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BERGMAN COMPLETENESS OF HYPERCONVEX MANIFOLDS

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Abstract. We proved that any hyperconvex manifold has a complete Bergman metric.

§1. Introduction

Let M be an $n-$dimensional complex manifold. Let $\mathcal{H}$ denote the Hilbert space of holomorphic $n-$forms on M such that $|\int_M f \wedge \bar{f}| < \infty$. Let $h_0, h_1, \cdots$ be a complete orthonormal basis for $\mathcal{H}$. Then the $2n-$form defined on $M \times M$ given by $K_M = \sum_{j=0}^{\infty} h_j \wedge \bar{h}_j$ is called the Bergman kernel form of M. Let $z = (z_1, \cdots, z_n)$ be a local coordinate system in M and let $K_M(z) = K_M^*(z) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ where K_M^* is a locally defined function. Then $\beta := \partial\bar{\partial} \log K_M^*$ is a well-defined Hermitian form of bidegree (1,1), whenever K_M^* is nonzero. We call β the Bergman metric if it is everywhere positive definite. Let us recall

Definition. A complex manifold M is called hyperconvex if there exists a strictly plurisubharmonic (psh) function $\rho : M \to [-1, 0)$ such that $\{x \in M : \rho(x) < c\}$ is relatively compact in M for every $c < 0$.

The purpose of this note is to show the following

Theorem 1. *Every hyperconvex manifold has a complete Bergman metric.*

Theorem 1 was conjectured by S. Kobayashi [11]. In the special case of bounded hyperconvex domains $\Omega \subset \mathbf{C}^n$, it suffices to show that the volume of $\{g_\Omega(\cdot, y) < -1\}$ tends to zero as $y \to \partial\Omega$, where $g_\Omega(\cdot, y)$ denotes the

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pluricomplex Green function of Ω (cf. Chen [3] and Herbort [8] independently), and this property was verified by Blocki-Pflug [2] (independently Herbort [8]). The case of hyperconvex Riemann surfaces was shown in [4].

Combining with a theorem of Ohsawa-Sibony [13], we obtain

Corollary 2. *Every bounded pseudoconvex domain with C^2 boundary in a complex manifold with positive holomorphic bisectional curvature (eg.* $\mathbf{P}^n$*) is Bergman complete.*

Greene-Wu [7] proved the the existence of a bounded smooth strictly psh exhaustion function under the following curvature condition. Hence

Corollary 3. *Let M be a complete Kähler manifold with a pole o such that its sectional curvature* K *is non-positive and in addition satisfies*

$$\mathrm{K} \le -\frac{1+\epsilon}{r^2 \log r}$$

for some constant $\epsilon > 0$ outside a compact subset of M, where r denotes the distance function based at o. Then M has a complete Bergman metric.

In [7], Bergman completeness has been shown in the case when M is a simply-connected complete Kähler manifold such that the sectional curvature is suitably negatively pinched, for instance, pinched between negative constants. Their result was extended in [4] by only assuming that the curvature is bounded from above by $-A/r^2$.

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§2. Proof of Theorem 1

Let $g_M(\cdot, y)$ be the pluricomplex Green function on M, i.e.,

$$g_M(x, y) = \sup\{u(x)\}$$

where the superum is taken over all negative functions $u \in PSH(M)$ satisfying the property that the function $u - \log|z|$ is bounded from above in a deleted neighborhood of y for some holomorphic local coordinates z centered at y, that is, $z(y) = 0$. Since M is hyperconvex, $g_M(\cdot, y)$ is non-trivial (cf. [4]). Set $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$. It is easy to see that $dd^c = 2i\partial\bar{\partial}$. As in [2], the following inequality of Blocki is again crucial.

