fibers of ${}^{a}T^*M$, which survives at ${}^{\partial}E_1^{i,j}$. Then a also survives at ${}^{\sigma}E_1^{s,i+j-s}(A_{\partial,\sigma}(M))$. The identification of both ${}^{\partial}E_1$ and ${}^{\sigma}E_1$ with forms on ${}^{a}T^*M$ is realized by the map χ. From the definition, $d_1[a] = \chi([b(a)]_{[i-1]})$ is represented by the part of $b(a)$ of degree $[-i+1]$ in t. From the structure of the product on the boundary algebras, this is exactly the part of pure homogeneity $s-1$ of $b(a)$, i.e., it is equal to $d_1^a[a]$. The claim follows from Proposition 4.5.

Note that d_1 and $(*_a)^{-1}d_v *_a$ are operators with polynomial coefficients in the fibers of ${}^{a}T^*M_{|t=0}$. We claim that we can recover d_1 from its action on homogeneous forms, i.e., ${}^{\partial}E_1^{i,j}(A_{\partial,\sigma}(M))$. Indeed, we can retrieve the coefficients of a differential operator with polynomial coefficients from its action on polynomials.

We can describe the homogeneity with respect to t of the operator $*_a$ as follows:

$$\begin{aligned} t^{-i}\Lambda^k({}^{a}T^*M_{|t=0}) &\oplus t^{-i}dt \wedge \Lambda^{k-1}({}^{a}T^*M_{|t=0}) \\ &\xrightarrow{*_a} t^{-i+k-n}\Lambda^{2n-k}({}^{a}T^*M_{|t=0}) \oplus t^{-i+k-n-2}dt \wedge \Lambda^{2n-k+1}({}^{a}T^*M_{|t=0}). \end{aligned} \tag{27}$$

The following Proposition is a consequence of (27) and Proposition 5.2.

Proposition 5.3. *Conjugation by $*_a^M$ on ${}^\partial E_1$ induces the following identifications for the E_2 terms:*

$$\begin{aligned}
{}^\partial E_2^{i,k}(I_\partial(M)) &\cong t^{k-n} H^{2n-i-k}_{\mathbb{S}}({}^a T^*M_{|t=0}) \\
&\quad \oplus t^{k-n-2}dt \otimes H^{2n-i-k+1}_{\mathbb{S}}({}^a T^*M_{|t=0}); \qquad (28)\\
{}^\partial E_2^{i,k}(A_\partial(M)) &\cong t^{k-n} H^{2n-i-k}_{S}({}^a T^*M_{|t=0}) \\
&\quad \oplus t^{k-n-2}dt \otimes H^{2n-i-k+1}_{S}({}^a T^*M_{|t=0});\\
{}^\partial E_2^{i,k}(A_{\partial,\sigma}(M)) &\cong t^{k-n} H^{2n-i-k}({}^a S^*M_{|t=0} \times S^1) \qquad (29)\\
&\quad \oplus t^{k-n-2}dt \otimes H^{2n-i-k+1}({}^a S^*M_{|t=0} \times S^1);
\end{aligned}$$

We can now compute the effect of derivations on these E_2 terms. As in Proposition 4.7, the operator e_D acts on $E_2(I_\partial(M))$, $E_2(A_{\partial,\sigma}(M))$ and $E_2(A_\partial(M))$ by contraction with the vector field $t\frac{d}{dt}$. Hence the splitting from Proposition 5.3 is given by the null-space and image of e_D.

Proposition 5.4. *At the level of $E_2^{i,k}$, the operator L_D equals $k-n$ on* $\ker(e_D)$ *and $k-n-1$ on $im(e_D)$.*

Proof. Imitate the proof of Proposition 4.8 to obtain that L_D acts as the Lie derivative $\mathcal{L}_{\mathcal{R}_M + t\frac{d}{dt}}$. Hence, e_D has eigenvalues $k-n$ and $k-n-1$ on $E_2^{i,k}$, corresponding to the splitting from Proposition 5.3. This splitting is given by the image and null-space of e_D.

Corollary 5.5. *The spectral sequences from Proposition 5.3 degenerate at E_2.*

Proof. By naturality, the boundary maps in the spectral sequences commute with the maps e_D and L_D. In particular, they preserve the null-space and the image of e_D. On $\ker(e_D) \cap {}^\partial E_2^{i,k}$, L_D acts as multiplication by $k-n$. This shows that for $s \geq 2$, the map $d_s : \ker(e_D) \cap {}^\partial E_2^{i,k} \to \ker(e_D) \cap {}^\partial E_2^{i+s-1,k-s}$ must vanish. Similarly, d_s vanishes on the image of e_D.

Corollary 5.6. *Let $\mathcal{O}M$ denote the orientation bundle of M. Then*

$$HH_k(I_\partial(M)) \cong \Big(H^{n-k}(M, \mathcal{O}M) \oplus H^{n-k+1}(M, \mathcal{O}M)\Big) \otimes \mathcal{L}.$$

Proof. The convergence in the sense of Section 4.3 of the spectral sequence is easy, since there is only one filtration to control. In particular there is no residual homology (i.e., in filtration $-\infty$). Since we deal with vector spaces over $\mathcal{L}$, we see that $HH_k(I_\partial(M))$ is isomorphic to the direct sum of its filtration quotients, which by Corollary 5.5 are just the E_2 terms computed in Proposition 5.3. In fact, the isomorphism is canonical, corresponding to the splitting in eigenvalues for L_D. Finally, we use the Thom isomorphism to integrate the cohomology along the fibers, picking up coefficients in the orientation bundle along the way.

§6. The homology of the boundary ideal

In this section, $E(I_\partial(X))$ will represent the spectral sequence with respect to the filtration T_j of the Hochschild complex of $I_\partial(X)$. We define an algebra map $\phi : \Psi_a^{-\infty}(M) \to \Psi_a^{-\infty}(X)$, and show that it induces an isomorphism $E_1(I_\partial(M)) \cong E_1(I_\partial(X))$. This will imply that ϕ induces an isomorphism in Hochschild homology.

There exists a natural fibration $X_a^2 \xrightarrow{\pi} M_a^2$. The fibers of this fibration are $F \times F$, where F is the fiber of $X \to M$. Choose a smooth family dv of densities of volume 1 in each fiber of $X \to M$.

Definition 6.1. Define $\phi : \Psi_a^{-\infty}(M) \to \Psi_a^{-\infty}(X)$ by

$$A \mapsto (\pi^* A) \otimes dv_R,$$

where dv_R is the density dv in the second term of the fiber $F \times F$.

It is immediate that ϕ is a map of algebras, from the structure of the triple spaces. Indeed, X_a^3 fibers naturally over M_a^3, with fiber F^3 and the assertion follows from (4).

6.1. A connection on the vertical algebra

Consider the following diagram of fibrations:

(30)
$$\begin{tikzcd} X \arrow[d, "\pi"] & \pi^*({}^aT^*M_{|t=0}) \arrow[l, "p"'] \arrow[d, "\pi"'] \\ M & {}^aT^*M_{|t=0} \arrow[l, "p"'] \end{tikzcd}$$

Fix a connection in $X \to M$ and lift it through p to a connection in $\pi^*({}^aT^*M_{|t=0}) \to {}^aT^*M_{|t=0}$. Over a point $x \in M$, the restricted fibration $p^{-1}\pi^{-1}(x) \to p^{-1}(x)$ is canonically isomorphic to the product

$X_x \times {}^aT^*_xM_{|t=0}$, and the induced connection is the canonical trivial connection on the product.

Let $\Gamma_{S_{\rm cl}}(\Lambda^1({}^aT^*M_{|t=0}))$ denote the space of 1-forms on ${}^aT^*M_{|t=0}$ with symbolic coefficients in the fibers. The connection defines a derivation

$$\mathcal{S}(\pi^*({}^aT^*M_{|t=0})) \xrightarrow{\nabla} \Gamma_{S_{\rm cl}}(\Lambda^1({}^aT^*M_{|t=0})) \otimes \mathcal{S}(\pi^*({}^aT^*M_{|t=0})).$$

Recall the identification of $\Psi_a^{-\infty,[0]}(X)$, via Fourier transform, with $\mathcal{S}(X \times_M X \times_M {}^aT^*M_{|t=0}, \Omega_R)$. Let

$$\pi_R, \pi_L : X \times_M X \times_M {}^aT^*M_{|t=0} \to X \times_M {}^aT^*M_{|t=0}$$

be the projections on the first, respectively the second X factor. The action of $\Psi_a^{\mathbb{Z},[0]}$ on $\Psi_a^{-\infty,[0]}$ is $\pi_L^*(C^\infty(X))$-linear. After Fourier transform, $\Psi_a^{\mathbb{Z},[0]}$ acts $\mathcal{S}({}^aT^*M_{|t=0})$-linearly on $\mathcal{S}(X \times_M {}^aT^*M_{|t=0})$:

$$\Psi_a^{\mathbb{Z},[0]} \otimes \mathcal{S}(X \times_M {}^aT^*M_{|t=0}) \ni (A, f) \mapsto \pi_{R*}(A\pi_L^*(f)) \in \mathcal{S}(X \times_M {}^aT^*M_{|t=0}).$$

This action is faithful. We extend by duality the derivation ∇ to $\Psi_a^{\mathbb{Z},[0]}$:

$$(\nabla A)(f) = \nabla(Af) - A(\nabla(f)).$$

In local coordinates, one sees easily that ∇ preserves $\Psi_a^{-\infty,[0]}$.

