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FORMATION AND CONSTRUCTION OF A SHOCK WAVE FOR 3-D COMPRESSIBLE EULER EQUATIONS WITH THE SPHERICAL INITIAL DATA

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Abstract. In this paper, the problem on formation and construction of a shock wave for three dimensional compressible Euler equations with the small perturbed spherical initial data is studied. If the given smooth initial data satisfy certain nondegeneracy conditions, then from the results in [22], we know that there exists a unique blowup point at the blowup time such that the first order derivatives of a smooth solution blow up, while the solution itself is still continuous at the blowup point. From the blowup point, we construct a weak entropy solution which is not uniformly Lipschitz continuous on two sides of a shock curve. Moreover the strength of the constructed shock is zero at the blowup point and then gradually increases. Additionally, some detailed and precise estimates on the solution are obtained in a neighbourhood of the blowup point.

§1. Introduction

In this paper, we are concerned with the development of singularities of solutions to the following three dimensional compressible Euler equations with smooth spherical initial data

$$(1.1)\qquad \begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u+pI)=0,\\ \partial_t\Big(\rho e+\dfrac{1}{2}\rho u^2\Big)+\operatorname{div}\Big(\big(\rho e+\dfrac{1}{2}\rho u^2+p\big)u\Big)=0,\\ \rho|_{t=0}=\bar\rho+\varepsilon\rho_0(x),\quad u|_{t=0}=\varepsilon u_0(x),\quad S|_{t=0}=\bar S, \end{cases}$$

where $u=(u_1,u_2,u_3)$ is the velocity, ρ the density, p the pressure, e the internal energy, I the 3×3 unit matrix, and S the specific entropy. Moreover, the pressure function $p=p(\rho,S)$ and the internal energy function

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$e = e(\rho, S)$ are smooth in their arguments, in particular, $\partial_\rho p(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$. With respect to the initial data in (1.1), $\bar{\rho} > 0$ and $\bar{S}$ are constants, $\varepsilon > 0$ is a small parameter, $\rho_0(x)$ and $u_0(x)$ are in $C^\infty(\mathbb{R}^3)$ with supports in the ball $B(0, M)$ of radius $M > 0$ centered at the origin. In what follows, we assume that $u_0(x) = w_0(x)x$, with a smooth function $w_0(x)$, and that $w_0(x)$ and $\rho_0(x)$ are functions of $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Under the above assumptions, we know by [23] and [24] that the lifespan T_ε of a smooth solution to (1.1) satisfies:

$$\begin{aligned}(1.2)\quad &\lim_{\varepsilon\to 0} \varepsilon \ln T_\varepsilon = \tau_0 \\ &\equiv -\frac{2\bar{c}}{(\bar{\rho}c'(\bar{\rho}, \bar{S}) + \bar{c}) \min\limits_{|q|\le M} \left[q^2\partial_q w_0(q) + 3q w_0(q) + \frac{\bar{c}}{\bar{\rho}}(q\partial_q \rho_0(q) + \rho_0(q))\right]},\end{aligned}$$

where $c(\rho, S) = \sqrt{\partial_\rho p(\rho, S)}$, $\bar{c} = c(\bar{\rho}, \bar{S})$; and $\tau_0 > 0$ as long as $\rho_0(x) \not\equiv 0$ or $u_0(x) \not\equiv 0$. Therefore, (1.2) implies that the nontrivial smooth solution of (1.1) must blow up in finite time. To better understand the physical process of development of singularities from smooth flow and the evolution of singularities starting from the blowup point, we are motivated to deduce more precise estimates of a solution and its derivatives in a neighbourhood of the blowup point.

Now we briefly mention some remarkable works on the hyperbolic conservation laws in one space dimension, since this will be helpful to understand our motivation better in this paper. For 1-D hyperbolic systems of conservation laws, there is an extensive literature treating the global existence and uniqueness of weak solutions with (small) initial data in BV spaces (see [3], [9], [10], [16], [19] and the references therein). For instance, the results of [3] and [19] yield uniqueness, continuous dependence and global stability of entropy-admissible weak solutions for general $n \times n$ systems with small initial data. In the case of the system (1.1) with spherical structure, we get a 3×3 system of conservation laws with source terms, by use of polar coordinates. However, these source terms involve singularities at $r = 0$, and this fact causes many difficulties when one wants to study the global existence and uniqueness of weak solutions. Even the global existence of spherical weak entropy solutions of (1.1) seems to be open. Here we should point out that when (1.1) does not contain the third equation (i.e., the equation for the entropy), the authors in [7] consider the case where $p(\rho) = A\rho^\gamma$, $1 \le \gamma \le 5/3$, and prove the global existence of a weak entropy solution outside a core centered at the origin. Anyway, whether global solutions exist or not, all above results do not give detailed properties of a classical solution near blowup points. From the viewpoint of understanding the physical process of the appearance of singularities, it is an interesting problem to give a clear picture on the generation of singularities at a blowup point, in particular, that of the singularity of the type of shock.

As in [5], the above problem is called formation and construction of a shock. For scalar equations, this problem has been completely solved early (see [6], [8], [20] and so on). It is well known that in this case the formation of a shock is caused by the squeeze of characteristics. For 2×2 p-systems of gas dynamics the same fact is also true (see [4] and [17]) under certain conditions of nondegeneracy on the initial data or the blowup point. One of the basic ideas in [4] and [17] is to introduce the Riemann invariants so that the p-system can be diagonalized and subsequently to analyze the structure of singularities at a blowup point as in the case of 1-D scalar equations.

For $n \times n$ $(n \geq 3)$ systems in one space dimension, it is well known (see [1], [11], [12], [18]) that if the system is strictly hyperbolic and genuinely nonlinear with respect to a characteristic family and if the initial data are smooth and satisfy some condition of nondegeneracy, then the corresponding smooth solution blows up only at one spatial point at the blowup time. We constructed in [5] a weak entropy solution near the blowup point. In contradiction to the case of 2×2 p-systems as treated in [4] and [17], one cannot effectively use of the Riemann invariants. One of the new ideas in [5] is to find a new transformation of an $n \times n$ system by which the corresponding solution of the resulting system becomes more singular in one specific direction than the others. Using this new form of the system as well as Alinhac's result (see [1]) on the blowup system analysis, we carried out in [5] delicate analysis and constructed a shock starting from a blowup point.

A problem arises naturally: Consider 1-D systems of conservation laws with source terms. Suppose the first-order derivatives of a smooth solution blow up, while the solution itself remains bounded. Will the shock be formed from the blowup point and propagate as shown in [4], [5] and [17]? For 1-D compressible isentropic Euler equations, we have shown in [5] that a new shock is formed and propagates from the blowup point. Do similar phenomena occur for the multidimensional system (1.1) with spherical structure? To this last question, we will give an affirmative answer in this paper. In proving our assertion, one of the main difficulties is that the derivatives of a solution blow up like $(T_\varepsilon - t)^{-1}$, with T_ε the blowup time, are not locally integrable in space-time variables. In this sense, our problem is different from the usual Riemann problem. In fact, for the Riemann problem, the initial data are discontinuous and piecewise smooth; and the derivatives are locally bounded (except the appearance of a measure) around singularities. In addition to the methods developed in [5], we need some new ingredients to overcome the above-mentioned difficulty. Firstly, we need a result on extending a solution across the blowup time so that we can not only analyze the blowup mechanism at the blowup point, but also describe in detail the behavior of the derivatives of the solution. Secondly, in the case where a shock arises, we have to carry out more detailed computation than in the case treated in [5], which is needed by the appearance of singular source terms after transforming (1.1), in order to prove the convergence of approximate solutions around the blowup time. Here we should note that the approximate solutions are not uniformly Lipschitzian in their domains of definition.

Our paper is organized as follows. In Section 2, we first prove that a solution of (1.1) blows up with respect to the third eigenvalue and that there exists a unique blowup point under certain assumption of nondegeneracy on the initial data. Secondly, we transform (1.1) to a new form and give a precise description on formation (and construction) of a shock. In Section 3, we construct approximate solutions near the blowup point with the aid of a specific iteration scheme and prove the existence of a solution with a shock starting from the blowup point.

§2. Analysis on the blowup mechanism and main theorem

Now we study the blowup mechanism of smooth solution to (1.1) and extend the solution of (1.1) across the blowup time. We will also distinguish a direction along which the first order derivatives of ρ and u will blow up.

Firstly, it is easy to see that as far as smooth solutions are concerned, (1.1) is equivalent to the following system:

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho u) = 0, \\
\partial_t u + u\nabla u + \dfrac{\nabla p}{\rho} = 0, \\
\partial_t S + u\nabla S = 0, \\
\rho|_{t=0} = \bar{\rho} + \varepsilon\rho_0(r), \quad u|_{t=0} = \varepsilon w_0(r)x, \quad S|_{t=0} = \bar{S},
\end{cases}
\tag{2.1}
$$

where ∇ denotes $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

Here we emphasize that the systems (1.1) and (2.1) are not equivalent for the weak solution. When the smooth solution of (1.1) blows up and the shock wave is formed, then across the shock the entropy S will become a function of (t, x) plus a constant.

When the initial velocity in (2.1) is radial, the smooth solution to (2.1) has such a form in $t < T_\varepsilon$: $\rho(t,x) = \rho(t,r)$, $u(t,x) = \nabla\omega(t,r)$ and $S(t,x) = \bar{S}$, where $\omega(t,r)$ is a potential function of velocity.

From the second equation in (2.1), we get

$$\partial_t \nabla\omega + \nabla\Big(\frac{1}{2}|\nabla\omega|^2\Big) = -\nabla h(\rho), \tag{2.2}$$

where $h'(\rho) = c^2(\rho, \bar{S})/\rho$, $h(\bar{\rho}) = 0$. Hence $\partial_t\omega + \frac{1}{2}|\nabla\omega|^2 = -h(\rho)$.

Notice that $h'(\rho) > 0$ for $\rho > 0$. By the implicit function theorem we see that

$$\rho = h^{-1}\Big(-\big(\partial_t\omega + \frac{1}{2}|\nabla\omega|^2\big)\Big), \quad \bar{\rho} = h^{-1}(0). \tag{2.3}$$

Substituting (2.3) into the first equation in (2.1), we have

$$\begin{cases} \partial_t^2\omega - c^2\Big(h^{-1}\Big(-\big(\partial_t\omega + \frac{1}{2}|\nabla\omega|^2\big)\Big), \bar{S}\Big)\triangle\omega \\ \qquad + 2\displaystyle\sum_{k=1}^{3}\partial_k\omega\partial_t\partial_k\omega + \sum_{i,k=1}^{3}\partial_i\omega\partial_k\omega\partial_{ik}^2\omega = 0, \\ \omega(0,r) = \varepsilon\displaystyle\int_M^r s w_0(s)ds, \\ \partial_t\omega(0,r) = -\varepsilon\dfrac{\bar{c}^2}{\bar{\rho}}\rho_0(r) + \varepsilon^2 g(r,\varepsilon), \end{cases} \tag{2.4}$$

where

$$g(r,\varepsilon) = -\int_M^r\Big[\int_0^1 \frac{d}{d\rho}\Big(\frac{c^2(\rho,\bar{S})}{\rho}\Big)\Big|_{\rho=\bar{\rho}+\theta\varepsilon\rho_0(s)}d\theta\Big]\rho_0(s){\rho'}_0(s)ds - \frac{1}{2}r^2(w_0(r))^2.$$

For notational convenience, without loss of generality we assume $\bar{c} = 1$ in this paper.

By the results in [22] and [23], we know that the solution of (2.4) only blows up for $t \le T_\varepsilon$ in the domain $D = \{(t,r) : e^{\tau_0/2\varepsilon} \le t \le T_\varepsilon,\ -4M \le r - t \le M\}$, which is close to the surface of forward light cone (in fact, the blowup point lies in $t = T_\varepsilon$). In light of the standard process of disposing the problem on the nonlinear wave equations with small initial data, as in [11], [13] or [14], we introduce the normal transformation $\sigma = r - t$ and the slow time variable $\tau = \varepsilon \ln t$ to rewrite the equation (2.4). Let $\omega(t,r) = \frac{\varepsilon}{r} G(\tau,\sigma)$. Then a direct computation yields an equation on G in the domain D

$$\partial^2_{\sigma\tau} G + p(G,\nabla G)\partial^2_\sigma G + \varepsilon e^{-\tau/\varepsilon} q(G,\nabla G)\partial^2_\tau G + e^{-\tau/\varepsilon} r(G,\nabla G) = 0, \tag{2.5}$$

where

$$\begin{aligned} p(G,\nabla G) &= \frac{t(c^2(\rho,\bar S) - (1-\partial_r\omega)^2)}{2\varepsilon(1-\partial_r\omega)} \\ &= (1+\bar\rho c'(\bar\rho,\bar S))\partial_\sigma G + e^{-\tau/\varepsilon} O(\sigma, e^{-\tau/\varepsilon}, G, \nabla_{\sigma,\tau} G), \\ q(G,\nabla G) &= -\frac{1}{2(1-\partial_r\omega)} \\ &= -\frac{1}{2} + e^{-\tau/\varepsilon} O(\sigma, e^{-\tau/\varepsilon}, G, \nabla_{\sigma,\tau} G), \\ r(G,\nabla G) &= \frac{1}{2}(\partial_\tau G - 2(\partial_\sigma G)^2) + e^{-\tau/\varepsilon} O(\sigma, e^{-\tau/\varepsilon}, G, \nabla_{\sigma,\tau} G). \end{aligned}$$

Here the notation "$O(\sigma, e^{-\tau/\varepsilon}, G, \nabla_{\sigma,\tau} G)$" denotes generic smooth functions in its arguments.

To study the blowup mechanism of solutions to (2.5), as in [2] and [22], we introduce a transformation:

$$\tau = \tau, \quad \sigma = \varphi(\tau, y), \tag{2.6}$$

which is singular only at the blowup point. $\varphi(\tau,y)$ is unknown and will be determined together with the solution G of (2.5). Let $G(\tau,\varphi(\tau,y)) = m(\tau,y)$, $(\partial_\sigma G)(\tau,\varphi(\tau,y)) = v(\tau,y)$. Then (2.5) is reformulated as follows

$$\frac{\partial_y v}{\partial_y \varphi} I_1 + I_2 = 0, \tag{2.7}$$

where

$$\begin{aligned} I_1 &= \frac{t(c^2(\rho,\bar S) - (\frac{\varepsilon}{t}\partial_\tau\varphi - \partial_r\omega + 1)^2)}{2\varepsilon(1-\partial_r\omega)} \\ &= 2\partial_\tau\varphi - 2(1+\bar\rho c'(\bar\rho,\bar S))v + e^{-\tau/\varepsilon} q_1(e^{-\tau/\varepsilon}, \varphi, m, v, \partial_\tau\varphi, \partial_\tau m), \\ I_2 &= -2\partial_\tau v + e^{-\tau/\varepsilon} q_2(e^{-\tau/\varepsilon}, \varphi, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v, \partial^2_\tau\varphi, \partial^2_\tau m), \end{aligned}$$

and the functions q_i $(i = 1, 2)$ are smooth.