PROPOSITION 4. (cf.[1]) *Let Ω be a smooth bounded domain in a complex manifold M. Assume that $u, v \in C^\infty(\overline{\Omega})$ are non-positive psh functions such that $u = 0$ on $\partial\Omega$. Then*

$$\int_\Omega |u|^n (dd^c v)^n \le n! \|v\|_\infty^{n-1} \int_\Omega |v| (dd^c u)^n.$$

Proof. Note that

$$\begin{aligned}\int_\Omega (-u)^n (dd^c v)^n &= n \int_\Omega (-u)^{n-1} du \wedge d^c v \wedge (dd^c v)^{n-1} \\ &= n \int_\Omega (-u)^{n-1} dv \wedge d^c u \wedge (dd^c v)^{n-1} \\ &= n \int_\Omega (-u)^{n-1} (-v) dd^c u \wedge (dd^c v)^{n-1} \\ &\quad + n(n-1) \int_\Omega (-u)^{n-2} v du \wedge d^c u \wedge (dd^c v)^{n-1} \\ &\le n \|v\|_\infty \int_\Omega (-u)^{n-1} dd^c u \wedge (dd^c v)^{n-1}\end{aligned}$$

where the first and third equalities follow from Stokes' theorem and the second one from the fact that the $(1,1)$ parts of $du \wedge d^c v$ and $dv \wedge d^c u$ coincide, the inequality follows from $du \wedge d^c u = 2i\partial u \wedge \bar{\partial} u \ge 0$. The desired inequality is obtained by repeating the argument $n-1$ times.

LEMMA 5. *Let M be a hyperconvex manifold. For any $y \in M$, the following inequality holds:*

$$\int_M |g_M(\cdot, y)|^n (dd^c \rho)^n \le n! (2\pi)^n |\rho(y)|.$$

Proof. For any positive integer j, we set $M_j = \{x \in M : \rho(x) < 1/j\}$. Let $y \in M$ and let $g_{M_j}(\cdot, y)$ denote the pluricomplex Green function on M_j for all sufficiently large j. Since M_j is hyperconvex, for any fixed j, the function $\max\{g_{M_j}(\cdot, y), -k\} + \frac{1}{k}(\rho - 1/j)$ is a continuous strictly psh function and approaches to zero at ∂M_j for each integer $k > 0$ (cf. [6]). According to a well-known theorem of Richberg [14], there is a smooth psh function $\{g_{j,k}\}$ on M_j such that

$$\left| g_{j,k} - \max\{g_{M_j}(\cdot, y), -k\} - \frac{1}{k}(\rho - 1/j) \right| < \frac{1}{2k} |\rho - 1/j|,$$

which implies $g_{j,k} < 0$, $g_{j,k}(x) \to 0$ as $x \to \partial M_j$ and $g_{j,k} \to g_{M_j}(\cdot, y)$ as $k \to \infty$. Hence we can take a sequence of positive numbers $\{\lambda_k\}$ with $\lambda_k \to 0$ as $k \to \infty$ such that

$$M_{j-1} \subset\subset M_{j,k} := \{x \in M_j : g_{j,k}(x) < -\lambda_k\} \subset\subset M_j$$

and $M_{j,k}$ has a smooth boundary by Sard's theorem. By Proposition 4, we have

$$\begin{aligned}\int_{M_{j-1}} |g_{j,k} + \lambda_k|^n (dd^c\rho)^n &\le \int_{M_{j,k}} |g_{j,k} + \lambda_k|^n (dd^c\rho)^n \\ &\le n! \int_{M_{j,k}} |\rho| (dd^c g_{j,k})^n \\ &\le n! \int_{M_j} |\rho| (dd^c g_{j,k})^n.\end{aligned}$$

According to [6], letting $k \to \infty$, we obtain

$$\begin{aligned}\int_{M_{j-1}} |g_{M_j}(\cdot, y)|^n (dd^c\rho)^n &\le n! \int_{M_j} |\rho| (dd^c g_{M_j}(\cdot, y))^n \\ &= n!(2\pi)^n |\rho(y)|\end{aligned}$$

where the equality follows from $(dd^c g_{M_j}(\cdot, y))^n = (2\pi)^n \delta_y$ on M_j. The desired inequality is then obtained by letting $j \to \infty$ since $g_{M_j}(\cdot, y) \searrow g_M(\cdot, y)$.