Let $\nabla^t = \nabla + \frac{\partial}{\partial t}dt$ be the extension of ∇ to $\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)$. Then ∇^t commutes with multiplication by $\mathcal{S}({}^aT^*M_{|t=0}) \otimes \mathcal{L}$, in the following sense: Let $A \in \Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)$, $V \in \Gamma_{\mathcal{S}}(T({}^aT^*M_{|t=0})) \otimes \mathcal{L} \oplus \mathcal{S}({}^aT^*M_{|t=0}) \otimes \mathcal{L}\frac{\partial}{\partial t}$ and $g \in \mathcal{S}({}^aT^*M_{|t=0}) \otimes \mathcal{L}$. Then

$$\nabla^t{}_{gV}(A) = g\nabla^t{}_V(A). \tag{31}$$

By duality, ∇^t is a derivation on $\Psi_a^{\mathbb{Z},[0]} \otimes \mathcal{L}$.

6.2. The analog of χ

The product in $I_\partial(X)$ is a deformation of the vertical product. It follows that $E_0(I_\partial(X)) \cong E_0(\Psi_a^{-\infty,[\mathbb{Z}]}(X))$. Moreover, this isomorphism commutes with d_0. By using the Künneth formula for continuous Hochschild homology [18], we obtain

$$\begin{aligned} E_1^{i,j}(I_\partial(X)) &\cong E_1^{i,j}(\Psi_a^{-\infty,[\mathbb{Z}]}(X)) \\ &= HH_{i+j}(\Psi_a^{-\infty,[\mathbb{Z}]}(X))_{[i]} \\ &= t^{-i} \otimes HH_{i+j}(\Psi_a^{-\infty,[0]}) \oplus t^{-i+1}dt \otimes HH_{i+j-1}(\Psi_a^{-\infty,[0]}). \end{aligned}$$

There exists a $\mathcal{S}({}^aT^*M_{|t=0})$–valued trace functional on the vertical ideal $\Psi_a^{-\infty,[0]}$, defined by push-forward under the projection map π of the restriction to the diagonal of each fiber of kernels in $\Psi_a^{-\infty,[0]}$. Actually, Tr_{V} extends to $\Psi_a^{-m-\epsilon,\mathbb{Z}}(X)$ for $\epsilon > 0$:

$$A \overset{\mathrm{Tr}_{\mathrm{V}}}{\mapsto} \int_{\Delta \times_M {}^aT^*M_{|t=0}/{}^aT^*M_{|t=0}} A_{|\Delta \times_M {}^aT^*M_{|t=0}}.$$

Let $A = A_0 \otimes A_1 \otimes \ldots \otimes A_k \in C_k(\Psi_a^{-\infty,[\mathbb{Z}]}(X))$. The following definition was inspired by [18]. Define $\chi_a : C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L}) \to \Lambda_{\mathcal{S}}({}^aT^*M_{|t=0}) \otimes \Lambda\mathcal{L}$ by

$$\chi_a(A) := \mathrm{Tr}_{\mathrm{V}}(A_0 \nabla^t A_1 \wedge \ldots \wedge \nabla^t A_k). \tag{32}$$

For $V_1, \ldots, V_k \in \Gamma_S(T({}^aT^*M_{|t=0})) \otimes \mathcal{L} \oplus S_{\mathrm{cl}}({}^aT^*M_{|t=0}) \otimes \mathcal{L}\frac{\partial}{\partial t}$, we have

$$\chi_a(A)(V_1, \ldots, V_k) = \mathrm{Tr}_{\mathrm{V}} \left(\sum_{\sigma \in S_k} A_0 \nabla^t{}_{V_{\sigma(1)}}(A_1) \ldots \nabla^t{}_{V_{\sigma(k)}}(A_k) \right).$$

Note that $\chi_a(A)$ can be defined for any $A = A_0 \otimes A_1 \otimes \ldots \otimes A_k \in (\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))^{\otimes k+1}$, such that the real part of the order of $A_0 \nabla^t A_1 \wedge \ldots \wedge \nabla^t A_k$ is strictly less than $-m$.

Proposition 6.2. *If $A \in F_{-m-k-\epsilon} C_k(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$, then $\chi_a \circ bA = 0$.*

Proof. From the Leibniz identity for ∇^t, it follows that

$$\chi_a(b(A)) = (-1)^k \mathrm{Tr}_{\mathrm{V}}[A_k A_0 \nabla^t A_1 \wedge \ldots \wedge \nabla^t A_{k-1} \\ - A_0 \nabla^t A_1 \wedge \ldots \wedge \nabla^t (A_{k-1}) A_k].$$

Now use the fact that Tr_{V} vanishes on commutators of operators of total order less than $-m - \epsilon$.

In particular, χ_a descends to the Hochschild homology of the vertical ideal $I_\partial(X)$.

Proposition 6.3. *For $A, B \in \Psi_a^{-\infty,[0]}(M) \otimes \mathcal{L}$,*

$$\nabla^t(\phi(A))\phi(B) = \phi(d(A)B).$$

Proof. Straightforward; however, it is not true that $\nabla^t(\phi(A)) = \phi(d(A))$!

PROPOSITION 6.4. *The map χ_a is related to χ through $\chi_a \circ \phi = \chi$. Hence, χ_a is surjective.*

Proof. Let $A = A_0 \otimes A_1 \otimes \ldots \otimes A_k \in C_k(\Psi_a^{-\infty,[0]}(M) \otimes \mathcal{L})$. Then

$$\begin{aligned}
\chi_a(\phi(A)) &= \mathrm{Tr_V}[\phi(A_0)\nabla^t\phi(A_1) \wedge \ldots \wedge \nabla^t\phi(A_k)]\\
&= \mathrm{Tr_V}[\nabla^t\phi(A_1) \wedge \ldots \wedge (\nabla^t\phi(A_k))\phi(A_0)]\\
&= \mathrm{Tr_V}[\nabla^t\phi(A_1) \wedge \ldots \wedge \nabla^t\phi(A_{k-1}) \wedge \phi(d(A_k)A_0)]\\
&= \ldots = \mathrm{Tr_V}[\phi(d(A_1) \wedge \ldots \wedge d(A_k)A_0)]\\
&= d(A_1) \wedge \ldots \wedge d(A_k)A_0\\
&= \chi(A).
\end{aligned}$$

We have used Proposition 6.3 and commutativity of the trace.

6.3. The E_1 term

Since χ is an isomorphism on Hochschild homology [7], by Proposition 6.4, χ_a must be surjective. In order to prove that χ_a is an isomorphism at E_1 we need to prove injectivity. Recall the decomposition $b = \sum_{i=0}^{k}(-1)^i b_i$ of the Hochschild differential on C_k.

Let g be a metric on M and r_g the injectivity radius of (M, g). Let ψ be a cut-off function supported inside $(-r_g, r_g)$ with $\psi(0) = 1$. For two points x_1, x_2 in M at distance less than r_g, let $\tau_{x_2}^{x_1}$ denote the parallel transport along the shortest geodesic from x_1 to x_2 in $X \to M$, with respect to the connection ∇, and also in ${}^a T^*M_{|t=0} \to M$ with respect to ∇^{LC}. Fix again a vertical density dv on the fibers of $X \to M$.

Let $P_i = (\xi_i, u_i, v_i)$ denote a typical point in ${}^a T^*M_{|t=0} \times_M X \times_M X$ over a base point $x_i \in M$, $i = 0, \ldots, k$.h The topological chain space $C_k(\Psi_a^{-\infty,[\mathbb{Z}]}(X))$ is just

$$\mathcal{S}(({}^a T^*M_{|t=0} \times_M X \times_M X)^{k+1}) \otimes \mathcal{L}(t_0, \ldots, t_k) \otimes dv_0 \otimes \ldots \otimes dv_k.$$

Let a be a cycle in $C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L})$,

$$a = A(t_0, \xi_0, u_0, v_0, \ldots, t_k, \xi_k, u_k, v_k)dv_0 \otimes \ldots \otimes dv_k.$$

For $l \in \{0, \ldots, k+1\}$, we say that a satisfies the property $\mathcal{P}_l$ if the following two conditions are satisfied:

1. $b_j(a) = 0$ for $j = 0, \ldots, l-1$.