Inspired by the notion of blowup system for a quasilinear wave equation in [2], we here define the blowup system corresponding to (2.5) as

$$\begin{cases} I_1 = 0,\ I_2 = 0,\ I_3 = \partial_y m - v\partial_y \varphi = 0, \\ \varphi\Big(\frac{\tau_0}{2}, y\Big) = y,\ m\Big(\frac{\tau_0}{2}, y\Big) = G\Big(\frac{\tau_0}{2}, y\Big),\ v\Big(\frac{\tau_0}{2}, y\Big) = (\partial_\sigma G)\Big(\frac{\tau_0}{2}, y\Big). \end{cases} \tag{2.8}$$

Obviously, if (2.8) is solved in the class of smooth functions, then (2.5) is also solved in the domain where the transformation (2.6) is invertible. In particular, when the function $\varphi(\tau, y)$ satisfies the following nondegeneracy conditions at some point $(\tau_\varepsilon = \varepsilon \ln T_\varepsilon, y_\varepsilon)$:

$$\partial_y \varphi(\tau_\varepsilon, y_\varepsilon) = 0,\quad \partial_y^2 \varphi(\tau_\varepsilon, y_\varepsilon) = 0,\quad \partial_y^3 \varphi(\tau_\varepsilon, y_\varepsilon) > 0,\quad \partial_{y\tau}^2 \varphi(\tau_\varepsilon, y_\varepsilon) < 0,$$

and the function v has the property $\partial_y v(\tau_\varepsilon, y_\varepsilon) \neq 0$, one can get a complete description on the blowup mechanism of smooth solution to (2.4) at the blowup point $(T_\varepsilon, r_\varepsilon = T_\varepsilon + \varphi(\tau_\varepsilon, y_\varepsilon))$. Indeed, a simple computation implies that the solution $\omega(t, r)$ and its first order derivatives are continuous at the blowup point, while the second order derivatives of $\omega(t, r)$ blow up like $1/(T_\varepsilon - t)$. Furthermore, we can give an extension property of solution to (2.8).

LEMMA 2.1. *Let*

$$\bar{D} = \Big\{(\tau, y) : \frac{\tau_0}{2} \le \tau \le 2\tau_0,\ -4M \le y \le M\Big\}.$$

Then the blowup system (2.8) *has a smooth solution* (φ, m, v) *in* $\bar{D}$ *for small* $\varepsilon > 0$*, satisfying the estimates*

$$|\varphi|_{C^k(\bar{D})} + |m|_{C^k(\bar{D})} + |v|_{C^k(\bar{D})} \le C_k,\quad k = 1, 2, 3, \dots,$$

with constants $C_k > 0$ *independent of* ε*. In particular, suppose that the function*

$$F(q) = q^2 \partial_q w_0(q) + 3q w_0(q) + \frac{\bar{c}}{\bar{\rho}}(q\partial_q \rho_0(q) + \rho_0(q))$$

satisfies a nondegeneracy condition at a unique minimum point, i.e., that there exists a unique point q_0 *such that* $F(q_0) = \min F(q)$*,* $F'(q_0) = 0$ *and* $F''(q_0) > 0$*. Then we have*

$$\partial_y \varphi(\tau, y) \ge 0$$

in $\bar{D}_1 = \{(\tau, y) : \tau_0/2 \le \tau \le \tau_\varepsilon,\ -4M \le y \le M\}$ *of* $\bar{D}$.

Moreover there exists a unique point $(\tau_\varepsilon, y_\varepsilon)$ *such that*

$$\partial_y \varphi(\tau, y) = 0 \Longleftrightarrow \begin{cases} (\tau, y) = (\tau_\varepsilon, y_\varepsilon), & \partial_y^2 \varphi(\tau_\varepsilon, y_\varepsilon) = 0, \\ \partial_y^3 \varphi(\tau_\varepsilon, y_\varepsilon) > 0, & \partial_{y\tau}^2 \varphi(\tau_\varepsilon, y_\varepsilon) < 0 \end{cases}$$

and $\partial_y v(\tau_\varepsilon, y_\varepsilon) \neq 0$.

Remark 2.1. By (1.2), we know that τ_ε actually satisfies

$$\lim_{\varepsilon \to 0} \tau_\varepsilon = -\frac{2}{(\bar{\rho} c'(\bar{\rho}, \bar{S}) + 1) \min_q F(q)}.$$

Hence the effect of $F(q)$ is very silimar to that of initial data for Burger's equation.

Proof. The proof is given in Theorem 2 of [22], so we omit it. ∎

Based on Lemma 2.1, we can determine the blowup direction of (ρ, u, S) and construct a 3-shock starting from the blowup point $(T_\varepsilon, r_\varepsilon = T_\varepsilon + \varphi(\tau_\varepsilon, y_\varepsilon))$. Motivated by the physical background we set $u(t, x) = \tilde{u}(t, r)\frac{x}{r}$. From the system (1.1), we get a conservation law system on $(\rho(t, r), \tilde{u}(t, r), S(t, r))$ with the source terms

(2.9)

$$\begin{cases} \partial_t \rho + \partial_r(\rho \tilde{u}) = -\dfrac{2\rho \tilde{u}}{r}, \\ \partial_t(\rho \tilde{u}) + \partial_r(\rho \tilde{u}^2 + p) = -\dfrac{2\rho \tilde{u}^2}{r}, \\ \partial_t\Big(\rho e + \dfrac{1}{2}\rho \tilde{u}^2\Big) + \partial_r\Big(\big(\rho e + \dfrac{1}{2}\rho \tilde{u}^2 + p\big)\tilde{u}\Big) = -\dfrac{2}{r}\Big(\rho e + \dfrac{1}{2}\rho \tilde{u}^2 + p\Big)\tilde{u}, \\ \rho(0, r) = \bar{\rho} + \varepsilon \rho_0(r), \quad \tilde{u}(0, r) = \varepsilon w_0(r), \quad S(0, r) = \bar{S}. \end{cases}$$

Here we should notice that the blowup point of (1.1) is far away from $r = 0$ and the new shock will be constructed near the blowup point. Hence the factor $1/r$ is not a singularity in our study.

A simple computation yields that (2.9) has three distinct eigenvalues $\lambda_1(t, r) = \tilde{u} - c(\rho, S)$, $\lambda_2(t, r) = \tilde{u}$ and $\lambda_3(t, r) = \tilde{u} + c(\rho, S)$. The corresponding left eigenvectors are $l_1 = (1, -\rho/c(\rho, S), 0)$, $l_2 = (0, 0, 1)$ and $l_3 = (1, \rho/c(\rho, S), 0)$, respectively. Now we give a detailed information on the blowup direction of $(\rho, \tilde{u}, S)$ at the blowup point.

LEMMA 2.2. *Under the nondegeneracy condition on $F(q)$ in Lemma 2.1, $l_3\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix}$ blows up at the blowup point $(T_\varepsilon, r_\varepsilon)$, while $l_1\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix}$ and $l_2\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix}$ are still continuous and bounded.*

Proof. By (2.3) and $u = \nabla\omega$, $S = \bar S$ for $t \le T_\varepsilon$, we can get

$$l_3\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix} = \frac{\rho}{c(\rho,\bar S)}\Big(\partial_r^2\omega - \frac{1}{c(\rho,\bar S)}(\partial_{tr}^2\omega + \partial_r\omega\partial_r^2\omega)\Big).$$

Noting $\omega = \frac{\varepsilon}{r}G(\varepsilon\ln t, r-t)$, then one has

$$\begin{aligned}l_3\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix} &= \frac{\varepsilon\rho}{rc(\rho,\bar S)}\bigg\{\Big[1 + \frac{1}{c(\rho,\bar S)} - \frac{\varepsilon}{rc(\rho,\bar S)}\Big(\partial_\sigma G - \frac{G}{r}\Big)\Big]\partial_\sigma^2 G\\ &\qquad - \frac{\varepsilon}{tc(\rho,\bar S)}\partial_{\sigma\tau}^2 G\bigg\} + h_1(\varepsilon, r, t, G, \partial_\sigma G, \partial_\tau G),\end{aligned}$$

where h_1 is a smooth function on its arguments.

Additionally,

$$\begin{aligned}&(\partial_\tau G)(\tau,\varphi(\tau,y)) = \partial_\tau m - v\partial_\tau\varphi, \quad (\partial_\sigma^2 G)(\tau,\varphi(\tau,y)) = \frac{\partial_y v}{\partial_y\varphi},\\ &(\partial_{\sigma\tau}^2 G)(\tau,\varphi(\tau,y)) = \partial_\tau v - \frac{\partial_y v}{\partial_y \varphi}\partial_\tau\varphi\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}l_3\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix} &= \frac{\varepsilon\rho}{rc(\rho,\bar S)}\Big[1 + \frac{1}{c(\rho,\bar S)} - \frac{\varepsilon}{rc(\rho,\bar S)}\Big(v - \frac{m}{r}\Big) + \frac{\varepsilon}{tc(\rho,\bar S)}\partial_\tau\varphi\Big]\frac{\partial_y v}{\partial_y\varphi}\\ &\quad + \tilde h_1(\varepsilon, r, t, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v),\end{aligned}$$

where $\tilde h_1$ is smooth.

Since $I_1 = 0$ yields

$$\partial_\tau\varphi = \frac{(c(\rho,\bar S) - 1)t}{\varepsilon} + \frac{t}{r}\Big(v - \frac{m}{r}\Big),$$

we have

$$l_3\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix} = \frac{2\varepsilon\rho}{rc(\rho,\bar S)}\frac{\partial_y v}{\partial_y\varphi} + \tilde h_1.$$

Then by Lemma 2.1, we know that $l_3\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix}$ blows up at the point $(T_\varepsilon, r_\varepsilon)$.

Similarly, by a direct computation we have

$$\begin{aligned}
l_1\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix} &= \frac{\varepsilon\rho}{rc(\rho,\bar S)}\Big[-1+\frac{1}{c(\rho,\bar S)}-\frac{\varepsilon}{rc(\rho,\bar S)}\Big(v-\frac{m}{r}\Big)+\frac{\varepsilon}{tc(\rho,\bar S)}\partial_\tau\varphi\Big]\frac{\partial_y v}{\partial_y\varphi}\\
&\quad + h_2(\varepsilon, r, t, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v)\\
&= \tilde h_2(r, t, m, v, \partial_\tau\varphi, \partial_\tau m, \partial_\tau v),\\
l_2\partial_r\begin{pmatrix}\rho\\ \tilde u\\ S\end{pmatrix} &= 0,
\end{aligned}$$

where h_2 and $\tilde h_2$ are smooth.

Therefore Lemma 2.2 is proved. ∎

Now we give a reduction on (2.9) so that each equation in the new system contains only the differentiation along the same direction. This reduction will bring us much convenience in order to obtain the convergence of our iterative scheme in the process of shock construction.

Lemma 2.3. *By an invertible transformation, the system* (2.9) *is reduced to*

$$\begin{cases}\partial_t w + A(w)\partial_r w = \dfrac{B(w)}{r},\\ w(0,r) = \varepsilon\bar w_0(r,\varepsilon),\end{cases} \tag{2.11}$$

in a neighborhood of $(\bar\rho, 0, \bar S)$. *Here* $A(w) = \begin{pmatrix}\lambda_1(w) & a(w) & 0\\ 0 & \lambda_2(w) & 0\\ 0 & -a(w) & \lambda_3(w)\end{pmatrix}$, $a(0)=0$, *and* $B(w) = \begin{pmatrix}b_1(w)\\ 0\\ b_3(w)\end{pmatrix}$ *is a smooth vector function. Moreover* (2.11) *is rewritten as*

$$\begin{cases}\partial_t w_1 + \lambda_1(w)\partial_r w_1 + \bar a(w)(\partial_t w_2 + \lambda_1(w)\partial_r w_2) = \dfrac{\bar b_1(w)}{r},\\ \partial_t w_2 + \lambda_2(w)\partial_r w_2 = 0,\\ \partial_t w_3 + \lambda_3(w)\partial_r w_3 - \bar a(w)(\partial_t w_2 + \lambda_3(w)\partial_r w_2) = \dfrac{\bar b_3(w)}{r},\end{cases} \tag{2.12}$$

where $\bar a(w)$ *and* $\bar b_i(w)$, $i=1,2$, *are smooth and* $\bar a(0) = \bar b_i(0) = 0$.

Proof. As in the case of 2×2 systems, we introduce the Riemann invariants as follows

$$\begin{cases} \alpha_1 = \tilde{u} - F(\rho, S), \\ \alpha_2 = S - \bar{S}, \\ \alpha_3 = \tilde{u} + F(\rho, S), \end{cases} \tag{2.13}$$

where $\partial_\rho F(\rho, S) = c(\rho, S)/\rho$ and $F(\bar{\rho}, \bar{S}) = 0$.

Obviously, (2.13) is invertible as long as $\rho > 0$ (in our discussion, ρ is a small perturbation of $\bar{\rho}$, hence $\rho > 0$ is fulfilled).

By a direct computation using (2.13), we get from (2.9)

$$\begin{cases} \partial_t \alpha_1 + \lambda_1 \partial_r \alpha_1 + q(\rho, S) \partial_r \alpha_2 = -\dfrac{2\tilde{u}^2}{r} + \dfrac{2\tilde{u} c(\rho, S)}{r}, \\ \partial_t \alpha_2 + \lambda_2 \partial_r \alpha_2 = 0, \\ \partial_t \alpha_3 + \lambda_1 \partial_r \alpha_3 + q(\rho, S) \partial_r \alpha_2 = -\dfrac{2\tilde{u}^2}{r} - \dfrac{2\tilde{u} c(\rho, S)}{r}, \end{cases}$$

where $q(\rho, S) = \partial_S p(\rho, S)/\rho - c(\rho, S) \partial_S F(\rho, S)$.