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Proof of Theorem 1. The existence of the Bergman metric of a hyperconvex manifold was shown in [4]. Take a smooth function χ on $\mathbf{R}$ such that $\chi = 1$ on $(-\infty, -1]$ and $\chi = 0$ on $[0, \infty)$. Let $f \in \mathcal{H}$ and $\{y_k\}_{k=1}^{\infty}$ be a sequence of points which has no adherent point in M. Set

$$\begin{aligned}\eta_k &= \chi\left(-\log(-g_M(\cdot, y_k) + 1) + \log 2\right) f. \\ \varphi_k &= 2n g_M(\cdot, y_k) - \log(-g_M(\cdot, y_k) + 1)\end{aligned}$$

Let us first proceed the proof under the assumption that η_k, φ_k are smooth and φ_k is strictly psh. By the well-known L^2 estimates (cf. [5], [12]), we can solve the equation $\overline{\partial} u_k = \overline{\partial}\eta_k$ in such a way that

$$\left|\int_M u_k \wedge \bar{u}_k e^{-\varphi_k}\right| \le \int_M |\overline{\partial}\eta_k|^2_{\partial\bar\partial\varphi_k} e^{-\varphi_k} dV_{\varphi_k}$$
$$\le C_1 \left|\int_{A_k} f \wedge \overline{f}\right|$$

since

$$\partial\bar\partial\varphi_k \ge \frac{\partial g_M(\cdot, y_k)\bar\partial g_M(\cdot, y_k)}{(-g_M(\cdot, y_k)+1)^2}.$$

Here C_1 is a constant depending only on $\sup|\chi'|$ and $A_k = \{x \in M : g_M(\cdot, y_k) < -1\}$. The general case follows from a standard limiting procedure as follows: By a similar argument as in the proof of Lemma 5, one can approximate $g_M(\cdot, y_k)$ by a sequence of negative smooth strictly psh functions on M and solve the $\bar\partial-$equation with $g_M(\cdot, y_k)$ replaced by such functions, then take a limit.

Hence the function $F_k = \eta_k - u_k$ is holomorphic on M which satisfies $F_k(y_k) = f(y_k)$ and $\left|\int_M F_k \wedge \overline{F}_k\right| \le C_2 \left|\int_{A_k} f \wedge \overline{f}\right|$. It follows that

$$\frac{f(y_k)\wedge \bar{f}(y_k)}{K_M(y_k)} \le C_2 \left|\int_{A_k} f \wedge \overline{f}\right|. \tag{1}$$

For any $\epsilon > 0$, there is a relative compact subset M_ϵ so that

$$\left|\int_{M\setminus M_\epsilon} f \wedge \bar{f}\right| < \epsilon. \tag{2}$$

By Lemma 5, we have

$$\int_{M_\epsilon \cap A_k} (dd^c\rho)^n \le \int_M |g_M(\cdot, y_k)|^n (dd^c\rho)^n$$
$$\le n!(2\pi)^n |\rho(y_k)|.$$

This shows that one can choose a k_ϵ such that for all $k > k_\epsilon$,

$$\left|\int_{M_\epsilon\cap A_k} f\wedge\bar{f}\right| \le \sup_{M_\epsilon}\left|\frac{f\wedge\bar{f}}{(dd^c\rho)^n}\right| \cdot \int_{M_\epsilon\cap A_k}(dd^c\rho)^n < \epsilon \tag{3}$$

since $\rho(y_k) \to 0$ as $k \to \infty$. By (1)–(3) and the well-known Kobayashi's criterion [10], the Bergman metric on M is complete.

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