2. The function A is independent of $v_0, \ldots, v_{l-1}$.

PROPOSITION 6.5. *If a satisfies $\mathcal{P}_l$, then it is homologous to another cycle a', which satisfies $\mathcal{P}_{l+1}$.*

Proof. Define $h_l : C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L}) \to C_{k+1}(\Psi_a^{-\infty,[0]} \otimes \mathcal{L})$ by

$$\begin{aligned}
&h_l(a)(t_0, P_0, \ldots, t_{k+1}, P_{k+1})\\
&= \psi(d(x_l, x_{l+1}))e^{-\|\tau_{x_l}^{x_{l+1}}\xi_{l+1}-\xi_l\|^2} dv_l\\
&\quad \otimes a(t_0, P_0, \ldots, t_l, \xi_l, u_l, \tau_{x_l}^{x_{l+1}} v_{l+1}, \ldots, t_{k+1}, P_{k+1}).
\end{aligned}$$

Note that $h_l(a)$ has Schwartz behavior in both ξ_l and $\xi_{l+1} - \xi_l$, therefore also in ξ_{l+1}. Let a' be defined by the following equation:

$$\begin{aligned}
&b(h_l(a))\\
&= (-1)^l a + (-1)^{l+1}\psi(d(x_l, x_{l+1}))e^{-\|\tau_{x_l}^{x_{l+1}}\xi_{l+1}-\xi_l\|^2} dv_l\\
&\quad \otimes \left(\int_{X_{x_{l+1}}} A(t_0, P_0, \ldots, t_l, \xi_l, u_l, \tau_{x_l}^{x_{l+1}} z, t_{l+1}, \xi_{l+1}, z, v_{l+1}, \ldots)dz\right.\\
&\quad \left. - \int_{X_{x_{l+1}}} A(t_0, P_0, \ldots, t_l, \xi_l, u_l, z, t_l, \xi_l, z, \tau_{x_l}^{x_{l+1}} v_{l+1}, \ldots,)dz\right)\\
&\quad - h_l(b(a))\\
&= (-1)^l(a - a').
\end{aligned}$$

Since, by hypothesis, $b(a) = 0$, we have $h_l(b(a)) = 0$. In addition, a' satisfies $\mathcal{P}_l$ since a does. Thus $b_l(a') = 0$ and a' does not depend on v_l, hence a' satisfies $\mathcal{P}_{l+1}$.

By induction, every cycle $a \in C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L})$ is homologous to a cycle which satisfies $\mathcal{P}_k$.

DEFINITION 6.6. Let a be a cycle in $C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L})$, which satisfies $\mathcal{P}_k$. For $l \in \{0, \ldots, k+1\}$, we say that a satisfies $\mathcal{R}_l$ if the the function A is independent of $u_0, \ldots, u_{l-1}$.

PROPOSITION 6.7. *If a satisfies $\mathcal{R}_l$, then it is homologous to another cycle a' which satisfies $\mathcal{R}_{l+1}$.*

Proof. Define $q_l : C_k(\Psi_a^{-\infty,[\mathbb{Z}]}(X)) \to C_{k+1}(\Psi_a^{-\infty,[\mathbb{Z}]}(X))$ by

$$\begin{aligned} &q_l(a)(t_0, P_0, \ldots, t_{k+1}, P_{k+1}) \\ &:= \psi(d(x_l, x_{l+1}))e^{-\|\tau_{x_l}^{x_{l+1}}\xi_{l+1}-\xi_l\|^2} dv_l \\ &\quad \otimes a(t_0, P_0, \ldots, t_l, \xi_l, \tau_{x_l}^{x_{l+1}} u_{l+1}, \ldots, t_{k+1}, P_{k+1}). \end{aligned}$$

As above, $q_l(a)$ is well defined. Since a satisfies $\mathcal{P}_k$, it follows that

$$\begin{aligned} &b(q_l(a)) \\ &= (-1)^l a - (-1)^l \psi(d(x_l, x_{l+1}))e^{-\|\tau_{x_l}^{x_{l+1}}\xi_{l+1}-\xi_l\|^2} dv_l \\ &\quad \otimes \left(\int_{X_{x_{l+1}}} A(t_0, P_0, \ldots, t_l, \xi_l, \tau_{x_l}^{x_{l+1}} u_{l+1}, t_{l+1}, \xi_{l+1}, z, \ldots) dz \right) \\ &= (-1)^l (a - a'), \end{aligned}$$

where the last equality is the definition of a'. Note that a' satisfies $\mathcal{P}_k$ and $\mathcal{R}_l$ since a does, and moreover it is independent of u_l.

Therefore, every cycle $a \in C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L})$ is homologous to a cycle which satisfies $\mathcal{P}_k$ and $\mathcal{R}_{k+1}$.

COROLLARY 6.8. *If $a \in C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L})$ is a cycle, then $\chi_a(a)$ is independent of ∇.*

Proof. We have seen above that a is homologous to some a' which lies in the image of the map ϕ, i.e $a' = \phi(c)$. By Proposition 6.2, we have $\chi_a(a) = \chi_a(a')$. Finally, by Proposition 6.4, it follows that $\chi_a(a') = \chi(c)$, hence $\chi_a(a)$ is independent of ∇.

PROPOSITION 6.9. *The map*

$$\chi_a : E_1(I_\partial(X)) \cong HH_*(\Psi_a^{-\infty,[\mathbb{Z}]}(X)) \to \Lambda_{\mathcal{S}}({}^aT^*M_{|t=0}) \otimes \Lambda\mathcal{L}$$

is injective. Together with Proposition 6.4, this implies that χ_a is an isomorphism.

Proof. Let a be a cycle in $C_k(\Psi_a^{-\infty,[0]} \otimes \mathcal{L})$ so that $[a] \in \ker(\chi_a)$. We can assume that a satisfies $\mathcal{P}_k$ and $\mathcal{R}_{k+1}$. This means that

$$a \in \operatorname{im}(\phi : C_k(\Psi_a^{-\infty,[0]}(M) \otimes \mathcal{L}) \to C_k(\Psi_a^{-\infty,[0]}(X) \otimes \mathcal{L}).$$

Say $a = \phi(\bar{a})$. Then, by Proposition 6.4, we have

$$0 = \chi_a(a) = \chi_a(\phi(\bar{a})) = \chi(\bar{a}).$$

Since χ is an isomorphism, it follows that $\bar{a}$ is exact in $C_*(\Psi_a^{-\infty,[0]}(M) \otimes \mathcal{L})$. As ϕ is an algebra map, it commutes with the Hochschild differential, and so a is also exact.

THEOREM 6.10. *The map ϕ induces an isomorphism between the homology groups of the semi-classical and adiabatic smoothing ideals (see Subsection 2.2 and Section 5):*

$$\begin{aligned} HH_*(I_\partial(M)) &\overset{\phi}{\cong} HH_*(I_\partial(X)) \\ &\cong \Big(H^{n-k}(M, \mathcal{O}M) \oplus H^{n-k+1}(M, \mathcal{O}M)\Big) \otimes \mathcal{L}. \end{aligned}$$

Proof. Propositions 6.4 and 6.9 imply that χ_a is an isomorphism, hence the map ϕ induces an isomorphism between $E_1(I_\partial(M))$ and $E_1(I_\partial(X))$. The result follows using a general property of spectral sequences [14]. We only need to notice that the two spectral sequences are (almost tautologically) convergent.

The map ϕ and the derivation D commute. From Theorem 6.10 and the similar properties of $I_\partial(M)$, it follows that e_D acts on $E_\infty(I_\partial(X))$ by contraction with the vector field $t\frac{d}{dt}$. The space $E_\infty(I_\partial(X))$ splits as the direct sum of the null-space and the image of e_D. The effect of L_D on $\ker(e_D) \cap {}^\partial E_\infty^{i,k}(I_\partial(X))$, respectively on $im(e_D) \cap {}^\partial E_\infty^{i,k}(I_\partial(X))$, is multiplication by $k-n$, respectively $k-n-1$.

Notice that $HH_0(I_\partial(X))$ is always a 1-dimensional $\mathcal{L}$-vector space. Thus there exists essentially a unique trace functional on $I_\partial(X)$. In Subsection 7.3 we will show that this trace extends to a trace on $\Psi_a^{-\infty}(X)$ which for positive t is simply the operator trace.