By a linear transformation

$$\begin{cases} w_1 = \alpha_1 - q(\bar{\rho}, \bar{S}) \alpha_2, \\ w_2 = \alpha_2, \\ w_3 = \alpha_3 + q(\bar{\rho}, \bar{S}) \alpha_2, \end{cases} \tag{2.14}$$

one gets

$$\begin{cases} \partial_t w_1 + \lambda_1(w) \partial_r w_1 + a(w) \partial_r w_2 = -\dfrac{2\tilde{u}^2}{r} + \dfrac{2\tilde{u} c(\rho, S)}{r}, \\ \partial_t w_2 + \lambda_2(w) \partial_r w_2 = 0, \\ \partial_t w_3 + \lambda_3(w) \partial_r w_3 - a(w) \partial_r w_3 = -\dfrac{2\tilde{u}^2}{r} - \dfrac{2\tilde{u} c(\rho, S)}{r}, \\ w_1(0, r) = \varepsilon w_0(r) - F(\bar{\rho} + \varepsilon \rho_0(r), \bar{S}), \quad w_2(0, r) = 0, \\ w_3(0, r) = \varepsilon w_0(r) + F(\bar{\rho} + \varepsilon \rho_0(r), \bar{S}), \end{cases}$$

where $a(w) = -q(\bar{\rho}, \bar{S}) c(\rho, S) + q(\rho, S)$.

Obviously, (2.13) and (2.14) transform the point $(\rho, \tilde{u}, S) = (\bar{\rho}, 0, \bar{S})$ to the point $(w_1, w_2, w_3) = (0, 0, 0)$. Moreover $a(0) = 0$.

In addition, we see by a simple algebraic computation that (2.12) is obtained directly from (2.11). Hence Lemma 2.3 is proved. ∎

By Lemma 2.2, it is easy to see that $\partial_r w_3$ blows up at the blowup point $(T_\varepsilon, r_\varepsilon)$, while $\partial_r w_1$ and $\partial_r w_2$ are continuous at $(T_\varepsilon, r_\varepsilon)$. Hence we expect that a 3-shock will be formed from the blowup point. Our result can be stated as follows.

THEOREM 2.1. *Consider the system* (2.9) *and suppose that $F(q)$ satisfies the nondegeneracy condition as stated in Lemma* 2.1 *at only one point. Then for small $\varepsilon > 0$,* (2.11) *admits a weak entropy solution with a continuously differentiable shock curve $\Gamma : r = \phi(t)$ which starts from the blowup point $(T_\varepsilon, r_\varepsilon)$. The solution w is continuously differentiable in $([T_\varepsilon, T_\varepsilon+1]\times\mathbb{R})\backslash\Gamma$ and satisfies the Rankine-Hugoniot condition and entropy condition on Γ. Moreover, the estimates*

$$\begin{aligned}
&\phi(t) = r_\varepsilon + \lambda_3(T_\varepsilon, r_\varepsilon)(t-T_\varepsilon) + O((t-T_\varepsilon)^2),\\
&w_1(t,r) = w_1(T_\varepsilon, r_\varepsilon) + O\big((t-T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t-T_\varepsilon))^2\big)^{1/3},\\
&w_2(t,r) = O\big((t-T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t-T_\varepsilon))^2\big)^{1/2},\\
&w_3(t,r) = w_3(T_\varepsilon, r_\varepsilon) + O\big((t-T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t-T_\varepsilon))^2\big)^{1/6},
\end{aligned}$$

hold in

$$\Omega = \{(t,r) : T_\varepsilon < t \le T_\varepsilon + 1,\ r_\varepsilon - 2(T_\varepsilon + 1 - t) \le r \le r_\varepsilon + 2(T_\varepsilon + 1 - t)\}.$$

Thus, returning to (2.9) *we have*

$$\begin{aligned}
&\rho(t,r) = \rho(T_\varepsilon, r_\varepsilon) + O\big((t-T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t-T_\varepsilon)^2\big)^{1/6},\\
&\tilde{u}(t,r) = \tilde{u}(T_\varepsilon, r_\varepsilon) + O\big((t-T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t-T_\varepsilon)^2\big)^{1/6},\\
&S(t,r) = \bar{S} + O\big((t-T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t-T_\varepsilon)^2\big)^{1/2}
\end{aligned}$$

near $(T_\varepsilon, r_\varepsilon)$. Here, "$O$" stands for uniformly bounded quantities independent of ε.

Remark 2.2. Some weaker singularities of the solution of (2.9) may propagate into the domain $[T_\varepsilon, T_\varepsilon + 1] \times \mathbb{R}$ along 1-characteristics and 2-characteristics through $(T_\varepsilon, r_\varepsilon)$ although the solution itself is continuous there.

Remark 2.3. Under the assumption of Theorem 2.1, we know by [22] and [23] that the solution of (1.1) or (2.9) does not blow up away from a small neighbourhood of r_ε, provided that t is in $[T_\varepsilon, T_\varepsilon+1]$. Hence, in order to complete our construction of a shock wave for $t \in [T_\varepsilon, T_\varepsilon + 1]$, we need only study our problem in the set Ω as defined above.

Remark 2.4. The same method applies to (1.1) in two space dimensions with axisymmetric and nondegenerate initial data, and we obtain results similar to those stated in Theorem 2.1.

Remark 2.5. In view of the proof of Theorem 2.1 given in the next section, the results in Theorem 2.1 hold in the time interval $[T_\varepsilon, T_\varepsilon + A/\varepsilon]$, with an appropriate $A > 0$ depending only on the initial data of (1.1).

§3. Proof of Theorem 2.1

As remarked in the last section we need only to do analysis in the neighbourhood Ω of $(T_\varepsilon, r_\varepsilon)$. The solution w of (2.11) will be constructed in $t \geq T_\varepsilon$ by an iterative procedure. To this end, we will construct a sequence of approximate solutions $\{w^{(n)}(t,r)\}$ and a corresponding sequence $\{\phi^{(n)}(t)\}$ standing for the location of the approximate shocks, and show the convergence of these sequences. Here we choose the solution of blowup system (2.8) as the first approximation $w^{(0)}(t,r)$, while $\phi^{(0)}(t)$ is determined by an ordinary differential equation, which is derived from the Rankine-Hugoniot conditions. The advantage of this choice is that we can get a piecewise continuous solution of (2.11) which satisfies the entropy condition on $\phi^{(0)}(t)$ and a "good" estimate near the point $(T_\varepsilon, r_\varepsilon)$. Subsequently, the whole sequence $\{w^{(n)}(t,r)\}$ can be successively determined by the method of characteristics, and $\{\phi^{(n)}(t)\}$ can be determined by the R-H conditions correspondingly.

This section is arranged as follows: In Step 1, we give the first approximation of system (2.11) and some precise descriptions of the approximation as a preparation for further discussion. In Step 2, we will give an iterative scheme to construct the sequence $\{w^{(n)}(t,r)\}$ of approximate solutions, and establish estimates on $\{w^{(n)}\}$, $\{\partial_t w^{(n)}\}$ and $\{\partial_r w^{(n)}\}$. Step 3 is devoted to the proof of the convergence of all these sequences.

Step 1. First approximation

Denoting by $H(t,y) = t + \varphi(\varepsilon \ln t, y)$, by Lemma 2.1, we know that $H(t,y)$ satisfies

$$\partial_y H(t,y) = 0 \Longleftrightarrow \begin{cases} (t,y) = (T_\varepsilon, y_\varepsilon), & \partial_y^2 H(T_\varepsilon, y_\varepsilon) = 0, \\ \partial_y^3 H(T_\varepsilon, y_\varepsilon) > 0, & \partial_{yt}^2 H(T_\varepsilon, y_\varepsilon) < 0. \end{cases} \tag{3.1}$$

More precisely, by a similar treatment in [5] and [17], we can show the following two lemmas which describe some subtle properties of $H(t,y)$.

Lemma 3.1. *1) For $t \in (T_\varepsilon, T_\varepsilon+1]$ and in a small neighbourhood of y_ε, $\partial_y H(t,y)=0$ has two distinct real function roots $\eta_-^\varepsilon(t)$ and $\eta_+^\varepsilon(t)$ such that $\eta_+^\varepsilon(t) < y_\varepsilon < \eta_-^\varepsilon(t)$ and $\eta_\pm^\varepsilon(t) \in C^\infty(T_\varepsilon, T_\varepsilon+1]$.*

2) Set $r_-^\varepsilon(t) = H(t, \eta_-^\varepsilon(t))$ and $r_+^\varepsilon(t) = H(t,\eta_+^\varepsilon(t))$. Then

$r = H(t,y)$ has three distinct real roots $y_-^\varepsilon(t,r) < y_c^\varepsilon(t,r) < y_+^\varepsilon(t,r)$ if $r \in (r_+^\varepsilon(t), r_-^\varepsilon(t))$.

$r = H(t,y)$ has a unique real root $y_+^\varepsilon(t,r)$ if $r \geq r_-^\varepsilon(t)$.

$r = H(t,y)$ has a unique real root $y_-^\varepsilon(t,r)$ if $r \leq r_+^\varepsilon(t)$.

3) Denote

$$\Omega_+ = \{(t,r) \in \Omega : T_\varepsilon < t \leq T_\varepsilon + 1,\ r > r_+^\varepsilon(t)\},$$
$$\Omega_- = \{(t,r) \in \Omega : T_\varepsilon < t \leq T_\varepsilon + 1,\ r < r_-^\varepsilon(t)\}.$$

Then $y_\pm^\varepsilon(t,r) \in C^\infty(\Omega_\pm) \cap C(\bar{\Omega}_\pm)$.

Lemma 3.2. *Denoting $d_\varepsilon = (t-T_\varepsilon)^3 + (r - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon)(t-T_\varepsilon))^2$, we have*

$$|y_\pm^\varepsilon(t,r) - y_\varepsilon| < C d_\varepsilon^{1/6}, \quad |\partial_r y_\pm^\varepsilon(t,r)| \leq C d_\varepsilon^{-1/3},$$
$$|\partial_\ell y_\pm^\varepsilon(t,r)| \leq C d_\varepsilon^{-1/6}, \quad |\partial_x^2 y_\pm^\varepsilon(t,r)| \leq C d_\varepsilon^{-5/6},$$

where ℓ is the third characteristics passing through $(T_\varepsilon, r_\varepsilon)$, and the generic constant C is independent of ε.

Based on Lemma 3.1 and Lemma 2.1, we can get two extensions of a solution of (2.11) across the blowup time T_ε.

In fact, let $\rho^*(t,y) = \rho(t, H(t,y))$ and $u^*(t,y) = \tilde{u}(t, H(t,y))$. Then by the definitions of $G(\tau,\sigma)$, $m(\tau,y)$ and $v(\tau,y)$ we have

$$\tag{3.2} \begin{cases} \rho^*(t,y) = h^{-1}(g(t,y)), \\ u^*(t,y) = \dfrac{\varepsilon}{H(t,y)}\Big(v(\varepsilon\ln t, y) - \dfrac{m(\varepsilon \ln t, y)}{H(t,y)}\Big), \end{cases}$$

where

$$g(t,y) = \frac{\varepsilon}{H(t,y)}\Big\{v(\varepsilon \ln t, y) - \frac{\varepsilon}{t}\Big[\partial_\tau m(\varepsilon\ln t, y) - v(\varepsilon \ln t, y)\partial_\tau\varphi(\varepsilon\ln t, y) \\ - \frac{\varepsilon}{2H(t,y)}\Big(v(\varepsilon\ln t, y) - \frac{m(\varepsilon\ln t,y)}{H(t,y)}\Big)^2\Big]\Big\}.$$

From (2.13), (2.14) and (3.2), we can define two vector valued functions $w^0_\pm(t,r) = (w^0_{1,\pm}(t,r), w^0_{2,\pm}(t,r), w^0_{3,\pm}(t,r))$ by

$$\begin{cases} w^0_{1,\pm}(t,r) = u^*(t, y^\varepsilon_\pm(t,r)) - F(\rho^*(t, y^\varepsilon_\pm(t,r)), \bar{S}), \\ w^0_{2,\pm}(t,r) = 0, \\ w^0_{3,\pm}(t,r) = u^*(t, y^\varepsilon_\pm(t,r)) + F(\rho^*(t, y^\varepsilon_\pm(t,r)), \bar{S}). \end{cases} \tag{3.3}$$

Note that $w^0_\pm(t,r)$ are the smooth solutions of (2.11) in $\Omega_\pm$ respectively. Therefore, they are both extensions of solutions of (2.11).

Now we define the first approximate shock curve $\phi^0(t)$ starting from the point $(T_\varepsilon, r_\varepsilon)$. Since we have chosen the entropy $S \equiv \bar{S}$, we hope that $\phi^0(t)$ can be determined by the corresponding Rankine-Hugoniot conditions for the first two equations in (2.9), that is,

$$\begin{cases} [\rho](\phi^0(t))' = [\rho\tilde{u}], \\ [\rho\tilde{u}](\phi^0(t))' = [\rho\tilde{u}^2 + p(\rho, \bar{S})]. \end{cases} \tag{3.4}$$

Hence $\phi^0(t)$ should satisfy the following ordinary differential equation

$$\begin{cases} \dfrac{d\phi^0(t)}{dt} = \tilde{\lambda}_3(t, \phi^0(t)), \\ \phi^0(T_\varepsilon) = r_\varepsilon, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} \tilde{\lambda}_3(t,r) = \Big\{ \int_0^1 c^2\Big(\theta G\Big(\frac{w^0_{3,+} - w^0_{1,+}}{2}, \bar{S}\Big) + (1-\theta) G\Big(\frac{w^0_{3,-} - w^0_{1,-}}{2}, \bar{S}\Big)\Big) d\theta \\ - \frac{(w^0_{1,+} + w^0_{3,+} - w^0_{1,-} - w^0_{3,-})^2}{48} \Big\}^{1/2} + \frac{w^0_{1,+} + w^0_{3,+} + w^0_{1,-} + w^0_{3,-}}{4}, \end{aligned}$$

and $G\big(\frac{w_3 - w_1}{2}, \bar{S}\big)$ is the inverse function of ρ in (2.13) and (2.14) with $S = \bar{S}$.

As in [5, Lemma 3.2], we have

Lemma 3.3. *The equation* (3.5) *has a solution* $\phi^0(t) \in C^\infty[T_\varepsilon, T_\varepsilon + 1]$. *Moreover,* $\phi^0(t)$ *satisfies* $r^\varepsilon_+(t) < \phi^0(t) < r^\varepsilon_-(t)$, *and*

$$\phi^0(t) = r_\varepsilon + \lambda_3(T_\varepsilon, r_\varepsilon)(t - T_\varepsilon) + O((t - T_\varepsilon)^2); \quad t \in [T_\varepsilon, T_\varepsilon + 1]$$

Here O *represents generic quantities independent of* ε.