§7. The residue functional

7.1. Definition of the residue

Since $Q^z \in \Psi_a^{z,0}(X)$, multiplication by Q^z descends to $\Psi_a^{\mathbb{Z},[0]}$ and extends to chains in $C_*(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$ by $Q^z(a_0 \otimes \ldots \otimes a_h) = (Q^z a_0) \otimes \ldots \otimes a_h$. We denote by $*_a^M$ the operator $*_a$ on ${}^aT^*M$. Let $\mathcal{O}M$ be the orientation bundle of M.

Definition 7.1. For $z \in C$, $\Re(z)$ sufficiently negative, define

$$F(z) : C_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)) \to \Lambda^{n-h}(M, \mathcal{O}M) \otimes \mathcal{L} \oplus \Lambda^{n-h+1}(M, \mathcal{O}M) \otimes \mathcal{L}dt$$
$$A \mapsto \int_{{}^aT^*M_{|t=0}/M} *_a^M(\chi_a(Q^z A)).$$

Proposition 7.2. *For $\Re(z) < h-N-i$, the dt-free part, $\imath_{\frac{d}{dt}} dt \wedge F(z)$, of $F(z)$ is well-defined and holomorphic on $F_iC_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$. For $\Re(z) < h-N-i-1$ the dt part, $\imath_{\frac{d}{dt}} F(z)$, of $F(z)$ is well-defined and holomorphic on $F_iC_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$.*

Proof. From the definition, for $V \subset U$ open sets in M and V relatively compact in U, $F(z)(A)_{|V}$ depends only on $A_{|(\pi^{-1}(U))^{h+1}}$. Hence $F(z)$ is local in M.

Lemma 7.3. *The operator $F(z)$ is independent of ∇.*

Proof. Work over a coordinate patch $U \subset M$. Let (x_i, ξ_j), $(x_i, \tilde{\xi}_j)$ be local coordinates adapted to the cotangent structure in T^*M and ${}^aT^*M_{|t=0}$ over U. The map ϕ_a takes the form $(x_i, \xi_j) \mapsto (x_i, t\xi_j)$. Let Id be the local isomorphism $(x_i, \xi_j) \mapsto (x_i, \xi_j)$. Let $*^M$ be Brylinski's duality operator on $\Lambda(T^*M)$. We denote by $*^U$ the conjugate of $*^M$ by Id. On h-forms,

$$\imath_{\frac{d}{dt}} dt \wedge (*_\phi^U + t^{-1} dt \otimes \imath_{\mathcal{R}_M} *_\phi^U) = t^{h-n} *^U \imath_{\frac{d}{dt}} dt \wedge . \tag{33}$$

Using this and the fact that the integral along the fiber vanishes on forms that are not multiples of $d\xi_1 \wedge \dots d\xi_n$, we get

$$\begin{aligned} \imath_{\frac{d}{dt}} dt \wedge F(z)(A) &= \int_{U \times \mathbb{R}^n/U} t^{h-n} *^U \imath_{\frac{d}{dt}} dt \wedge \chi_a(Q^z A) \\ &= t^{h-n} \int_{U \times \mathbb{R}^n/U} *^U \operatorname{Tr}_V(\widehat{Q^z a_0} \nabla_\xi \hat{a}_1 \wedge \dots \wedge \nabla_\xi \hat{a}_h), \end{aligned} \tag{34}$$

where $\nabla_\xi = \sum \nabla_{\partial_{\xi_i}} d\xi_i$. Recall that we Fourier-transform the kernels in the horizontal tangent directions. Since the connection is lifted to $X \times_M X \times_M {}^aT^*M \to {}^aT^*M$ from $X \times_M X \to M$, it follows that $\nabla_{\partial_{\xi_i}}$ is actually independent of ∇. This proves that the dt-free part of $F(z)$ is independent of ∇. The other case follows by replacing formula (33) with (36).

Using Lemma 7.3, we can assume that we work in local coordinates and that formula (34) holds. From the properties of symbols, the assumption on the order of A, and the assumption $\Re(z) + i - h < -N$, it follows that

$$\widehat{Q^z a_0} \nabla_\xi \hat{a}_1 \wedge \ldots \wedge \nabla_\xi \hat{a}_h \in d\xi^h F_{z+i-h} \Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X) \tag{35}$$

and hence is of trace class (recall that all operators of order less than $-m$ are of vertical trace class). Taking the trace increases homogeneity by m, hence

$$\mathrm{Tr}_{\mathrm{V}}(\widehat{Q^z a_0} \nabla_\xi \hat{a}_1 \wedge \ldots \wedge \nabla_\xi \hat{a}_h) \in d\xi^h S^{z+i-h+m}({}^a T^* M_{|t=0}) \otimes \mathcal{L}$$

and so

$$*^M \mathrm{Tr}_{\mathrm{V}}(\widehat{Q^z a_0} \nabla_\xi \hat{a}_1 \wedge \ldots \wedge \nabla_\xi \hat{a}_h) \in \Lambda^{2n-h}_{[z+i-h+N]}({}^a T^* M_{|t=0}) \otimes \mathcal{L}.$$

Hence for $\Re(z) + i - h + N < 0$, this form is of negative homogeneity and therefore integrable along the fibers of ${}^a T^* M_{|t=0}/M$. This proves the first assertion. The other one is similar and uses the identity

$$\imath_{\frac{d}{dt}}(*^U_\phi + t^{-1} dt \otimes \imath_{\mathcal{R}_M} *^U_\phi) = t^{h-n-1} *^U \imath_{\frac{d}{dt}} + t^{h-n-2} \alpha \wedge *^U \imath_{\frac{d}{dt}} dt \wedge \tag{36}$$

instead of (33). Here α is the pull-back via Id^{-1} of the canonical form on T^*M, and has homogeneity 1.

Remark 7.4. From (34), we see that $\imath_{\frac{d}{dt}} dt \wedge F(z)$ shifts the t-degree by $h - n$ on h-chains.

Proposition 7.5. *Let $A_z \in F_i C_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$ be an entire family of h-chains in filtration i. Then $F(z)(A_z)$ is meromorphic, with at most simple poles at the real integers.*

Proof. Denote by $\mathcal{H}(\mathcal{B})$ the holomorphic functions in the band $\mathcal{B} = \{-1 < \Re(z) < 1\}$. We shall show that $F(z)(A_z)$ has at most a simple pole at 0 in the band $\mathcal{B}$. Fix a total symbol map for adiabatic operators, i.e., a map $q : \Psi_a(X) \to S({}^a T^* X)[t^{-1}]$, which extends the symbol map. Let A_z^j be the component of order $z + j$ of $q(\widehat{Q^z a_0}(z) \otimes \hat{a}_1(z) \ldots \otimes \hat{a}_h(z))$. By Proposition 7.2, only a finite number of A_z^j's are significant for the poles of $F(z)(A_z)$ in $\mathcal{B}$. Fix a metric $|r| = |(\xi, \eta)|$ on ${}^a T^* X$. Choose B_z^j such that $q(B_z^j) = A_z^j$. Let ψ be a cut-off function, $\psi(r) \equiv 0$ for small r, $\psi(r) \equiv 1$ for $r \geq 1$.

Lemma 7.6. *The following identity holds modulo a holomorphic family of forms on* $\Lambda_{\mathcal{S}}({}^aT^*M)$:

$$\int_{{}^aT^*M_{|t=0}/M} *^M_a \operatorname{Tr}_V(\chi_a(B^j_z)) \equiv \int_{{}^aT^*X_{|t=0}/M} *^X_a \psi(\xi,\eta)\chi(A^j_z).$$

Proof. Both terms are local, so we can prove the statement in local coordinates. The left-hand side is independent of the connection. We claim that the part in $*^X_a\chi(A^j_z)$ which is a multiple of the fiber volume form dV of ${}^aT^*X_{|t=0}/M$, is also independent of ∇. Indeed,

$$*^X_a = (1 + t^{-1}dt \otimes \imath_{\mathcal{R}_M})\phi_a^{*-1} * \phi_a^*,$$

where ϕ_a is given by (10). Let μ denote a monomial form in local coordinates. First, if μ does not contain a multiple of dV, then neither does $(1 + t^{-1}dt \otimes \imath_{\mathcal{R}_M})\phi_a^{*-1}\mu$. Secondly, if μ contains any $dx, d\eta, dy$, then $*\mu$ does not contain multiples of dV [4]. Finally, if μ contains some $dx, d\eta, dy$, then so do all monomials in $\phi^*_a\mu$. Therefore, only those μ that contain only $d\xi$'s survive in $\int_{{}^aT^*X_{|t=0}/M} \psi(\xi,\eta) *^X_a \mu$. Moreover, the result is seen to be independent of the map A in (10), which can therefore be assumed of the form $A(x, y, \xi, \eta) = \xi$. Then, modulo $\mathcal{H}(\mathbb{C}, \Lambda_{\mathcal{S}}({}^aT^*M))$, we have

$$\operatorname{Tr}_V(\chi_a(B^j_z)) = \int_{{}^aT^*X_{|t=0}/{}^aT^*M_{|t=0}} \psi(\xi,\eta)\chi(A^j_z)\omega^m_{X/M}.$$