Define the function $w^0(t,r) = (w_1^0(t,r), w_2^0(t,r), w_3^0(t,r))$ by

$$w_i^0(t,r) = \begin{cases} w_{i,+}^0(t,r), & r > \phi^0(t) \\ w_{i,-}^0(t,r), & r < \phi^0(t) \end{cases} \qquad i = 1, 3,$$
$$w_2^0(t,r) \equiv 0$$

in Ω. Obviously, $w^0(t,x)$ is a solution of (2.11) in $\Omega_\pm$ respectively. But it is not a weak solution of (2.11) because it does not satisfy the Rankine-Hugoniot condition along the curve $\gamma : r = \phi^0(t)$. We will use an iterative scheme to construct a shock starting from the point $(T_\varepsilon, r_\varepsilon)$ for the system (2.11) by modifying the location of curve γ as well as the solution on both sides of γ. In the process of the forthcoming iteration, $(w^0(t,r), \phi^0(t))$ will be chosen as the first approximation of the iterative scheme.

LEMMA 3.4. *In the domain $\Omega\backslash\gamma$, we have the following.*

1) $w_3^0(t,r)$ *satisfies the estimates*:

$$\begin{cases} |w_3^0(t,r) - w_3^0(T_\varepsilon, r_\varepsilon)| \le C\varepsilon d_\varepsilon^{1/6}, \\ |\partial_\ell w_3^0(t,r)| \le C\varepsilon d_\varepsilon^{-1/6}, \\ |\partial_r w_3^0(t,r)| \le C\varepsilon d_\varepsilon^{-1/3}, \\ |\partial_r^2 w_3^0(t,r)| \le C\varepsilon d_\varepsilon^{-5/6}. \end{cases} \tag{3.6}$$

2) $w_1^0(t,r)$ *satisfies the estimates*:

$$\begin{cases} |w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon)| \le C\varepsilon d_\varepsilon^{1/3}, \\ |\partial_t w_1^0(t,r)| \le C\varepsilon, \\ |\partial_r w_1^0(t,r)| \le C\varepsilon, \\ |\partial_r^2 w_1^0(t,r)| \le C\varepsilon d_\varepsilon^{-1/2}. \end{cases} \tag{3.7}$$

Proof. It is enough to prove the lemma in the domain Ω_+.

For the simplicity to write, from (2.13) and (2.14) we denote by $w_1^*(t,y) = u^*(t,y) - F(\rho^*(t,y), \bar{S})$ and $w_3^*(t,y) = u^*(t,y) + F(\rho^*(t,y), \bar{S})$. Then $w_{i,\pm}^0(t,r) = w_i^*(t, y_\pm^\varepsilon(t,r))$ for $i = 1, 3$.

1) Thanks to the existence and regularity in Lemma 2.1, one has

$$
\begin{aligned}
&w_3^0(t,r)-w_3^0(T_\varepsilon,r_\varepsilon)=w_3^*(t,y_+^\varepsilon(t,r))-w_3^*(T_\varepsilon,y_\varepsilon)\\
&\qquad=\partial_t w_3^*(T_\varepsilon,y_\varepsilon)(t-T_\varepsilon)+\partial_y w_3^*(T_\varepsilon,y_\varepsilon)(y_+^\varepsilon(t,r)-y_\varepsilon)\\
&\qquad\quad+O\big(\varepsilon(t-T_\varepsilon)^2+\varepsilon(y_+^\varepsilon(t,r)-y_\varepsilon)^2\big),\\
&\partial_\ell w_3^0(t,r)=\partial_t w_3^*(t,y_+^\varepsilon(t,r))+\partial_y w_3^*(t,y_+^\varepsilon(t,r))\partial_\ell y_+^\varepsilon(t,r),\\
&\partial_r w_3^0(t,r)=\partial_y w_3^*(t,y_+^\varepsilon(t,r))\partial_r y_+^\varepsilon(t,r),\\
&\partial_r^2 w_3^0(t,r)=\partial_y^2 w_3^*(t,y_+^\varepsilon(t,r))(\partial_r y_+^\varepsilon(t,r))^2+\partial_y w_3^*(t,y_+^\varepsilon(t,r))\partial_r^2 y_+^\varepsilon(t,r).
\end{aligned}
$$

Hence (3.6) follows from Lemma 2.1 and Lemma 3.2.

2) Firstly, we claim that

$$
\partial_y w_1^*(T_\varepsilon,y_\varepsilon)=0. \tag{3.8}
$$

In fact, by a direct computation from (3.2) one has

$$
\begin{aligned}
\partial_y u^*(t,y)&=\partial_y H\Big(-\frac{u^*}{H}+\frac{\varepsilon m}{H^3}\Big)+\frac{\varepsilon}{H}\Big(\partial_y v-\frac{\partial_y m}{H}\Big),\\
\partial_y \rho^*(t,y)&=\frac{\rho^*}{c^2(\rho^*,\bar S)}\partial_y g,
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_y g(t,y)=\partial_y H\Big\{-\frac{g}{H}+\frac{\varepsilon^2}{2H^3}\Big(v-\frac{m}{H}\Big)^2-\frac{\varepsilon^2 m}{H^4}\Big(v-\frac{m}{H}\Big)\Big\}\\
&+\frac{\varepsilon}{H}\Big\{\partial_y v-\frac{\varepsilon}{t}(\partial_{y\tau}^2 m-\partial_y v\partial_\tau\varphi-v\partial_{y\tau}^2\varphi)-\frac{\varepsilon}{H}\Big(v-\frac{m}{H}\Big)\Big(\partial_y v-\frac{\partial_y m}{H}\Big)\Big\}.
\end{aligned}
$$

Since $\partial_y H(T_\varepsilon,y_\varepsilon)=0$, $\partial_y\varphi(\tau_\varepsilon,y_\varepsilon)=0$ and $\partial_y m(\tau_\varepsilon,y_\varepsilon)=0$, we get

$$
\partial_y w_1^*(T_\varepsilon,y_\varepsilon)=\frac{\varepsilon}{H(T_\varepsilon,y_\varepsilon)}(I+II),
$$

where

$$
\begin{aligned}
I&=\frac{\varepsilon}{T_\varepsilon c(\rho^*(T_\varepsilon,y_\varepsilon),\bar S)}(\partial_{y\tau}^2 m(\tau_\varepsilon,y_\varepsilon)-v(\tau_\varepsilon,y_\varepsilon)\partial_{y\tau}^2\varphi(\tau_\varepsilon,y_\varepsilon)),\\
II&=\frac{\partial_y v(\tau_\varepsilon,y_\varepsilon)}{c(\rho^*(T_\varepsilon,y_\varepsilon),\bar S)}\Big(c(\rho^*(T_\varepsilon,y_\varepsilon),\bar S)-1\\
&\qquad+\frac{\varepsilon}{H(T_\varepsilon,y_\varepsilon)}\Big(v(\tau_\varepsilon,y_\varepsilon)-\frac{m(\tau_\varepsilon,y_\varepsilon)}{H(T_\varepsilon,y_\varepsilon)}\Big)-\frac{\varepsilon}{T_\varepsilon}\partial_\tau\varphi(\tau_\varepsilon,y_\varepsilon)\Big).
\end{aligned}
$$

Taking the first order derivative in τ on two sides of $I_3 = 0$ in (2.8), and using $\partial_y \varphi(\tau_\varepsilon, y_\varepsilon) = 0$, one has

$$\partial^2_{y\tau} m(\tau_\varepsilon, y_\varepsilon) - v(\tau_\varepsilon, y_\varepsilon)\partial^2_{y\tau}\varphi(\tau_\varepsilon, y_\varepsilon) = 0,$$

that is, $I = 0$.

Additionally, $I_1 = 0$ in (2.8) implies

$$\frac{\varepsilon}{T_\varepsilon}\partial_\tau \varphi(\tau_\varepsilon, y_\varepsilon) = c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S}) - 1 + \frac{\varepsilon}{H(T_\varepsilon, y_\varepsilon)}\Big(v(\tau_\varepsilon, y_\varepsilon) - \frac{m(\tau_\varepsilon, y_\varepsilon)}{H(T_\varepsilon, y_\varepsilon)}\Big).$$

This leads us to $II = 0$. Hence (3.8) is proved.

Secondly, we claim that

$$\partial^2_y w_1^*(T_\varepsilon, y_\varepsilon) = 0. \tag{3.9}$$

Indeed, by $\partial_y H(T_\varepsilon, y_\varepsilon) = \partial^2_y H(T_\varepsilon, y_\varepsilon) = 0$ and $\partial_y\varphi(\tau_\varepsilon, y_\varepsilon) = \partial_y w(\tau_\varepsilon, y_\varepsilon) = \partial^2_y m(\tau_\varepsilon, y_\varepsilon) = 0$, we have

$$\partial^2_y u^*(T_\varepsilon, y_\varepsilon) = \frac{\varepsilon}{H(T_\varepsilon, y_\varepsilon)}\partial^2_y v(\tau_\varepsilon, y_\varepsilon).$$

Using $I_1 = 0$ and $I_3 = 0$ again in (2.8), one has

$$\begin{aligned}
\partial_y g(T_\varepsilon, y_\varepsilon) &= \frac{\varepsilon^2 c^2(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}{H^2(T_\varepsilon, y_\varepsilon)}(\partial_y v(\tau_\varepsilon, y_\varepsilon))^2,\\
\partial^2_y g(T_\varepsilon, y_\varepsilon) &= \frac{\varepsilon c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}{H(T_\varepsilon, y_\varepsilon)}\partial^2_y v(\tau_\varepsilon, y_\varepsilon)\\
&\qquad + \frac{\varepsilon^2 \partial_\rho c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})\rho^*(T_\varepsilon, y_\varepsilon)}{H^2(T_\varepsilon, y_\varepsilon)c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}(\partial_y v(\tau_\varepsilon, y_\varepsilon))^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial^2_y w_1^*(T_\varepsilon, y_\varepsilon) &= \partial^2_y u^*(T_\varepsilon, y_\varepsilon) + \frac{\partial_\rho c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})\rho^*(T_\varepsilon, y_\varepsilon)}{c^4(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}(\partial_y g(\tau_\varepsilon, y_\varepsilon))^2\\
&\qquad - \frac{\partial^2_y g(T_\varepsilon, y_\varepsilon)}{c(\rho^*(T_\varepsilon, y_\varepsilon), \bar{S})}\\
&= 0.
\end{aligned}$$

Now we prove (3.7).

By (3.8), (3.9) and Taylor's formula, we get

$$\begin{aligned}\partial_y w_1^*(t, y_+^\varepsilon(t,r)) &= \partial_{ty}^2 w_1^*(T_\varepsilon, y_\varepsilon)(t-T_\varepsilon)\\ &\quad + O\big(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon(t,r)-y_\varepsilon)^2\big),\\ \partial_y^2 w_1^*(t, y_+^\varepsilon(t,r)) &= \partial_t\partial_y^2 w_1^*(T_\varepsilon, y_\varepsilon)(t-T_\varepsilon) + \partial_y^3 w_1^*(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon(t,r)-y_\varepsilon)\\ &\quad + O\big(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon(t,r)-y_\varepsilon)^2\big).\end{aligned}$$

By Lemma 2.1 and Lemma 3.2 one has

$$|\partial_y w_1^*(t, y_+^\varepsilon(t,r))| \le C\varepsilon d_\varepsilon^{1/3}, \quad |\partial_y^2 w_1^*(t, y_+^\varepsilon(t,r))| \le C\varepsilon d_\varepsilon^{1/3}. \tag{3.10}$$

Additionally,

$$\begin{aligned}&w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon) = \partial_t w_1^*(T_\varepsilon, y_\varepsilon)(t-T_\varepsilon) + \partial_y w_1^*(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon(t,r)-y_\varepsilon)\\ &\qquad + O\big(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(t-T_\varepsilon)(y_+^\varepsilon(t,r)-y_\varepsilon)^2 + \varepsilon(y_+^\varepsilon(t,r)-y_\varepsilon)^3\big),\\ &\partial_t w_1^0(t,r) = \partial_t w_1^*(t, y_+^\varepsilon(t,r)) + \partial_y w_1^*(t, y_+^\varepsilon(t,r))\partial_t y_+^\varepsilon(t,r),\\ &\partial_r w_1^*(t,r) = \partial_y w_1^*(t, y_+^\varepsilon(t,r))\partial_r y_+^\varepsilon(t,r),\\ &\partial_r^2 w_1^0(t,r) = \partial_y^2 w_1^*(t, y_+^\varepsilon(t,r))(\partial_r y_+^\varepsilon(t,r))^2 + \partial_y w_1^*(t, y_+^\varepsilon(t,y))\partial_r^2 y_+^\varepsilon(t,r).\end{aligned}$$

Combining these with (3.10) and Lemma 3.2, we see that (3.7) holds. □

Denoting the jump of $w_i^0(t,x)$ on γ by $[w_i^0]$, which equals $w_i^0(t, \phi^0(t) + 0) - w_i^0(t, \phi^0(t) - 0)$, we have

Lemma 3.5. *The jump of w_i^0 $(i = 1, 3)$ satisfies*

$$|[w_1^0]| \le C_0\varepsilon(t-T_\varepsilon)^{1/2}, \quad |[w_3^0]| \le C_0\varepsilon(t-T_\varepsilon)^{3/2}.$$

Proof. Using the estimates of $\phi^0(t)$ on γ, we have $d_\varepsilon = (t-T_\varepsilon)^3 + (\phi^0(t) - r_\varepsilon - \lambda_3(T_\varepsilon, r_\varepsilon))(t-T_\varepsilon)^2 \sim (t-T_\varepsilon)^3$. Therefore Lemma 3.4 1) implies

$$\begin{aligned}|[w_3^0]| &\le |w_3^0(t, \phi^0(t)+0) - w_3^0(T_\varepsilon, r_\varepsilon)| + |w_3^0(t, \phi^0(t)-0) - w_3^0(T_\varepsilon, r_\varepsilon)|\\ &\le C_0\varepsilon(t-T_\varepsilon)^{1/2}.\end{aligned}$$

Now we show that Lemma 3.5 holds for $[w_1^0]$.