$\square$

Lemma 7.7. *Let* $\mu_j(z) \in \Lambda_{[j]}({}^aT^*X \setminus 0)$ *be entire in* z. *Then*

$$\int_{{}^aT^*X_{|t=0}/M} \psi(r)r^z\mu_j(z)$$

has only one simple pole at $z = -j$, *and*

$$\operatorname{Res}_{z=-j} \int_{{}^aT^*X_{|t=0}/M} \psi(r)r^z\mu_j(z) = -\int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}}\mu_j(0).$$

Proof. Let $\mathcal{R}$ be the radial vector field on ${}^aT^*X$. In polar coordinates,

$$\int_{{}^aT^*X_{|t=0}/M} \psi(r)r^z\mu_j(z) = \int_0^\infty \psi(r)r^{z+j-1}dr \int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}}\mu_j(z).$$

The first factor equals $-\frac{1}{z+j}$ (mod $\mathcal{H}(\mathbb{C})$). The second one is entire.

From Lemma 7.6, $F(z)(A)$ has the same poles in $\mathcal{B}$ as a finite sum

$$\sum_j \int_{{}^aT^*X_{|t=0}/M} \psi(r) r^z \mu_j(z),$$

where $\mu_j(z)$ is entire in z and of homogeneity j in the fibers of ${}^aT^*X_{|t=0}$. By Lemma 7.7, the proposition follows.

Definition 7.8. For $A \in C_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$, define

$$R(A) = \mathrm{Res}_{z=0}(F(z)(A)).$$

Definition 7.9. Let $A = A_0 \otimes \ldots \otimes A_k \in C_k(\Psi_a^{\mathbb{Z},[0]} \otimes \mathcal{L})$. Define

$$e_{Q^z}(A) = (-1)^n z Q^z \left(\frac{Q^{-z} A_k Q^z - A_k}{z} \right) A_0 \otimes A_1 \otimes \ldots \otimes A_{k-1}.$$

If $A \in C_k(\Psi_a(X))$, define $e_{Q^z}(A)$ by the same expression, where now all products are in $\Psi_a(X)$. Note that $e_{Q^z} = (-1)^n z Q^z e_{D_z}$ (see (13)). By direct computation, the following identity holds on $C_*(\Psi_a^{\mathbb{Z},[0]} \otimes \mathcal{L})$ and on $C_*(\Psi_a(X))$.

$$Q^z b = b Q^z + z Q^z e_{D_z}. \tag{37}$$

Proposition 7.10. *The identity $R \circ b = 0$ holds on $C_k(\Psi_a^{\mathbb{Z},[0]} \otimes \mathcal{L})$.*

Proof. Using Lemma 37,

$$\begin{aligned} R(b(A)) = \mathrm{Res}_{z=0} \int_{{}^aT^*M_{|t=0}/M} &*_a^M \mathrm{Tr}_{\mathrm{V}}(\chi_a(b(Q^z A))) \\ &+ \mathrm{Res}_{z=0}(z F(z)(e_{Q^z}(A))). \end{aligned} \tag{38}$$

Note that $e_{Q^z}(A)$ is holomorphic, including at $z = 0$. By Proposition 7.5, the second term vanishes. By Proposition 6.2, the first integrand vanishes for small $\Re(z)$, so it extends to be identically zero by analytic continuation.

It follows that R induces a map

$$R : HH_k(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)) \rightarrow \Lambda(M, \mathcal{O}M) \otimes \Lambda\mathcal{L}. \tag{39}$$

We claim that on this space, $R \circ e_D = \imath_{t\frac{d}{dt}} R$. Indeed, $\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)$ is a module over $\mathcal{L} \otimes Polyn({}^aT^*M)$, and hence $HH_k(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$ is a module over $HH_k(\mathcal{L} \otimes Polyn({}^aT^*M))$, i.e., over the ring of differential forms with polynomial coefficients. Then, by exactly the same reasoning as in Chapter 4, we can prove that formula (23) holds with χ replaced by χ_a. Note that the claim is false at chain level.

This observation shows that the map (39) takes $im(\alpha), im(\beta)$ onto forms without dt, respectively multiples of dt.

PROPOSITION 7.11. *Let $A \in F_{h-N}HH_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)) \cap im(\alpha)$. Then*

$$R(A) = \imath_{\frac{d}{dt}} dt \wedge R(A) = -\int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}} *_a^X \chi(\sigma(A)).$$

Let $A \in F_{h-N-1}C_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$. Then $R(A)$ is a multiple of dt and

$$R(A) = -\int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}} *_a^X \chi(\sigma(A)).$$

Proof. We know that in the first case, $R(A)$ does not contain dt. Apply Proposition 7.2 with $i = h - N - 1$, respectively $i = h - N - 2$. It follows that $\imath_{\frac{d}{dt}} dt \wedge F(z)$, respectively $F(z)$, is holomorphic for such chains around $z = 0$, which implies that $\imath_{\frac{d}{dt}} dt \wedge R(A)$, respectively $R(A)$ depend only on $\sigma(A)$. The result follows from Lemmas 7.6 and 7.7.

Remark 7.12. This Proposition implies that we can improve Remark 7.4 as follows: The map (39) shifts t-degree by $h - n$ when applied to $HH_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)) \cap im(\alpha)$, and by $h-n-1$ when applied to $HH_h(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)) \cap im(\beta)$.

From Proposition 7.2, it follows that R vanishes on $\Psi_a^{-\infty,[\mathbb{Z}]}(X)$, so we think about it as being defined on symbols, with a natural extension to vertical operators. Using the projection $T_j A_\sigma(X) \to \Psi_a^{\mathbb{Z},[0]}$, we can extend the definition of R to $C_*(A_\sigma(X))$.

PROPOSITION 7.13. *Let $A \in F_{h-N}C_h(A_\sigma(X))$ or $F_{h-N-1}C_h(A_\sigma(X))$ represent a class $[A]$ in $HH_h(A_\sigma(X))$. Then $R(A)$ is de Rham closed in M. If A is Hochschild exact, then $R(A)$ is de Rham exact.*

Proof. We have seen in Theorem 4.6 that the differential form

$$[A]_{E_2} = *_a^X \chi(\sigma(A))$$

is closed and of homogeneity 0. Hence $\mathcal{L}_{\mathcal{R}}[A]_{E_2} = 0$. Using (22), this implies $(\imath_{\mathcal{R}} d_v + d_v \imath_{\mathcal{R}})[A]_{E_2} = 0$, hence $\imath_{\mathcal{R}}[A]_{E_2}$ is closed. Proposition 7.11 shows that

$$d_v R(a) = d_v \int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}}[A]_{E_2} = \int_{{}^aS^*X_{|t=0}/M} d_v \imath_{\mathcal{R}}[A]_{E_2} = 0.$$

If $[A] = 0$, then $[A]_{E_2} = d_v\beta$ is exact, and, since d_v preserves homogeneity, we can assume that β is also homogeneous of homogeneity 0. Therefore $\mathcal{L}_{\mathcal{R}}\beta = 0$. This implies $\imath_{\mathcal{R}}[A]_{E_2} = \imath_{\mathcal{R}} d_v \beta = -d_v \imath_{\mathcal{R}}\beta$, and so

$$R(a) = \int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}}[A]_{E_2} = -d_v \int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}}\beta.$$

□

7.2. The boundary map

Let b_0 denote the differential map in $C_*(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$. We will derive a formula for the boundary map δ of the short exact sequence (1) in terms of the presentations found in Chapters 4 and 5 for Hochschild homology as de Rham cohomology groups. As with all spectral sequences, we can in principle observe only the top part, say $\delta_0 : HH(A_\sigma(X))_{[i]} \to HH(I_\partial(X))_{[i]}$, of δ. This might be zero on some element even though δ itself does not vanish on that element.

Remark 7.14. The boundary map commutes with e_D, L_D, $t^{-1}dt\wedge$, α and β, since these operations are maps of complexes on $C_*(\Psi_a(X), b)$.

This means that δ preserves the dt and dt-free parts, and the t-homogeneity in the presentation by cohomology spaces.

PROPOSITION 7.15. *If* $\dim M \neq 0$ *then* $\delta_0 = 0$.