Since

$$\begin{aligned}w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon) &= \partial_t w_1^*(T_\varepsilon, y_\varepsilon)(t-T_\varepsilon)\\ &\quad + O\big(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(t-T_\varepsilon)(y_+^\varepsilon - y_\varepsilon)^2 + \varepsilon(y_+^\varepsilon - y_\varepsilon)^3\big)\end{aligned}$$

in Ω_+, and since

$$\begin{aligned} w_1^0(t,r) - w_1^0(T_\varepsilon, r_\varepsilon) &= \partial_t w_1^*(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) \\ &\quad + O\big(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(t-T_\varepsilon)(y_-^\varepsilon - y_\varepsilon)^2 + \varepsilon(y_-^\varepsilon - y_\varepsilon)^3\big) \end{aligned}$$

in Ω_-, we have

$$\begin{aligned} |[w_1^0]| &= |w_1^0(t,\phi(t)+0) - w_1^0(T_\varepsilon, r_\varepsilon) - \{w_1^0(t,\phi(t)-0) - w_1^0(T_\varepsilon, r_\varepsilon)\}| \\ &= |O(\varepsilon d_\varepsilon^{1/2})| \le C_0 \varepsilon (t-T_\varepsilon)^{3/2}. \qquad \square \end{aligned}$$

Step 2. The iterative scheme

Next we improve the approximation sequence successively. Denote the unknown shock curve by $r = \phi(t)$. Then the slope of shock $\sigma(t) = \phi'(t)$ must satisfy the Rankine-Hugoniot conditions:

$$\begin{cases} \sigma[\rho] = [\rho\tilde{u}], \\ \sigma[\rho\tilde{u}] = [\rho\tilde{u}^2 + P(\rho,S)], \\ \sigma\Big[\rho e(\rho,S) + \dfrac{1}{2}\rho\tilde{u}^2\Big] = \Big[(\rho e(\rho,S) + \dfrac{1}{2}\rho\tilde{u}^2 + P(\rho,S))\tilde{u}\Big], \end{cases} \tag{3.11}$$

and the entropy condition for 3-shock.

If we denote the inverse of the transformations given in (2.13) and (2.14) by

$$(\rho, \tilde{u}, S) = \Big(q(w), \frac{w_1 + w_3}{2}, \bar{S} + w_2\Big),$$

then (3.11) is equivalent to

$$\begin{cases} \sigma[\rho_1(w)] - [F_1(w)] = 0, \\ \sigma[\rho_2(w)] - [F_2(w)] = 0, \\ \sigma[\rho_3(w)] - [F_3(w)] = 0, \end{cases} \tag{3.12}$$

where

$$\begin{gathered} \rho_1(w) = q(w), \quad \rho_2(w) = \frac{w_1 + w_3}{2}\rho_1(w), \\ \rho_3(w) = \rho_1(w)\Big(e(\rho_1(w), \bar{S} + w_2) + \frac{1}{2}\Big(\frac{w_1+w_3}{2}\Big)^2\Big), \end{gathered}$$

and

$$\begin{aligned} F_1(w) &= \rho_2(w), \quad F_2(w) = \Big(\frac{w_1+w_3}{2}\Big)^2 \rho_1(w) + P(\rho_1(w), \bar{S} + w_2), \\ F_3(w) &= \rho_2(w)\Big[e(\rho_1(w), \bar{S} + w_2) + \frac{\rho_2^2(w)}{2\rho_1^2(w)} + \frac{P(\rho_1(w), \bar{S} + w_2)}{\rho_1(w)}\Big]. \end{aligned}$$

The entropy condition for 3-shock is written as

$$\lambda_3(w_-(t)) < \sigma(t) < \lambda_3(w_+(t)), \quad \lambda_2(w_-(t)) < \sigma(t), \tag{3.13}$$

where

$$\begin{aligned} w_\pm(t) &= (w_{1,\pm}(t), w_{2,\pm}(t), w_3(t,\pm)) \\ &= (w_1(t,\phi(t)\pm 0), w_2(t,\phi(t)\pm 0), w_3(t,\phi(t)\pm 0)). \end{aligned}$$

Now we claim that for small ε, $(w_{1,-}(t), w_{2,-}(t))$ is uniquely determined from $(w_{1,+}(t), w_{2,+}(t), w_{3,\pm}(t), \sigma(t))$ by two of three equalities in (3.12). This assertion is important because by the entropy condition (3.13) we need the boundary value $(w_{1,-}(t), w_{2,-}(t))$ in order to solve $w_{1,-}(t,r)$ and $w_{2,-}(t,r)$ in the domain Ω_-.

LEMMA 3.6. *$(w_{1,-}(t), w_{2,-}(t))$ is determined by the equations*

$$\sigma[\rho_1(w)] - [F_1(w)] = 0, \quad \sigma[\rho_2(w)] - [F_2(w)] = 0.$$

Proof. By Lemma 2.3 and the assumption on $\bar{c} = 1$, we know that

$$\left(\frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}(0)\right)^{-1} \left(\frac{\partial(F_1,F_2,F_3)}{\partial(w_1,w_2,w_3)}(0)\right) = \mathrm{diag}\{-1,0,1\},$$

that is,

$$\left(\frac{\partial(F_1,F_2,F_3)}{\partial(\rho_1,\rho_2,\rho_3)}(0) - I\right) \begin{pmatrix} \frac{\partial\rho_1}{\partial w_1}(0) & \frac{\partial\rho_1}{\partial w_2}(0) \\ \frac{\partial\rho_2}{\partial w_1}(0) & \frac{\partial\rho_2}{\partial w_2}(0) \\ \frac{\partial\rho_3}{\partial w_1}(0) & \frac{\partial\rho_3}{\partial w_2}(0) \end{pmatrix} = \begin{pmatrix} -2\frac{\partial\rho_1}{\partial w_1}(0) & -\frac{\partial\rho_1}{\partial w_2}(0) \\ -2\frac{\partial\rho_2}{\partial w_1}(0) & -\frac{\partial\rho_2}{\partial w_2}(0) \\ -2\frac{\partial\rho_3}{\partial w_1}(0) & -\frac{\partial\rho_3}{\partial w_2}(0) \end{pmatrix}.$$

Additionally, a direct computation shows that

$$\frac{\partial(\rho_1,\rho_2)}{\partial(w_1,w_2,w_3)}(0) =$$
$$\begin{pmatrix} -\bar{\rho}/2 & -\partial_S p(\bar{\rho},\bar{S}) \\ -\bar{\rho}/2 & 0 \\ -\frac{\bar{\rho}}{2}(e(\bar{\rho},\bar{S}) + \bar{\rho}\partial_\rho e(\bar{\rho},\bar{S})) & \bar{\rho}\partial_S e(\bar{\rho},\bar{S}) - \partial_S p(\bar{\rho},\bar{S})(e(\bar{\rho},\bar{S}) + \bar{\rho}\partial_\rho e(\bar{\rho},\bar{S})) \end{pmatrix}$$

has rank 2. Hence by the implicit function theorem we see that $(w_{1,-}(t), w_{2,-}(t))$ is determined by the two equations $\sigma[\rho_1(w)] - [F_1(w)] = 0$ and $\sigma[\rho_2(w)] - [F_2(w)] = 0$. ∎

Consequently, from Lemma 3.6, (3.12) is equivalent to:

$$
\begin{cases} \sigma[\rho_1(w)] - [F_1(w)] = 0, \\ \sigma[\rho_2(w)] - [F_2(w)] = 0, \\ \sigma = \tilde{\lambda}_3\Big(\displaystyle\int_0^1 (\partial_{\rho_i} F_j)(\theta\rho(w_+(t)) + (1-\theta)\rho(w_-(t)))d\theta\Big), \end{cases} \tag{3.14}
$$

where $\tilde{\lambda}_3$ is the third eigenvalue of matrix $\big(\int_0^1 (\partial_{\rho_i} F_j)(\theta\rho(w_+(t)) + (1-\theta)\rho(w_-(t)))d\theta\big)_{i,j=1}^3$, with $\rho(w_\pm(t)) = (\rho_1(w_\pm(t)), \rho_2(w_\pm(t)), \rho_3(w_\pm(t)))$.

Based on the above preparations we now construct the weak entropy solution of (2.11) by using an approximation procedure. To avoid difficulties caused by unknown shock curve, which may change its location in the process of iteration, we introduce a coordinate transformation

$$
\begin{cases} z = r - \phi(t), \\ t = t, \end{cases} \tag{3.15}
$$

which transforms the (unknown) shock to $z = 0$, $t = t$.

Under these new coordinates, the blowup point is $(T_\varepsilon, 0)$ and the system (2.12) is written as the following form:

$$
\begin{cases} \partial_t w_1 + (\lambda_1 - \sigma(t))\partial_z w_1 + \bar{a}(w)(\partial_t w_2 + (\lambda_1 - \sigma(t))\partial_z w_2) = \dfrac{\bar{b}_1(w)}{z + \phi(t)}, \\ \partial_t w_2 + (\lambda_2 - \sigma(t))\partial_z w_2 = 0, \\ \partial_t w_3 + (\lambda_3 - \sigma(t))\partial_z w_3 - \bar{a}(w)(\partial_t w_2 + (\lambda_3 - \sigma(t))\partial_z w_2) = \dfrac{\bar{b}_3(w)}{z + \phi(t)}, \\ w_i(t, z)|_{t=T_\varepsilon} = w_i^0(T_\varepsilon, z + r_\varepsilon), \quad i = 1, 2, 3. \end{cases} \tag{3.16}
$$

Let

$$
\begin{aligned} \tilde{\Omega}_- &= \{(t, z) : T_\varepsilon \le t \le T_\varepsilon + 1,\ -2(T_\varepsilon + 1 - t) \le z < 0\}, \\ \tilde{\Omega}_+ &= \{(t, z) : T_\varepsilon \le t \le T_\varepsilon + 1,\ 0 < z \le 2(T_\varepsilon + 1 - t)\}. \end{aligned}
$$

When $\varepsilon > 0$ is small, $\tilde{\Omega}_- \cup \tilde{\Omega}_+$ is obviously located in the determinate region of $\{(T_\varepsilon, z) : -K \le z \le K\}$. In order to construct the weak entropy solution of (2.11) in the domain $\tilde{\Omega}_- \cup \tilde{\Omega}_+$ and prove Theorem 2.1, we take the following iterative scheme:

$$
\begin{cases}
\partial_t w_{1,+}^{n+1} + (\lambda_1(w_+^n) - \sigma^n(t))\partial_z w_{1,+}^{n+1} \\
\qquad + \bar{a}(w_+^n)(\partial_t w_{2,+}^n + (\lambda_1(w_+^n) - \sigma^n(t))\partial_z w_{2,+}^n) = \dfrac{\bar{b}_1(w_+^n)}{z + \phi^n(t)}, \\
\partial_t w_{2,+}^{n+1} + (\lambda_2(w_+^n) - \sigma^n(t))\partial_z w_{2,+}^{n+1} = 0, \\
\partial_t w_{3,\pm}^{n+1} + (\lambda_3(w_\pm^n) - \sigma^n(t))\partial_z w_{3,\pm}^{n+1} \\
\qquad - \bar{a}(w_\pm^n)(\partial_t w_{2,\pm}^n + (\lambda_3(w_\pm^n) - \sigma^n(t))\partial_z w_{2,\pm}^n) = \dfrac{\bar{b}_3(w_\pm^n)}{z + \phi^n(t)}, \\
w_{1,+}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{1,+}^0(T_\varepsilon, z + r_\varepsilon), \quad w_{2,+}^{n+1}(t,z)|_{t=T_\varepsilon} = 0, \\
w_{3,\pm}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{3,\pm}^0(T_\varepsilon, z + r_\varepsilon),
\end{cases}
\tag{3.17}
$$

and

$$
\begin{cases}
\partial_t w_{1,-}^{n+1} + (\lambda_1(w_-^n) - \sigma^n(t))\partial_z w_{1,-}^{n+1} \\
\qquad + \bar{a}(w_-^n)(\partial_t w_{2,-}^n + (\lambda_1(w_-^n) - \sigma^n(t))\partial_z w_{2,-}^n) = \dfrac{\bar{b}_1(w_-^n)}{z + \phi^n(t)}, \\
\partial_t w_{2,-}^{n+1} + (\lambda_2(w_-^n) - \sigma^n(t))\partial_z w_{2,-}^{n+1} = 0, \\
\sigma^n(t) = \tilde{\lambda}_3\Big(\displaystyle\int_0^1 (\partial_{\rho_i} F_j)(\theta\rho(w_+^n(t,0+)) + (1-\theta)\rho(w_-^n(t,0-)))d\theta\Big), \\
w_{1,-}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{1,-}^0(T_\varepsilon, z + r_\varepsilon), \\
w_{2,-}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{2,-}^0(T_\varepsilon, z + r_\varepsilon), \\
w_{1,-}^{n+1}(t,z)|_{z=0} = w_{1,-}^{n+1}(t,0-), \\
w_{2,-}^{n+1}(t,z)|_{z=0} = w_{2,-}^{n+1}(t,0-),
\end{cases}
\tag{3.18}
$$

where $w_{1,-}^{n+1}(t,0-)$ and $w_{2,-}^{n+1}(t,0-)$ are determined by the equations:

$$
\begin{cases}
\sigma^n[\rho_1(w^{n+1})] = [F_1(w^{n+1})], \\
\sigma^n[\rho_3(w^{n+1})] = [F_3(w^{n+1})].
\end{cases}
\tag{3.19}
$$

Here, $\phi^n(t) = T_\varepsilon + \int_{T_\varepsilon}^t \sigma^n(t)dt$.

By the entropy condition (3.13), (3.17) and (3.18) are solved by the characteristics method. Since $w_{2,+}^{n+1} \equiv \bar{S}$, (3.17) becomes

$$\begin{cases} \partial_t w_{1,+}^{n+1} + (\lambda_1(w_+^n) - \sigma^n(t))\partial_z w_{1,+}^{n+1} = \dfrac{\bar{b}_1(w_+^n)}{z + \phi^n(t)}, \\ \partial_t w_{3,+}^{n+1} + (\lambda_3(w_+^n) - \sigma^n(t))\partial_z w_{3,+}^{n+1} = \dfrac{\bar{b}_3(w_\pm^n)}{z + \phi^n(t)}, \\ \partial_t w_{3,-}^{n+1} + (\lambda_3(w_-^n) - \sigma^n(t))\partial_z w_{3,-}^{n+1} \\ \qquad - \bar{a}(w_-^n)(\partial_t w_{2,-}^n + (\lambda_3(w_-^n) - \sigma^n(t))\partial_z w_{2,-}^n) = \dfrac{\bar{b}_3(w_-^n)}{z + \phi^n(t)}, \\ w_{1,+}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{1,+}^0(T_\varepsilon, z + r_\varepsilon), \\ w_{3,\pm}^{n+1}(t,z)|_{t=T_\varepsilon} = w_{3,\pm}^0(T_\varepsilon, z + r_\varepsilon). \end{cases} \tag{3.20}$$

In order to estimate $\{w_\pm^n\}$ and $\{\sigma^n(t)\}$, we need the following lemma.

LEMMA 3.7. *There exist two smooth functions*

$$G_i(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-)), \quad i = 1, 2$$

such that

$$[w_i] = G_i(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-))[w_3]^3, \quad i = 1, 2. \tag{3.21}$$

Proof. The equality for $[w_2]$ is well known, since the change of entropy across a shock is a small quantity of third order of the strength of the shock (for example, see [15] and [21]).