Proof. If $M = \{pt\}$, then the adiabatic algebra becomes isomorphic to the algebra $\Psi_{\mathbb{R}_+}(X)$ of one-parameter families of pseudodifferential operators. In that case, $\delta = \delta_0$ since t is a "flat" parameter. If $\dim M \neq 0$, we claim that the vertical boundary map vanishes. This is essentially Lemma 8 from [18]. For convenience, we reproduce the proof. Let $A \in C_k(I_\sigma^{\mathbb{Z},[\mathbb{Z}]}(X))$ be a cycle. Let $\tilde{A} \in C_k(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))$ be an extension to the full vertical algebra. Then, using Proposition 6.9,

$$\delta_0(A) = \int_{{}^aT^*M_{|t=0}/M} *_a^M \chi_a(b_0\tilde{A}).$$

From Corollary 6.8, this is independent of ∇. The vector fields tangent to the fibers of ${}^aT^*M_{|t=0} \to M$ are outer derivations on the algebra $\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)$. By naturality, the boundary map commutes with the action L_V of such a vector field: $[\delta_0, L_V] = 0$. But $L_V(A)$ has order 1 less than A. After repeated applications, the order of $L_{V_1} \dots L_{V_s}\tilde{A}$ will be less than $-m-k-1$, so, by Proposition 6.2, $\chi_a(b(L_{V_1} \dots L_{V_s}\tilde{A})) = 0$. Hence, $L_{V_1} \dots L_{V_s}\delta_0(A) = 0$. Since $V_1, \dots, V_s$ were arbitrary and $\delta_0(A)$ is a Schwartz form on ${}^aT^*M_{|t=0}$, the proposition follows.

Therefore, the first significant part of δ is

$$\delta_1 : HH(A_\sigma(X))_{[i]} \to HH(I_\partial(X))_{[i-1]},$$

which decreases the t-filtration by 1.

Theorem 7.16. *The leading part of the boundary map is essentially integration along the fibers,*

$$\delta_1 = i\int_{{}^aS^*X_{|t=0}/M}.$$

Proof. From Remark 7.14, it is enough to prove the claim for a class $[a]$ represented by some $a \in T_jHH_k(A_\sigma(X)) \cap im(\alpha)$. Such a class can be represented by a chain $A \in T_jF_{k-n-m}C_k(A_\sigma(X))$. By H-unitality, we can find $\tilde{A} \in T_jC_k(\Psi_a(X))$ a pre-image of A, such that $b\tilde{A} \in C_{k-1}(I_\partial(X))$. Since $\delta_0(a) = 0$, it follows that there exists $\gamma \in T_jC_k(I_\partial(X))$ such that $b\tilde{A} - b\gamma \in T_{j-1}$. By replacing $\tilde{A}$ with $\tilde{A} - \gamma$, we can assume that $\delta(a) = [b\tilde{A}]$ is represented by a chain in T_{j-1}. We use now a choice of a vector space isomorphism between $A_\sigma(X)/I_\sigma(X)$ and its graded algebra $\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X)$. Let $A_{[j]}, A_{[j-1]}$ be the components of $\tilde{A}$ in $C_k(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))_{[j]}$, respectively $C_k(\Psi_a^{\mathbb{Z},[\mathbb{Z}]}(X))_{[j-1]}$, where the subscript denotes the negative of the t-degree.

Using (37), we see that

$$\delta_1(a) = \int_{{}^aT^*M_{|t=0}/M} *_a^M \chi_a(b\tilde{A})_{[j-1]}$$

$$= \left(\int_{{}^aT^*M_{|t=0}/M} *_a^M \chi_a(Q^z b\tilde{A})_{[j-1]} \right)_{|z=0}$$

$$= \left(\int_{{}^aT^*M_{|t=0}/M} *_a^M \chi_a(bQ^z\tilde{A})_{[j-1]} \right)_{|z=0} \tag{40}$$

$$+ \left(\int_{{}^aT^*M_{|t=0}/M} *_a^M \chi_a(zQ^z e_{D_z}\tilde{A})_{[j-1]} \right)_{|z=0}. \tag{41}$$

We shall prove that (40) vanishes and (41) equals the desired expression. In (40), we can replace $\tilde{A}$ by $A_{[j]} + A_{[j-1]}$ since we are looking only at the part of degree $j-1$ in t. Let $b = b_0 + b_1 + \dots$ be the expansion of b according to the t-degree. By Proposition 6.2, $\chi_a(b_0 Q^z A_{[j-1]})$ vanishes for small $\Re(z)$, so the analytic continuation of the corresponding part in (40) is 0 at $z = 0$.

PROPOSITION 7.17. *The identity $b_0(A^j) = 0$ implies that*

$$\int_{{}^aT^*M_{|t=0}/M} *_a^M \chi_a(b_1 Q^z A_{[j]}) - d_v \int_{{}^aT^*M_{|t=0}/M} *_a^M (\chi_a(Q^z A_{[j]})) \in z\mathcal{H}(\mathcal{B}) \tag{42}$$

thus $\int_{{}^aT^*M_{|t=0}/M} *_a^M \chi_a(b_1 Q^z A_{[j]})$ tends to 0 in $\Lambda^{2n-k+1}(M)/d\Lambda^{2n-k}(M)$ as $z \to 0$.

Proof. This would be clear if we knew that $b_0(Q^z A_{[j]}) = 0$. Indeed, in this case, for $\Re(z) < 0$,

$$\chi_a(b_1 Q^z A_{[j]}) = d_v(\chi_a(Q^z A_{[j]})) \tag{43}$$

(compare with Proposition 5.2). In the semi-classical limit case, this condition holds by commutativity of the vertical algebra $\Psi_a^{\mathbb{Z},[0]}(M)$. The proof in the general case consists of following an explicit proof of (43) and noting that whenever we have to commute Q^z, we get an error term in $z\mathcal{H}(\mathcal{B})$.

Lemma 7.18. *Let $C \in T_j F_{k-N} C_k(\Psi_a(X))$. If $A_{[j]} = b_0 C$, then*

$$\int_{{}^a T^* M_{|t=0}/M} *_a^M \chi_a(b_1 Q^z A_{[j]})_{[j-1]}$$

vanishes at $z = 0$.

Proof. Notice that

$$\begin{aligned} \chi_a(b_1 Q^z A_{[j]})_{[j-1]} &= \chi_a(b_1 Q^z b_0 C)_{[j-1]} \\ &= \chi_a(b_1 b_0 Q^z C + b_1 z Q^z e_{D_z} C). \end{aligned}$$

The last term belongs to $F_{k-N-2+z} C_{h-1}$ and is a multiple of z, hence like in Proposition 7.2, its integral vanishes at $z = 0$. From $b^2 = 0$, we deduce $b_1 b_0 = -b_0 b_1$. Since $b_1 Q^z C \in T_{j-1}$, it follows from Proposition 6.2 that $\int_{{}^a T^* M_{|t=0}/M} *_a^M \chi_a(b_0 b_1 Q^z C)_{[j-1]}$ vanishes for $\Re(z)$ sufficiently small, hence it is identically zero.

Note that the term (41) is by definition $R_{j-1}(e_{D_Q} \tilde{A})$. From Remark 7.12, this is a dt-free form of t-degree $-j + k - n$. From Propositions 4.10 and 7.11,

$$R(e_{D_Q} \tilde{A}) = i \int_{{}^a S^* X_{|t=0}/M} \imath_{\mathcal{R}}(r^{-1} dr \wedge [a]_{E_2}) = i \int_{{}^a S^* X_{|t=0}/M} [a]_{E_2}.$$

We have assumed that $a \in T_j$. We can also ask that $[a]_{E_2}$ be of pure homogeneity with respect to L_D. This homogeneity must be at least $-j + k - n$, and if it is bigger, then $[a]_{E_2}$ is the class of some $a' \in T_{j-1}$. So we can also assume that $[a]_{E_2}$ has homogeneity $-j + k - n$. Using again Remark 7.12, we conclude that $R_{j-1}(e_{D_Q} \tilde{A}) = R(e_{D_Q} \tilde{A}) = R(e_{D_Q} a)$. This ends the proof of Theorem 7.16.