To prove the first equality (3.21), we rewrite (3.12) as

(3.22)

$$\left(\frac{\partial(F_1,F_2,F_3)}{\partial(\rho_1,\rho_2,\rho_3)}(w_-(t,0-)) - \sigma I\right)\left(\frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\right)(w_-(t,0-))\begin{pmatrix}[w_1]\\ [w_2]\\ [w_3]\end{pmatrix}$$
$$= \tilde{B}\begin{pmatrix}[w_1]^2 & [w_1][w_2] & [w_1][w_3]\\ [w_1][w_2] & [w_2]^2 & [w_2][w_3]\\ [w_1][w_3] & [w_2][w_3] & [w_3]^2\end{pmatrix}.$$

Here $\tilde{B} = (\tilde{b}_{ij}(w_-(t,0-), w_+(t,0+)))_{i,j=1}^3$ is a 3×3 smooth function matrix.

Since

$$\sigma(t) = \lambda_3(w_-(t,0-)) + \sum_{i=1}^{3} g_i(w_-(t,0-))[w_i] + \sum_{i,j=1}^{3} g_{ij}(w_-(t,0-), w_+(t,0+))[w_i][w_j],$$

and since Lemma 2.3 implies

$$\left(\frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\right)^{-1}\Bigg|_{w=w_-(t,0-)} \left(\frac{\partial(F_1,F_2,F_3)}{\partial(\rho_1,\rho_2,\rho_3)}(w_-(t,0-)) - \sigma I\right) \times \frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\Bigg|_{w=w_-(t,0-)}$$
$$= \begin{pmatrix} \lambda_1(w_-(t,0-)) - \sigma & a(w_-(t,0-)) & 0 \\ 0 & \lambda_2(w_-(t,0-)) - \sigma & 0 \\ 0 & -a(w_-(t,0-)) & \lambda_3(w_-(t,0-)) - \sigma \end{pmatrix},$$

multiplying (3.22) by $\left(\frac{\partial(\rho_1,\rho_2,\rho_3)}{\partial(w_1,w_2,w_3)}\right)^{-1}\Big|_{w=w_-(t,0-)}$ gives

$$[w_1] = \sum_{i,j=1}^{3} Q_{ij}(w_-(t,0-))[w_i][w_j] + \sum_{i,j,k=1}^{3} Q_{ijk}(w_-(t,0-), w_+(t,0+))[w_i][w_j][w_k], \tag{3.23}$$

where Q_{ij} and Q_{ijk} are smooth.

Interchanging $w_-(t,0-)$ and $w_+(t,0+)$ in (3.23) gives

$$[w_1] = -\sum_{i,j=1}^{3} Q_{ij}(w_+(t,0+))[w_i][w_j] + \sum_{i,j,k=1}^{3} Q_{ijk}(w_+(t,0+), w_-(t,0-))[w_i][w_j][w_k]. \tag{3.24}$$

Summing up (3.23) and (3.24), we have

$$[w_1] = \sum_{i,j,k=1}^{3} \tilde{Q}_{ijk}(w_-(t,0-), w_+(t,0+))[w_i][w_j][w_k],$$

where $\tilde{Q}_{ijk}$ are smooth.

Set $[w_1] = \mu[w_2]^3$, and note that

$$w_{1,-}(t,0-) = w_{1,+}(t,0+) - [w_1],$$
$$[w_2] = G_2(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-))[w_3]^3.$$

Applying the implicit function theorem we see that if $[w_3]$ is small, then

$$\mu = G_1(w_{1,+}(t,0+), w_{3,+}(t,0+), w_{3,-}(t,0-))$$

for a smooth function G_1. This proves Lemma 3.7. ∎

Step 3. Estimates of $\{w_{\pm}^{n+1}(t,z)\}$ and $\{\sigma^n(t)\}$

In this section, we estimate $\{w_{\pm}^{n+1}(t,z)\}$ and $\{\sigma^n(t)\}$. In what follows, "N" represents a constant independent of n and ε, which may vary from line to line.

LEMMA 3.8. *Let $C_0 > 0$ be the constant given in Lemma 3.5. There exists a number $N > 0$ independent of ε such that, for all n*

$$w_{\pm}^n \in C^1(\tilde{\Omega}_{\pm} \setminus (T_\varepsilon, 0)), \tag{3.25}$$
$$|w_{3,\pm}^n - w_{3,\pm}^0| \le N\varepsilon(t - T_\varepsilon), \tag{3.26}$$
$$|\partial_z(w_{3,\pm}^n - w_{3,\pm}^0)| \le N\varepsilon((t - T_\varepsilon)^3 + z^2)^{-1/6}, \tag{3.27}$$
$$|\partial_t(w_{3,\pm}^n - w_{3,\pm}^0)| \le N\varepsilon((t - T_\varepsilon)^3 + z^2)^{-1/6}, \tag{3.28}$$
$$|w_{i,\pm}^n - w_{i,\pm}^0| \le N\varepsilon(t - T_\varepsilon)^{3/2}, \quad i = 1,2, \tag{3.29}$$
$$|\partial_z(w_{i,\pm}^n - w_{i,\pm}^0)| \le N\varepsilon(t - T_\varepsilon)^{1/2}, \quad i = 1,2, \tag{3.30}$$
$$|\partial_t(w_{i,\pm}^n - w_{i,\pm}^0)| \le N\varepsilon(t - T_\varepsilon)^{1/2}, \quad i = 1,2, \tag{3.31}$$

in $\tilde{\Omega}_-$ or in $\tilde{\Omega}_+$.

Proof. Obviously, (3.25)–(3.31) hold for $n = 0$. Now we prove the conclusion by induction. Assuming that these estimates hold for n, we prove that they hold also for $n+1$. The proof will be divided into six steps.

Part 1. Estimate of $\sigma^n(t)$

Suppose (3.25)–(3.31) are true for n. By the expression for $\sigma^n(t)$ given in (3.18) and the mean value theorem, we have

$$|\sigma^n(t) - \sigma^0(t)| \le C_N\varepsilon(t - T_\varepsilon)$$

in $[T_\varepsilon, T_\varepsilon + 1]$. Here $C_N > 0$ depends only on N.

Part 2. Estimates of $w_{3,\pm}^{n+1}$, $w_{1,+}^{n+1}$ and $w_{2,+}^{n+1}$

We estimate only $w_{3,-}^{n+1}$; estimates of the others are completely parallel, or even simpler.

The function $v(t,z) = w_{3,-}^{n+1} - w_{3,-}^0$ satisfies

$$
\left\{
\begin{aligned}
&\partial_t v + (\lambda_3(w_-^n) - \sigma^n)\partial_z v = (\lambda_3(w_-^0) - \lambda_3(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{3,-}^0 \\
&\qquad + \bar{a}(w_-^n)\big\{\partial_t(w_{2,-}^n - w_{2,-}^0) + (\lambda_3(w_-^n) - \sigma^n)\partial_z(w_{2,-}^n - w_{2,-}^0) \\
&\qquad - (\lambda_3(w_-^0) - \lambda_3(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{2,-}^0\big\} \\
&\qquad + (\bar{a}(w_-^n) - \bar{a}(w_-^0))(\partial_t w_{2,-}^0 + (\lambda_3(w_-^0) - \sigma^0)\partial_z w_{2,-}^0) \\
&\qquad + \frac{\bar{b}_3(w_-^n)}{z+\phi^n(t)} - \frac{\bar{b}_3(w_-^0)}{z+\phi^0(t)}, \\
&v(T_\varepsilon, z) = 0.
\end{aligned}
\right.
\tag{3.32}
$$

Noting

$$
\begin{aligned}
&\bar{a}(w_-^n)\big\{\partial_t(w_{2,-}^n - w_{2,-}^0) + (\lambda_3(w_-^n) - \sigma^n)\partial_z(w_{2,-}^n - w_{2,-}^0)\big\} \\
&\quad = (\partial_t + (\lambda_3(w_-^n) - \sigma^n)\partial_z)(\bar{a}(w_-^n)(w_{2,-}^n - w_{2,-}^0)) \\
&\qquad - \left\{\sum_{j=1}^{3}(\partial_{w_j}\bar{a})(w_-^n)(\partial_t w_{j,-}^n + (\lambda_3(w_-^n) - \sigma^n)\partial_z w_{j,-}^n)\right\}(w_{2,-}^n - w_{2,-}^0)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\bar{b}_3(w_-^n)}{z+\phi^n(t)} - \frac{\bar{b}_3(w_-^0)}{z+\phi^0(t)} \\
&\quad = \frac{\bar{b}_3(w_-^n) - \bar{b}_3(w_-^0)}{z+\phi^n(t)} + \frac{\bar{b}_3(w_-^0)}{(z+\phi^n(t))(z+\phi^0(t))}\int_{T_\varepsilon}^{t}(\sigma^n(t) - \sigma^0(t))dt
\end{aligned}
$$

in view of the induction hypothesis, $\bar{a}(0) = 0$ and Lemma 3.4, we can apply the method of characteristics to derive

$$
\begin{aligned}
|v(t,y)| &\le |\bar{a}(w_-^n)(w_{2,-}^n - w_{2,-}^0)| + C_N\varepsilon^2\int_{T_\varepsilon}^{t}(1 + \sqrt{s - T_\varepsilon})ds \\
&\le C_N\varepsilon^2(t - T_\varepsilon)
\end{aligned}
$$

with $C_N > 0$ depending only on N. Hence (3.26) holds for $n+1$, whenever ε is small.

Similarly, we can show

$$
|w_{1,+}^{n+1} - w_{1,+}^0| \le C_N\varepsilon^2(t - T_\varepsilon)^{3/2}.
$$

Part 3. Estimates of $w_{1,-}^{n+1}$ and $w_{2,-}^{n+1}$

It is enough to estimate $w_{1,-}^{n+1}$. The function $v(t,z) = w_{1,-}^{n+1} - w_{1,-}^0$ satisfies

$$\text{(3.33)}\quad \begin{cases} \partial_t v + (\lambda_1(w_-^n) - \sigma^n)\partial_z v = (\lambda_1(w_-^0) - \lambda_1(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{1,-}^0 \\ \qquad - \bar{a}(w_-^n)\big\{\partial_t(w_{2,-}^n - w_{2,-}^0) + (\lambda_1(w_-^n) - \sigma^n)\partial_z(w_{2,-}^n - w_{2,-}^0) \\ \qquad - (\lambda_1(w_-^0) - \lambda_1(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{2,-}^0\big\} \\ \qquad - (\bar{a}(w_-^n) - \bar{a}(w_-^0))(\partial_t w_{2,-}^0 + (\lambda_1(w_-^0) - \sigma^0)\partial_z w_{2,-}^0) \\ \qquad + \dfrac{\bar{b}_1(w_-^n)}{z + \phi^n(t)} - \dfrac{\bar{b}_1(w_-^0)}{z + \phi^0(t)}, \\ v(T_\varepsilon, z) = 0, \quad v(t,z)|_{z=0} = w_{1,-}^{n+1}(t,0-) - w_{1,-}^0(t,0-). \end{cases}$$

Let $\xi = \xi(t,z,s)$ be the backward characteristics of (3.33) through the point (t,z) in the domain $\tilde{\Omega}_-$. If the characteristics $\xi = \xi(t,z,s)$ intersects the z-axis, then as in Part 2, we have $|v(t,z)| \le C_M\varepsilon^2(t - T_\varepsilon)^{3/2}$. If the characteristics $\xi = \xi(t,z,s)$ intersects the t-axis at $(s,0)$ with $s > T_\varepsilon$, then we have to estimate $w_{1,-}^{n+1}(t,0-)$. Firstly, by using the induction hypothesis and the method of characteristics we have

$$\text{(3.34)}\qquad |v(t,z)| \le |w_{1,-}^{n+1}(s,0-) - w_{1,-}^0(s,0-)| + C_N\varepsilon^2(t - T_\varepsilon)^{3/2}.$$

Secondly, by Lemma 3.7 we know

$$\text{(3.35)}\qquad [w_1^{n+1}] = G_1(w_{1,+}^{n+1}(s,0+), w_{3,+}^{n+1}(s,0+), w_{3,-}^{n+1}(s,0-))[w_3^{n+1}]^3.$$

Since

$$|w_{1,-}^{n+1}(s,0-) - w_{1,-}^0(s,0-)| \le |[w_1^{n+1}]| + |w_{1,+}^{n+1}(s,0+) - w_{1,+}^0(s,0+)| + |[w_1^0]|$$

and

$$|[w_3^{n+1}]| \le |w_{3,+}^{n+1}(s,0+) - w_{3,+}^0(s,0+)| + |w_{3,-}^{n+1}(s,0-) - w_{3,-}^0(s,0-)| + |[w_3^0]|,$$

it follows from (3.34), (3.35) and Part 2 that

$$\text{(3.36)}\qquad |v(t,z)| \le C_0\varepsilon(t - T_\varepsilon)^{3/2} + C_N\varepsilon^2(t - T_\varepsilon)^{3/2} \le N\varepsilon(t - T_\varepsilon)^{3/2}$$

for small ε.