Recall the decomposition of $T_i/T_{-\infty} HH_k(A_\sigma(X))$ as eigenspaces of L_D. Let a be of pure type with respect to this decomposition. Assume a is dt-free. Since $*_\phi$ increases the t degree at most by $k - n$, it follows that $[a]$ belongs to $T_{n+i-h} H^{2N-k}(S^* X \times S^1)$ with $h \leq k$ Hence $L_{\frac{d}{dt}}$ acts on $[a]$ as multiplication with $h - i - n$. Assume that $\delta_1(a) = 0$. This means that $\delta(a) \in T_{i-s} HH_{k-1}(I_\partial(X))$ for some $s \geq 2$ minimal with this property. Let $\delta(a)_s$ be the image of $\delta(a)$ in $T_{i-s}/T_{i-s-1} HH_{k-1}(I_\partial(X)) \cong E_2^{i+s, k-1-i-s}(I_\partial(X))$. $L_{\frac{d}{dt}}$ acts as multiplication with $s + k - i - 1 - n$ on this space. But from $h \leq k$ and $s \geq 2$ we get $h - i - n < s + k - i - 1 - n$. Since δ and $L_{\frac{d}{dt}}$ commute, we deduce

PROPOSITION 7.19. *Let* $a \in T_i HH_k(A_\sigma(X))$. *If* $\delta_1(a) = 0$, *then there exists some* $\tilde{a}$ *with* $a - \tilde{a} \in T_{i-1}HH_k(A_\sigma(X))$ *and* $\delta(\tilde{a}) = 0$.

Theorem 7.16 and Proposition 7.19 completely characterize the boundary map

$$\delta : HH(A_\partial(X)) \to HH(I_\partial(X)).$$

7.3. Traces on the adiabatic algebras

From the long exact sequence derived from the short exact sequence

$$0 \to \Psi_a^{-\infty,-\infty}(X) \hookrightarrow \Psi_a^{-\infty}(X) \to I_\partial(X) \to 0, \tag{44}$$

the ideals $I_\partial(X)$ and $\Psi_a^{-\infty}(X)$ have the same homology, except in dimensions 0 and 1. Consider the following map:

$$\mathrm{Tr} : \Psi_a^{-\infty}(X) \to C^\infty(\mathbb{R}_+)[t^{-1}], \quad A \mapsto \int_{\Delta_a/\mathbb{R}_+} A_{|\Delta_a}.$$

LEMMA 7.20. *The map* Tr *generates* $HH^0(\Psi_a^{-\infty,\mathbb{Z}}(X))$ *as a module over* $C^\infty(\mathbb{R}_+)[t^{-1}]$.

Proof. Let first $A \in \dot{C}^\infty(\mathbb{R}_+, \Psi^{-\infty}(X)) \cong \Psi_a^{-\infty,-\infty}(X)$ be a rapidly vanishing family of smoothing operators. For any $t > 0$, the map $A \mapsto \mathrm{Tr}(A)(t)$ is a trace. Choose any local embedding of TX in X^2. Cut A off near the diagonal and pull it back to TX as a compactly supported section in the fiber density bundle. The pull-back of the fiber density bundle to the 0-section of TX is just the density bundle $\Omega(TX)$. In local coordinates, and pulling back via the canonical map ϕ_a, we get

$$\mathrm{Tr}(A) = \frac{1}{(2\pi)^n}\int_{T^*X} \hat{A}\omega^N = \frac{1}{(2\pi)^n}\int_{{}^aT^*X} \hat{A}\omega_a^N,$$

where ω, ω_a are the symplectic form, respectively the adiabatic symplectic form. This formula extends to $\Psi_a^{-\infty}(X)$, and is an extension of the map

$$\begin{aligned} I_\partial(X) \ni A \mapsto & \frac{1}{(2\pi)^n}\int_{{}^aT^*M_{|t=0}} \mathrm{Tr}_V(\hat{A})(\omega_a^M)^n \\ &= \frac{1}{(2\pi)^n}\int_{{}^aT^*M_{|t=0}} *_a\chi_a(A) \in \mathcal{L}, \end{aligned}$$

which, by Theorem 6.10 and Propositions 6.9 and 5.3, is a generator over $\mathcal{L}$ of $HH^0(I_\partial(X))$. This implies that the boundary map

$$HH_1(I_\partial(X)) \to HH_0(\Psi_a^{-\infty,-\infty}(X))$$

vanishes.

Recall that e_D is an isomorphism from $im(\beta) \subset HH_1(I_\partial(X))$ to $HH_0(I_\partial(X))$. A similar statement is valid for $\Psi_a^{-\infty,-\infty}(X)$. From this, it follows that $\mathrm{Tr} \circ e_D$ is a Hochschild cochain which generates $im(\beta) \subset HH_1(\Psi_a^{-\infty}(X))$, and that the other boundary map of (44) also vanishes.

Since we have the explicit $C^\infty(\mathbb{R}_+)[t^{-1}]$-valued cochain Tr on $\Psi_a^{-\infty}(X)$, we can imitate the construction of the residue trace to get explicit cocycles on $\Psi_a(X)$. Namely, we claim that the maps

$$\begin{aligned} \Psi_a(X) \to C^\infty(\mathbb{R}_+)[t^{-1}] \qquad & A \mapsto \mathrm{Res}_{z=0} \mathrm{Tr}(Q^z A) \\ C_1(\Psi_a(X)) \to C^\infty(\mathbb{R}_+)[t^{-1}] \qquad & A \otimes B \mapsto \mathrm{Res}_{z=0} \mathrm{Tr}(Q^z D(B) A) \end{aligned}$$

are cocycles on $\Psi_a(X)$. Since e_D is a map on homology, it is enough to prove the claim for the first map. We have seen that the residue is well-defined. We want to show that it vanishes on commutators. We have

$$\mathrm{Res}_{z=0} \mathrm{Tr}(Q^z[A,B]) = \mathrm{Res}_{z=0} \mathrm{Tr}([Q^z A, B] + zQ^z \frac{Q^{-z} B Q^z - B}{z} A) = 0,$$

since the first term vanishes for small $\Re(z)$, and the second term has no residue at zero by Proposition 7.5.

For $t > 0$, the first map is just Wodzicki's residue trace on X, hence it equals $\int_{{}^aS^*X} (\hat{A}_{-N}) \imath_{\mathcal{R}} \omega_a^N$. By continuity, this holds down to $t = 0$. We note here that for $t > 0$, $\omega_a^N = (\phi_a^{-1})^*(\omega^N)$ belongs to $T_{-n}\Lambda^2({}^aT^*X)$. Indeed, from (10), $\det(\phi_a) = t^n$. Therefore, the residue trace of a smooth adiabatic operator is a Laurent function in t of degree $-n$.

7.4. Properties of the residue functional

We can extend Proposition 7.13 to arbitrary orders. We first treat the semi-classical limit case. Recall that the subscript i means taking the part in T_i/T_{i-1}, i.e., the coefficient of t^{-i} in the Laurent expansion at $t = 0$.

Proposition 7.21. *If $A \in T_i C_k(A_\sigma(M))$ is a boundary, then $R_i(A)$ is d_v-exact.*

Proof. The spectral sequence ${}^\partial E_2$, which computes $HH(A_\sigma(M))$ using the T_i filtration, degenerates at E_2, so we can assume $A = b(P)$ with $P \in T_{i-1}C_{k+1}(A_\sigma(M))$. The semi-classical limit vertical algebra is commutative so $(Q^z P)_{i-1}$ is b-closed (this is not true in the general adiabatic setting). The projection $\Psi_a \to S_a$ is surjective; let $\tilde{P}$ be a pre-image of P. Using

(37), we get:

$$\begin{aligned}R_i(A) &= R_i(bP)\\ &= \mathrm{Res}_{z=0}\int_{{}^aT^*M_{|t=0}/M} *_a^M(\chi_a(Q^z b\tilde{P}))_i\\ &= \mathrm{Res}_{z=0}\int_{{}^aT^*M_{|t=0}/M} *_a^M(\chi_a(b(Q^z\tilde{P}))_i\\ &\quad + \mathrm{Res}_{z=0}\int_{{}^aT^*M_{|t=0}/M} *_a^M(\chi_a(zQ^z e_{Q^z}\tilde{P}))_i.\end{aligned}$$

By Proposition 7.5 and because of the z factor, the last term vanishes since $e_{Q^z}\tilde{P} \in T_i$ is entire. As for the first term, we claim it is exact.

Lemma 7.22. *For $\Re(z)$ sufficiently small, we have*

$$\int_{{}^aT^*M_{|t=0}/M} *_a^M\chi_a(b(Q^zP))_i = d_v\left(\int_{{}^aT^*M_{|t=0}/M} *_a^M(\chi_a(Q^zP))_{i-1}\right). \tag{45}$$

Proof. We notice that Q^zP survives at ${}^\partial E_1^{i,k-i}(\Psi_a^{z+Z}(M))$ Then (45) is the computation of $d_1[Q^zP]$ in the spectral sequence of the symbol algebra of order $z+\mathbb{Z}$. The identity $d_1 = *_a^{-1}d_v *_a$ (see (26)) holds in this context. The proof of Proposition 7.17 applies to prove the result.