Part 4. Estimates of $|\nabla(w^{n+1}_{3,\pm} - w^0_{3,\pm})|$

The function $v(t,z) = \partial_z(w^{n+1}_{3,-} - w^0_{3,-})$ satisfies

(3.37)
$$\begin{cases} \partial_t v + (\lambda_3(w^n_-) - \sigma^n(t))\partial_z v + \partial_z(\lambda_3(w^n_-))v = \bar{a}(w^n_-)\big\{\partial^2_{tz}(w^n_{2,-} - w^0_{2,-}) \\ \qquad + (\lambda_3(w^n_-) - \sigma^n)\partial^2_z(w^n_{2,-} - w^0_{2,-})\big\} \\ \qquad + (\lambda_3(w^n_-) - \lambda_3(w^0_-) - \sigma^n + \sigma^0)\partial^2_z w^0_{2,-}\big\} \\ \qquad + (\lambda_3(w^0_-) - \lambda_3(w^n_-) + \sigma^n - \sigma^0)\partial^2_z w^0_{3,-} \\ \qquad - \displaystyle\sum_{j=1}^{3}\big\{(\partial_{w_j}\lambda_3)(w^n_-)\partial_z w^n_{j,-} - (\partial_{w_j}\lambda_3)(w^0_-)\partial_z w^0_{j,-}\big\}\partial_z w^0_{3,-} \\ \qquad + \displaystyle\sum_{j=1}^{3}\big\{(\partial_{w_j}\bar{a})(w^n_-)\partial_z w^n_{j,-}\{\partial_t(w^n_{2,-} - w^0_{2,-}) \\ \qquad + (\lambda_3(w^n_-) - \sigma^n)\partial_z(w^n_{2,-} - w^0_{2,-}) \\ \qquad - (\lambda_3(w^0_-) - \lambda_3(w^n_-) + \sigma^n - \sigma^0)\partial_z w^0_{2,-}\big\} \\ \qquad - ((\partial_{w_j}\bar{a})(w^n_-)\partial_z w^n_{j,-} - (\partial_{w_j}\bar{a})(w^0_-)\partial_z w^0_{j,-}) \\ \qquad \times (\partial_t w^0_{2,-} + (\lambda_3(w^0_-) - \sigma^0)\partial_z w^0_{2,-})\big\} \\ \qquad + \displaystyle\sum_{j=1}^{3}\bar{a}(w^n_-)\big\{\partial_{w_j}\lambda_3(w^n_-)\partial_z w^n_{j,-}\partial_z w^n_{2,-} - \partial_{w_j}\lambda_3(w^0_-)\partial_z w^0_{j,-}\partial_z w^0_{2,-}\big\} \\ \qquad + \partial_z\Big(\dfrac{\bar{b}_3(w^n_-)}{z+\phi^n(t)}\Big) - \partial_z\Big(\dfrac{\bar{b}_3(w^0_-)}{z+\phi^0(t)}\Big), \\ v(T_\varepsilon, z) = 0. \end{cases}$$

Let $\xi^{n+1} = \xi^{n+1}(t,z,s)$ be the backward characteristics of (3.37) through the point (t,z), that is, the solution of the equation

$$\begin{cases} \dfrac{d\xi^{n+1}}{ds} = \lambda_3(w^n_-(s,\xi^{n+1})) - \sigma^n(s), \quad T_\varepsilon \le s \le t \\ \xi^{n+1}|_{s=t} = z. \end{cases}$$

As in the proofs of Lemmas 8.1 and 8.3 in [17], we can show that there exists a constant C independent of n and ε such that

$$(s - T_\varepsilon)^3 + (\xi^{n+1})^2 \ge C((t - T_\varepsilon)^3 + z^2) \tag{3.38}$$

and

$$\Big|\int_{T_\varepsilon}^{t}(\partial_z(\lambda_3(w^n_-)))(s,\xi^{n+1})ds\Big| \le \ln\frac{3}{2} + C\varepsilon\sqrt{t - T_\varepsilon} < \frac{1}{2}. \tag{3.39}$$

The reasons for which we can use the methods in [17] are the following: Firstly, we have $\partial_{w_3}\lambda_3(0) \neq 0$. Secondly, λ_3 and σ satisfy a relation which is similar to that given in Lemma 3.1 of [17] (see (3.50) below). These two conditions are the only keys for proving Lemmas 8.1 and 8.3 in [17] for p-system of 2×2 equations.

Combining (3.38) and (3.39) with Lemma 3.4, $\bar{b}_1(0) = 0$ and the induction hypothesis and then integrating (3.37), we obtain

$$\begin{aligned}
&|v(t,z)| \leq |\bar{a}(w_-^n)\partial_z(w_{2,-}^n - w_{2,-}^0)(t,z)| \\
&+ \int_{T_\varepsilon}^t |(\partial_z(\lambda_3(w_-^n)))(s,\xi^{n+1})||v(s,y)|ds + C_M\varepsilon^2 \int_{T_\varepsilon}^t \Big\{ \frac{s - T_\varepsilon}{((t-T_\varepsilon)^3 + z^2)^{5/6}} \\
&+ \frac{\sqrt{s - T_\varepsilon}}{((t-T_\varepsilon)^3 + z^2)^{1/3}} + \frac{1}{((t-T_\varepsilon)^3 + z^2)^{1/2}} \Big\} ds \\
&\leq C_M\varepsilon^2((t-T_\varepsilon)^3 + z^2)^{-1/6} + \int_{T_\varepsilon}^t |(\partial_z(\lambda_3(w_-^n)))(s,\xi^{n+1})||v(s,y)|ds.
\end{aligned}$$

By (3.39) and Gronwall's inequality, we see that (3.37) holds for small ε; and (3.28) follows from (3.33).

Part 5. Estimates on $|\nabla(w_{1,+}^{n+1} - w_{1,+}^0)|$

The function $v(t,z) = \partial_z(w_{1,+}^{n+1} - w_{1,+}^0)$ satisfies

$$(3.40) \qquad \begin{cases}
\partial_t v + (\lambda_1(w_+^n) - \sigma^n(t))\partial_z v + \partial_z(\lambda_1(w_+^n))v \\
\qquad + \displaystyle\sum_{j=1}^3 \{\partial_{w_j}\lambda_1(w_+^n)\partial_z w_{j,+}^n - \partial_{w_j}\lambda_1(w_+^0)\partial_z w_{j,+}^0)\partial_z w_{1,+}^0\} \\
\qquad = \partial_z\Big(\dfrac{\bar{b}_1(w_+^n)}{z + \phi^n(t)}\Big) - \partial_z\Big(\dfrac{\bar{b}_1(w_+^0)}{z + \phi^0(t)}\Big), \\
v(T_\varepsilon, z) = 0.
\end{cases}$$

For the backward characteristics $\xi_1^{n+1} = \xi_1^{n+1}(t,z,s)$ of (3.40) through the point (t,z), we have

$$(3.41) \qquad \xi_1^{n+1} \geq z + \frac{t-s}{2}$$

if ε is small.

By the characteristics method and $\bar{b}_1(0)=0$ we get

$$
\begin{aligned}
|v(t,z)| &\le C_N\varepsilon\int_{T_\varepsilon}^{t}\frac{|v(s,\xi_1^{n+1})|}{((s-T_\varepsilon)^3+(\xi_1^{n+1})^2)^{1/3}}ds\\
&+C_N\varepsilon^2\int_{T_\varepsilon}^{t}\Big\{\frac{\sqrt{s-T_\varepsilon}}{((s-T_\varepsilon)^3+(\xi_1^{n+1})^2)^{1/3}}+\frac{s-T_\varepsilon}{((s-T_\varepsilon)^3+(\xi_1^{n+1})^2)^{1/2}}\Big\}ds.
\end{aligned}
$$

Substituting (3.41) into the above inequality, we have

$$
|v(t,z)|\le C_N\varepsilon^2\sqrt{t-T_\varepsilon}+C_N\varepsilon\int_{T_\varepsilon}^{t}\frac{|v(s,\xi_1^{n+1})|}{(t-s)^{2/3}}ds.
$$

Hence Gronwall's inequality implies

$$
|v(t,y)|\le C_N\varepsilon^2\sqrt{t-T_\varepsilon}.
$$

Part 6. Estimates on $|\nabla(w_{1,-}^{n+1}-w_{1,-}^0)|$ and $|\nabla(w_{2,-}^{n+1}-w_{2,-}^0)|$

We estimate only $|\partial_t(w_{1,-}^{n+1}-w_{1,-}^0)|$ and $|\partial_z(w_{1,-}^{n+1}-w_{1,-}^0)|$. Firstly, the function $v(t,z)=\partial_t(w_{1,-}^{n+1}-w_{1,-}^0)$ satisfies
(3.42)

$$
\begin{cases}
\partial_t v+(\lambda_1(w_-^n)-\sigma^n(t))\partial_z v+\partial_t(\lambda_1(w_-^n)-\sigma^n(t))v\\
\qquad +\bar{a}(w_-^n)\big\{\partial_t^2(w_{2,-}^n-w_{2,-}^0)+(\lambda_1(w_-^n)-\sigma^n)\partial_{tz}^2(w_{2,-}^n-w_{2,-}^0)\big\}\\
\qquad +(\lambda_1(w_-^n)-\lambda_1(w_-^0)-\sigma^n+\sigma^0)\partial_{tz}^2 w_{2,-}^0\big\}\\
\qquad +(\bar{a}(w_-^n)-\bar{a}(w_-^0))\big\{\partial_t^2 w_{2,-}^0+(\lambda_1(w_-^0)-\sigma^0)\partial_{tz}^2 w_{2,-}^0\\
\qquad -(\lambda_1(w_-^0)-\lambda_1(w_-^n)+\sigma^n-\sigma^0)\partial_{tz}^2 w_{1,-}^0\\
\qquad +\sum_{j=1}^{3}\big\{(\partial_{w_j}\bar{a})(w_-^n)\partial_t w_{j,-}^n(\partial_t w_{2,-}^n+(\lambda_1)(w_-^n)-\sigma^n)\partial_z w_{2,-}^n)\\
\qquad -(\partial_{w_j}\bar{a})(w_-^0)\partial_t w_{j,-}^0(\partial_t w_{2,-}^0+(\lambda_1(w_-^0))-\sigma^0)\partial_z w_{2,-}^0\big\}\\
\qquad +\bar{a}(w_-^n)\partial_t(\lambda_1(w_-^n)-\sigma^n)\partial_z w_{2,-}^n-\bar{a}(w_-^0)\partial_t(\lambda_1(w_-^0)-\sigma^0)\partial_z w_{2,-}^0\\
\qquad =\partial_t\Big(\dfrac{\bar{b}_1(w_-^n)}{z+\phi^n(t)}\Big)-\partial_t\Big(\dfrac{\bar{b}_1(w_-^0)}{z+\phi^0(t)}\Big),\\
v(T_\varepsilon,z)=0.
\end{cases}
$$

We derive from Lemma 3.4, $\bar{b}_1(0)=0$, and the induction hypothesis that

$$
|w_{i,-}^n(t,z)-w_{i,+}^n(t,0+)|\le C_N\varepsilon((t-T_\varepsilon)^3+z^2)^{1/6},
$$

and

$$|w^n_{i,-}(t,z) - w^0_{i,-}(t,0-)| \le C_N\varepsilon((t-T_\varepsilon)^3 + z^2)^{1/6}$$

for $i = 1,2,3$. Hence, by the expressions of λ_3 and $\sigma^n(t)$,

$$|\lambda_3(w^n_-) - \sigma^n(t)| \le C_N\varepsilon((t-T_\varepsilon)^3 + z^2)^{1/6}. \tag{3.43}$$

Furthermore, by Lemma 3.4, the induction hypothesis and (3.43), we get

$$|\partial_t w^n_{3,-}(t,z)| \le \frac{C_N\varepsilon}{\sqrt{t-T_\varepsilon}}. \tag{3.44}$$

Let $\xi = \xi(t,z,s)$ be the backward characteristics of (3.42) through (t,z) in $\tilde{\Omega}_-$. Suppose first that this characteristics intersects the z-axis before meeting the t-axis. Then, integrating (3.42) along the characteristics and using the result in Part 1 as well as the induction hypothesis and $\bar{a}(0) = \bar{b}_1(0) = 0$, we get

$$\begin{aligned}|v(t,z)| &\le C_N\varepsilon^2(t-T_\varepsilon)^{1/2} \\ &\quad + \int_{T_\varepsilon}^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s)) - (\partial_t\sigma^n)(s)||v(s,\xi(t,z,s))|ds.\end{aligned}$$

In view of (3.44), the induction hypothesis and the expression of $\sigma^n(t)$, we obtain

$$\int_{T_\varepsilon}^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s)) - (\partial_t\sigma^n)(s)|ds \le C_N\varepsilon\sqrt{t-T_\varepsilon}.$$

So, by Gronwall's inequality, we conclude that

$$|v(t,z)| \le N\varepsilon(t-T_\varepsilon)^{1/2}$$

for small ε. When $\xi = \xi(t,z,s)$ intersects the t-axis at $(s,0)$ for some $s > T_\varepsilon$, integrating along the characteristics gives

$$\begin{aligned}|v(t,z)| &\le C_N\varepsilon^2(t-T_\varepsilon)^{1/2} + |(\partial_s(w^{n+1}_{1,-} - w^0_{1,-}))(s,0-)| \\ &\quad + \int_{T_\varepsilon}^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s)) - (\partial_t\sigma^n)(s)||v(s,\xi(t,z,s))|ds.\end{aligned} \tag{3.45}$$

We here estimate the term $|(\partial_s(w^{n+1}_{1,-} - w^0_{1,-}))(s,0-)|$ on the right side of (3.45).

Since

$$\begin{aligned}
|[w_3^{n+1}]| &\le |w_{3,+}^{n+1}(s,0+) - w_{3,+}^0(s,0+)| \\
&\qquad + |w_{3,-}^{n+1}(s,0-) - w_{3,-}^0(s,0+)| + |[w_3^0]| \le C_N\varepsilon(s-T_\varepsilon)^{1/2}, \\
|[\partial_s w_3^{n+1}] - [\partial_s w_3^0]| &\le \frac{C_N\varepsilon}{\sqrt{s-T_\varepsilon}}, \\
|[\partial_s w_3^{n+1}]| &\le \frac{C_N\varepsilon}{s-T_\varepsilon},
\end{aligned}$$

it follows from (3.35) and Lemma 3.4 that

$$|\partial_s(w_{1,-}^{n+1}(s,0-) - w_{1,-}^0(s,0-))| \le C_N\varepsilon^2(s-T_\varepsilon)^{1/2}$$

for small ε. We substitute this into (3.45) and apply Gronwall's inequality to get

$$|v(t,z)| \le C_N\varepsilon^2(t-T_\varepsilon)^{1/2}$$

for small ε. Finally, since $|\lambda_1(w_-^n) - \sigma^n| \ge 1/4$ if ε is small, we see from (3.33) that

$$|\partial_z(w_{1,-}^{n+1} - w_{1,-}^0)(t,z)| \le C_N\varepsilon^2(t-T_\varepsilon)^{1/2}.$$

Combining all of the above estimates we complete the proof of the lemma. ∎

Step 4. Convergence

The following result shows the contractivity of $\{\sigma^n\}$ on $[T_\varepsilon, T_\varepsilon+1]$ and of $\{w_{i,\pm}^n\}$ on $\tilde\Omega_\pm$, respectively.

LEMMA 3.9. *There exists a constant C_N independent of ε and n such that*

$$\|\sigma^n - \sigma^{n-1}\|_{L^\infty([T_\varepsilon,T_\varepsilon+1])} \le C_N \sum_{i=1}^3 \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}, \tag{3.46}$$

$$\begin{aligned}
&\|w_{3,\pm}^{n+1} - w_{3,\pm}^n\|_{L^\infty(\tilde\Omega_\pm)} + C_N \sum_{i=1,2} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} \\
&\quad \le (1-\varepsilon)\Big\{\|w_{3,\pm}^n - w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} + C_N \sum_{i=1,2} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\Big\},
\end{aligned} \tag{3.47}$$

if ε is small. Here,

$$\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} = \|w_{i,+}^n - w_{i,+}^{n-1}\|_{L^\infty(\tilde\Omega_+)} + \|w_{i,-}^n - w_{i,-}^{n-1}\|_{L^\infty(\tilde\Omega_-)}.$$

Proof. Firstly, (3.46) is obvious from Lemma 3.8 and the expression of $\sigma^n(t)$. Secondly, the function $v(t,z) = w_{3,-}^{n+1} - w_{3,-}^n$ satisfies

(3.48)
$$\begin{cases} \partial_t v + (\lambda_3(w_-^n) - \sigma^n)\partial_z v = (\lambda_3(w_-^{n-1}) - \lambda_3(w_-^n) + \sigma^n - \sigma^{n-1})\partial_z w_{3,-}^{n-1} \\ \qquad + \bar{a}(w_-^n)\big\{\partial_t(w_{2,-}^n - w_{2,-}^{n-1}) + (\lambda_3(w_-^n) - \sigma^n)\partial_z(w_{2,-}^n - w_{2,-}^{n-1}) \\ \qquad - (\lambda_3(w_-^{n-1}) - \lambda_3(w_-^n) + \sigma^n - \sigma^{n-1})\partial_z w_{2,-}^n\big\} \\ \qquad + (\bar{a}(w_-^n) - \bar{a}(w_-^{n-1}))(\partial_t w_{2,-}^{n-1} + (\lambda_3(w_-^{n-1}) - \sigma^{n-1})\partial_z w_{2,-}^{n-1}) \\ \qquad + \dfrac{\bar{b}_3(w_-^n)}{z + \phi^n(t)} - \dfrac{\bar{b}_3(w_-^{n-1})}{z + \phi^{n-1}(t)}, \\ v(T_\varepsilon, z) = 0. \end{cases}$$

The first term on the right side of (3.48) is most difficult to estimate, since it contains the nonintegrable function $\partial_z w_{3,-}^{n-1}$. To avoid this difficulty, we use

$$\begin{aligned} &(\partial_{w_k}\lambda_3)(w_-^n)\partial_z w_{3,-}^n \\ &\quad = \frac{(\partial_{w_k}\lambda_3)(w_-^n)}{(\partial_{w_3}\lambda_3)(w_-^n)}\left\{\partial_z(\lambda_3(w_-^n)) - \sum_{j=1,2}(\partial_{w_j}\lambda_3)(w_-^n)\partial_z w_{j,-}^n\right\} \end{aligned}$$

for $k = 1,2$, which is legitimate since $\partial_{w_3}\lambda_3(0) \neq 0$. Note that $\partial_z w_{j,-}^n$, $j = 1,2$, and $\partial_z(\lambda_3(w_-^n))$ can be estimated by (3.9) and Lemma 3.8. Now we set

$$(\lambda_3(w_-^n) - \lambda_3(w_-^{n-1}))\partial_z w_{3,-}^{n-1} = \sum_{i=1}^{7} J_i,$$

where

$$\begin{aligned} J_1 = \sum_{j,k=1}^{3}\bigg\{&\int_0^1\int_0^1(\partial^2_{w_j w_k}\lambda_3)(\theta_1(\theta w_-^n + (1-\theta)w_-^{n-1}) + (1-\theta_1)w_-^{n-1})\theta d\theta d\theta_1 \\ &\times (w_{k,-}^n - w_{k,-}^{n-1})(w_{j,-}^n - w_{j,-}^{n-1})\bigg\}\partial_z w_{3,-}^{n-1}, \end{aligned}$$

$$J_2 = \sum_{j=1}^{3}\{(\partial_{w_j}\lambda_3)(w_-^{n-1}) - (\partial_{w_j}\lambda_3)(w_-^n)\}\partial_z w_{3,-}^{n-1}(w_{j,-}^n - w_{j,-}^{n-1}),$$

$$J_3 = \sum_{j=1}^{3}(\partial_{w_j}\lambda_3)(w_-^n)(\partial_z w_{3,-}^{n-1} - \partial_z w_{3,-}^n)(w_{j,-}^n - w_{j,-}^{n-1}),$$

$$J_4 = \partial_z(\lambda_3(w_-^n))(w_{3,-}^n - w_{3,-}^{n-1}),$$
$$J_5 = -\sum_{k=1,2} (\partial_{w_k}\lambda_3)(w_-^n)\partial_z w_{k,-}^n (w_{3,-}^n - w_{3,-}^{n-1}),$$
$$J_6 = \sum_{k=1,2} \frac{(\partial_{w_k}\lambda_3)(w_-^n)}{(\partial_{w_3}\lambda_3)(w_-^n)}\partial_z(\lambda_3(w_-^n))(w_{k,-}^n - w_{k,-}^{n-1}),$$
$$J_7 = -\sum_{k,j=1,3} \frac{(\partial_{w_k}\lambda_3)(w_-^n)}{(\partial_{w_3}\lambda_3)(w_-^n)}\{(\partial_{w_j}\lambda_3)(w_-^n)\partial_z w_{j,-}^n\}(w_{k,-}^n - w_{k,-}^{n-1}).$$

Estimating each term gives

$$\begin{aligned}(3.49)\quad & |(\lambda_3(w_-^n) - \lambda_3(w_-^{n-1}))\partial_z w_{3,-}^{n-1}| \\ & \leq \Big(|\partial_z(\lambda_3(w_-^n))| + \frac{C_N\varepsilon}{\sqrt{t-T_\varepsilon}}\Big)|w_{3,-}^n - w_{3,-}^{n-1}| + C_N \sum_{i=1,2} |w_{i,\pm}^n - w_{i,\pm}^{n-1}|.\end{aligned}$$

We next estimate $(\sigma^n - \sigma^{n-1})\partial_z w_{3,-}^{n-1}$. Note that

$$(3.50)\qquad \sigma^n = \lambda_3(w_-(t,0-)) + \frac{1}{2}\sum_{k=1}^{3}(\partial_{w_k}\lambda_3)(w_-(t,0-))[w_k^n] + O([w^n]^2),$$

which is derived as in [15] or [21]. Then, similarly to the proof of (3.49), we can get

$$\begin{aligned}(3.51)\quad & |(\sigma^n - \sigma^{n-1})\partial_z w_{3,-}^{n-1}| \\ & \leq \Big(\frac{1}{2}|\partial_z(\lambda_3(w_-^n))| + \frac{C_N\varepsilon}{\sqrt{t-T_\varepsilon}}\Big)|w_{3,-}^n - w_{3,-}^{n-1}| + C_N \sum_{i=1,2} |w_{i,+}^n - w_{i,+}^{n-1}|.\end{aligned}$$

Using (3.49) and (3.51), we can proceed as in the proof of Lemma 3.8 (in particular, Part 4), to establish

$$\begin{aligned}\|w_{3,-}^{n+1} - w_{3,-}^n\|_{L^\infty(\tilde\Omega_-)} &\leq \Big(\ln\frac{3}{2} + C_N\varepsilon\sqrt{t-T_\varepsilon}\Big)\|w_{3,-}^n - w_{3,-}^{n-1}\|_{L^\infty(\tilde\Omega_-)} \\ &\quad + \Big(\frac{1}{2}\ln\frac{3}{2} + C_N\varepsilon\sqrt{t-T_\varepsilon}\Big)\|w_{3,\pm}^n - w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} \\ &\quad + C_N\sum_{i=1,2}\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|w_{3,+}^{n+1}-w_{3,+}^{n}\|_{L^\infty(\tilde\Omega_+)}&\le\Big(\ln\frac32+C_N\varepsilon\sqrt{t-T_\varepsilon}\Big)\|w_{3,+}^{n}-w_{3,+}^{n-1}\|_{L^\infty(\tilde\Omega_+)}\\&\quad+\Big(\frac12\ln\frac32+C_N\varepsilon\sqrt{t-T_\varepsilon}\Big)\|w_{3,\pm}^{n}-w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\\&\quad+C_N\sum_{i=1,2}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}.\end{aligned}$$

Adding these two estimates gives

(3.52)
$$\begin{aligned}\|w_{3,\pm}^{n+1}-w_{3,\pm}^{n}\|_{L^\infty(\tilde\Omega_\pm)}&\le\Big(2\ln\frac32+C_N\varepsilon\sqrt{t-T_\varepsilon}\Big)\|w_{3,\pm}^{n}-w_{3,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\\&\quad+C_N\sum_{i=1,2}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)},\end{aligned}$$

where the constant C_N of the last term of the right side can be made less than 1 provided ε is small enough. As in Part 3 of the proof of Lemma 3.8, we can also establish

$$\|w_{1,+}^{n+1}-w_{1,+}^{n}\|_{L^\infty(\tilde\Omega_+)}\le C_N\varepsilon(t-T_\varepsilon)\sum_{i=1}^{3}\|w_{i,\pm}^{n}-w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}. \tag{3.53}$$

Finally, consider the function $v(t,z)=w_{1,-}^{n+1}-w_{1,-}^{n}$, which satisfies

$$\begin{cases}\partial_t v+(\lambda_1(w_-^n)-\sigma^n)\partial_z v=(\lambda_1(w_-^{n-1})-\lambda_1(w_-^n)+\sigma^n-\sigma^{n-1})\partial_z w_{1,-}^n\\\qquad-\bar a(w_-^n)\big\{\partial_t(w_{2,-}^n-w_{2,-}^{n-1})+(\lambda_1(w_-^n)-\sigma^n)\partial_z(w_{2,-}^n-w_{2,-}^{n-1})\\\qquad+(\lambda_1(w_-^{n-1})-\lambda_1(w_-^n)+\sigma^n-\sigma^{n-1})\partial_z w_{2,-}^n\big\}\\\qquad-(\bar a(w_-^n)-\bar a(w_-^{n-1}))(\partial_t w_{2,-}^n+(\lambda_1(w_-^n)-\sigma^n)\partial_z w_{2,-}^n)\\\qquad+\dfrac{\bar b_1(w_-^n)}{z+\phi^n(t)}-\dfrac{\bar b_1(w_-^{n-1})}{z+\phi^{n-1}(t)},\\v(T_\varepsilon,z)=0,\quad v(t,z)|_{z=0}=w_{1,-}^{n+1}(t,0-)-w_{1,-}^n(t,0-).\end{cases}$$

If the backward characteristics $\xi=\xi(t,z,s)$ through (t,z) intersects the

z-axis before meeting the t-axis, we have

$$(3.54)\quad \begin{aligned} |v(t,z)| &\le |\bar{a}(w_-^n)(w_{2,-}^n - w_{2,-}^{n-1})| \\ &\quad + C_N\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \int_{T_\varepsilon}^t \Big(1 + \frac{1}{\sqrt{s-T_\varepsilon}}\Big) ds \\ &\le C_N\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)}. \end{aligned}$$

If $\xi = \xi(t,z,s)$ intersects the t-axis at $(s,0)$ for some $s \ge T_\varepsilon$, then
(3.55)

$$|v(t,z)| \le |w_{1,-}^{n+1}(s,0-) - w_{1,-}^n(s,0-)| + C_N\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^{n+1} - w_{i,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)}.$$

By (3.35), Lemma 3.7 and the above estimates, we get

$$\begin{aligned} |w_{1,-}^{n+1}(s,0-) - w_{1,-}^n(s,0-)| &\le |w_{1,+}^{n+1}(s,0+) - w_{1,+}^n(s,0+)| \\ &\quad + C_N\varepsilon\big(|w_{1,-}^{n+1}(s,0-) - w_{1,-}^n(s,0-)| + |w_{2,-}^{n+1}(s,0-) - w_{2,-}^n(s,0-)| \\ &\quad + |w_{3,\pm}^{n+1}(s,0\pm) - w_{3,\pm}^n(s,0\pm)|\big) \le C_N\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\Omega_\pm)}. \end{aligned}$$

Hence

$$|v(t,z)| \le C_N\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)}.$$

The term $|w_{2,-}^{n+1}(t,z) - w_{2,-}^n(t,z)|$ is estimated similarly (and even more simply), so the details are omitted.

Collecting terms now gives

$$\begin{aligned} &\|w_{3,\pm}^{n+1} - w_{3,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)} + \sum_{i=1,2} (C_N+1)\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \\ &\quad \le \Big(2\ln\frac{3}{2} + C_N\varepsilon\sqrt{t-T_\varepsilon} + C_N(C_N+1)\varepsilon\Big)\|w_{3,\pm}^n - w_{3,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \\ &\qquad + \frac{C_N + C_N(C_N+1)\varepsilon}{C_N+1} \sum_{i=1,2} (C_N+1)\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)}. \end{aligned}$$

Since $2\ln(3/2) < 1$ and $C_N/C_N+1 < 1$, replacing C_N+1 by C_N we conclude that Lemma 3.9 is valid if ε is small. ∎

Proof of Theorem 2.1. By Lemma 3.9 we see that there exist functions $\sigma(t) \in C[T_\varepsilon, T_\varepsilon + 1]$ and $w_{i,\pm}(t,z) \in C(\tilde{\Omega}_\pm)$ such that $\sigma^n(t) \to \sigma$ uniformly on $[T_\varepsilon, T_\varepsilon + 1]$ and $w^n_{i,\pm}(t,z) \to w_{i,\pm}(t,z)$ uniformly on $\tilde{\Omega}_\pm$, respectively, as $n \to \infty$. We can also prove that $\nabla_{t,z} w^n_{i,\pm}(t,z) \to \nabla_{t,z} w_{i,\pm}(t,z)$ uniformly on any closed subsets of $\tilde{\Omega}_\pm$, respectively. Moreover, by Lemma 3.6, (3.17) and (3.18), we see that $z \mapsto w^n_{i,\pm}(t,z)$ are equicontinuous in $z \in \tilde{\Omega}_\pm$, respectively, for each fixed $t \in (T_\varepsilon, T_\varepsilon + 1)$. Hence, the boundary values $w_{i,\pm}(t, 0\pm)$ exist. Moreover, due to the equivalence of (3.12) and (3.14), we conclude that these boundary values satisfy the Rankine-Hugoniot conditions on the shock curve

$$r = \phi(t) = T_\varepsilon + \int_{T_\varepsilon}^{t} \sigma(t)dt.$$

The entropy condition is also satisfied by Lemma 3.1 and the estimates given in Lemma 3.8. So the functions

$$w_i(t,z) = \begin{cases} w_{i,-}(t,z), & z < \phi(t), \\ w_{i,+}(t,z), & z > \phi(t), \end{cases}$$

define our desired weak entropy solution of (2.11). Estimates in Theorem 2.1 follow directly from Lemma 3.6, Lemma 3.8 and the convergence of approximate solutions. □

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