By the de Rham theorem, the space of exact forms is closed. A meromorphic family of forms, which takes exact values for z in an open set, must have exact residues. This finishes the proof of Proposition 7.21.

From Theorem 4.6 and Proposition 7.13 it follows that every Hochschild class in $T_iHH(A_\sigma(X))$ has a representative a such that $R_i(a)$ is closed. Together with Proposition 7.21, this proves:

Corollary 7.23. *The residue R_i descends as a map*

$$R : T_iHH_k(A_\sigma(M)) \to H^{n-k+j}(M) \otimes \Lambda^j\mathcal{L}.$$

Proposition 7.21 is a model for the general case. The main difference is that multiplication by Q^z destroys the cycle property.

Proposition 7.24. *Let $A \in T_iC_k(\Psi_a(X))$. If $bA \in T_{i-1}C_{k-1}(\Psi_a(X))$, then $R_{i-1}(bA) = dR_i(A)$, where d is the de Rham differential on M.*

Proof. $Q^z A_i$ is not a cycle in the vertical algebra, but, since $b(A_i) = 0$, it has the property that $b_0(Q^z A_i)$ is a holomorphic multiple of z. The proof of Proposition 7.17 goes through, with the following modification: replace congruences $\pmod{z\mathcal{H}(\mathcal{B})}$ by equalities of residues at $z = 0$.

It follows that if $A \in T_i C_k(\Psi_a(X))$ is Hochschild closed, then $R_i(A)$ is de Rham closed. Let now $A \in T_i F_j C_k(A_\sigma(X)) \cap im(\alpha)$. Assume that A is b-exact and that $j > k - N$. Then, there exists some $P \in T_{i-1} C_{k+1}(A_\sigma(X))$ such that $bP - A \in T_i F_{j-1} C_k(A_\sigma(X))$. This is a restatement of Proposition 4.15. By Proposition 7.24, $R_i(A) = R_i(A - bP)$ modulo exact forms. Inductively, $R_i(A) = R_i(A')$, where $A' \in T_i F_{k-N} C_k(\Psi_a(X))$. From Proposition 7.13, $R_i(A')$ is exact. As a consequence, we get the main result of this section:

THEOREM 7.25. *The residue R_i descends as a map on the Hochschild homology of adiabatic symbols with values in the cohomology of M (with twisted coefficients):*

$$T_i HH_k(A_\sigma(X)) \xrightarrow{R_i} H^{n-k}(M, \mathcal{O}M) t^{i+k-n} \oplus H^{n-k+1}(M, \mathcal{O}M) t^{i+k-n-2} dt.$$

On chains $A = A_0 \otimes \ldots \otimes A_k \in T_i C_k(\Psi_a(X))$, R_i is given by the explicit formula

$$R_i(A) = \mathrm{Res}_{z=0} \int_{{}^aT^*M_{|t=0}/M} *_a^M (\chi_a(Q^z A^{[i]})).$$

§8. An example

Consider a fibration $X \to S^1$ and let $A \in \Psi_a(X)$ be an elliptic adiabatic operator. Choose an inverse b of a modulo $\Psi_a^{-\infty}(X)$, then the Hochschild 1-chain $a \otimes b$ defines a cycle in $C_1(\Psi_a(X)/\Psi_a^{-\infty}(X))$. This cycle was introduced by Melrose and Nistor in [17]. The t-degree of this cycle is 0, in other words $a \otimes b \in T_0 C_1(A_\sigma(X))$.

We can therefore apply the functional R_0 to this cycle. We examine below the dt-free component $f(A) \in H^0(M)$ of $R_0(\mathrm{tr}(a \otimes b))$.

Let x, ξ be local coordinates in ${}^aT^*M_{|t=0}$ (this bundle is canonically trivial in our case). The adiabatic symplectic form has the form $\omega_a = t^{-1} d\xi \wedge dx$. Let α, β be the images of a, b in the suspended algebra of the fibers with parameters in the circle $M = S^1$ (i.e., in the graded algebra of $\Psi_a(X)$ associated to the filtration T_j). Following the definition, we get

$$f(A) = \left[\mathrm{Res}_{z=0} \int_{\mathbb{R}} \mathrm{Tr}_V(Q^z \alpha \partial_\xi \beta) d\xi \in \Lambda^0(S^1) \right]_{H^0(S^1)}.$$

Since we proved in Theorem 7.25 that $R_0(\mathrm{tr}(a \otimes b))$ is exact as a form on M, it follows that $\mathrm{Res}_{z=0} \int_{\mathbb{R}_\xi} \mathrm{Tr}_{\mathrm{V}}(Q^z \alpha \partial_\xi \beta) d\xi \in \Lambda^0(S^1)$ is constant as a function of $x \in S^1$. This constant equals $\mathrm{Tr}_{\mathrm{R}}(\alpha(x)\partial_\xi \beta(x))$ for all $x \in S^1$, where $\mathrm{Tr}_{\mathrm{R}} = \mathrm{Res}_{z=0} \overline{\mathrm{Tr}} \circ Q^z$ is the residue trace on the suspended algebra defined in [23]. This motivates the following definition:

Definition 8.1. Let Y be a closed manifold. Let $\alpha \in \Psi_{\mathrm{sus}(1)}(Y, \mathcal{E}, \mathcal{F})$ be an elliptic suspended operator in the sense of Melrose [16]. Define

$$f(\alpha) = \mathrm{Tr}_{\mathrm{R}} \left(\alpha \frac{\partial \beta}{\partial \xi} \right),$$

where β is an inverse of α modulo $\Psi^{-\infty}_{\mathrm{sus}(1)}(Y)$.

It is straightforward to see that f has the following properties:

1. $f(\alpha)$ is homotopy invariant inside elliptic suspended operators.
2. $f(\alpha_1 \alpha_2) = f(\alpha_1) + f(\alpha_2)$.
3. When $\alpha = D + i\xi$, where D is a positive first-order differential operator on Y, then $f(\alpha)$ equals the residue at the origin of the meromorphic function $\eta(D, s)$. From [1], this residue vanishes so $f(D + i\xi) = 0$
4. If α is differential, or more generally has even symbol in the sense of Gilkey [8], and $\dim(Y)$ is even, then $f(\alpha) = 0$.

The first two properties show that f depends only on the class in $K^0(T^*_{\mathrm{sus}(1)}Y) = K^1(T^*Y)$ of the principal symbol of α. For $\dim(Y)$ odd, this K group is generated rationally by the symbols of twisted signature operators, for which f was seen to vanish. Thus f is identically zero when $\dim(Y)$ is odd.

On even dimensional manifolds, Gilkey [8] has shown that $K^0(S^*Y)$ is generated rationally by self-adjoint polynomial symbols of even degree with values in GL. To such a symbol p of degree k we associate the class $p + i\xi(p^2 + \xi^{2k})^{\frac{2k-1}{2k}} \in K^1(T^*Y)$; in fact these classes generate $K^1(T^*Y, \mathbb{Q})$. Let $A \in \Psi^k_{\mathrm{sus}(1)}(Y)$ be the suspended operator defined by

$$A := P + i\xi(P^2 + \xi^{2k})^{\frac{2k-1}{2k}}$$

where P is an invertible self-adjoint differential operator on Y with symbol p (such an operator exists since we can add a small real constant to remove the possible eigenvalue 0). We set $Q := (A^*A)^{\frac{1}{2}}$ and $B := A^{-1}$. The operator A is not differential, nor is its symbol even, so we cannot conclude directly that $f(A) = 0$. However, for each fixed $\xi \in \mathbb{R}$, the operators A, Q and $\partial_\xi A$ preserve the eigenspaces of P. We decompose the trace defining $f(B)$ over these eigenspaces; we get

$$\int_{\mathbb{R}} \mathrm{Tr}_{\mathrm{V}}(Q^{-z} B \partial_\xi A) d\xi = \eta(P, z) h(z)$$

where $\eta(P, z)$ is the APS eta function of P, and $h(z)$ is the integral on $\mathbb{R}$ of a smooth function in ξ, depending holomorphically on z, which grows like $\xi^{-2-\frac{z(k+1)}{2}}$ as $|\xi| \to \infty$. This integrand can be made explicit, but we only need to notice that $h(z)$ is regular at $z = 0$. Together with the regularity of the eta function shown by Gilkey [8], this implies that $f(B)$, and so also $f(A)$, vanish when $\dim(Y)$ is even.

Therefore in this example the invariant f vanishes identically. Nevertheless, this vanishing is highly non-trivial. This fact suggests that the homological residues R_j introduced above may lead to subtle geometric and analytic properties of elliptic operators.

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Institutul de Matematică al Academiei Române
P. O. Box 1-764, RO-70700
Bucharest
Romania
moroianu@alum.mit.edu