LOCAL SPLITTING FAMILIES OF
HYPERELLIPTIC PENCILS, II

TATSUYA ARAKAWA AND   TADASHI ASHIKAGA

Abstract. We propose certain obstructions for the existence of hyperelliptic splitting families of degenerations of curves. Moreover we determine the complete system of hyperelliptic atomic fibers of genus 3.

Introduction

This paper is a continuation of [1]. We study splitting families of degenerations for hyperelliptic curves so that the original complicated singular fiber decomposes into several simple singular fibers by those local deformations.

The main aims of the present paper are the following.

Firstly we show that certain invariants for the singular fiber of a hyperelliptic pencil are conserved in the splitting families.

Secondly, we give a complete system $\mathcal{A}$ of hyperelliptic 'atomic fibers', i.e., the atoms of degenerations, of genus three.

The paper consists of 6 sections.

In §1, we show that an analytic splitting naturally induces a C^∞-splitting in the usual sense of topology (cf. [7]). Namely, let $\{\varphi_u : S_u \to \overline{\Delta}\}_{u \in \Delta}$ be a splitting family of a degeneration $\varphi_0 : S_0 \to \overline{\Delta}$ of curves over the complex one-dimensional closed disk $\overline{\Delta}$. Then there exists a diffeomorphism $h : S_0 \to S_u$ which fixes the fibers over the boundary $\partial\overline{\Delta}$. This is analogous to the C^∞ local triviality of smooth analytic families of compact complex manifolds.

In §2, firstly we correct the definition of the type II singular fibers in hyperelliptic pencils, which we defined in [1] but was incomplete.

Secondly as an application of the existence of splitting families, we show in Proposition 2.6 that any fixed component of the relative canonical linear

system of a germ of a hyperelliptic degeneration of genus 3 disappears via a finite times of local deformations.

In §3, we define a new invariant for the singular fiber of a hyperelliptic pencil, which we shall call the *Namba number* or *N-number*. We define it as the number of components of the classical link which appears in a certain natural way from the branch curve of the relative hyperelliptic involution of the degeneration.

In §4, firstly in Proposition 4.1, we show that the Horikawa index (cf. [1] §4) and the Euler contribution for the singular fiber of a hyperelliptic pencil are conserved in the splitting families. Our proof is based on the conservation of the local signature due to Endo [4] and Terasoma [14].

Secondly in Proposition 4.3, we show the similar conservation property of the N-number.

In §5, firstly in Proposition 5.1, we show that the conservation of the Horikawa index and the Euler contribution is not enough for the existence of a splitting family with the preassigned invariants. In the proof of this proposition, N-number plays a crucial role.

In the remaining part of §5 and in §6, we show that a complete system $\mathcal{A}$ of hyperelliptic 'atomic fibers' of genus three consists of the germs of types 0_0, I, $II(i)$ and $II(iii)$ defined in 2.2 (see Figure 1). More precisely, any germ (φ, F) of a hyperelliptic degeneration of genus 3 splits into the members of $\mathcal{A}$ via several splitting families and equisingular deformations, and any member of $\mathcal{A}$ does not split furthermore. By [1], (φ, F) splits stepwisely into some of germs of types 0_0, I, $II(i)$, $II(ii)$ and $II(iii)$. We show in Proposition 5.5 that any germ of type $II(ii)$ splits into germs of types I and $II(iii)$ after an equisingular deformation. We construct this splitting family by taking a cyclic quotient of a certain splitting family of the stable reduction of the original degeneration. There is another way of the construction by using the double cover of a certain splitting family of a genus two fibration (§5.6).

§1. Analytic splitting and C^∞ splitting

In this section, we prove that an analytic splitting of a degeneration of curves naturally induces a C^∞ splitting in the sense of Matsumoto [7].

1.1 Let Δ_ϵ (resp. $\overline{\Delta}_\epsilon$) be the open (resp. closed) disk of radius ϵ with center at the origin on the complex plane $\mathbf{C}$. We denote by $\partial\overline{\Delta}_\epsilon$ the boundary of $\overline{\Delta}_\epsilon$. Let S be a nonsingular complex surface and let $\varphi : S \to \Delta_\epsilon$ be a degeneration of curves of genus g with a unique singular fiber (i.e. a fiber with singularities) $F = \varphi^{-1}(0)$. Let M be a three-dimensional complex manifold and let $\Phi : M \to \Delta_\epsilon \times \Delta_\delta$ be a proper flat surjective holomorphic map such that $M_u = \Phi^{-1}(\Delta_\epsilon \times \{u\})$ is a smooth surface for any $u \in \Delta_\delta$. We call Φ an *analytic splitting family* of φ if the following conditions are satisfied:

(i) The restriction $\Phi_0 : M_0 \to \Delta_\epsilon \times \{0\}$ of Φ over $0 \in \Delta_\delta$ coincides with φ,

(ii) For any $u \in \Delta_\delta \setminus \{0\}$, the restriction $\Phi_u : M_u \to \Delta_\epsilon \times \{u\}$ has l singular fibers where l is an integer greater than or equal to 1 and is independent on u.

Then we say that the fiber germ $[\varphi, F]$ splits into the fiber germs $[\Phi_u, F_{u,1}], \cdots, [\Phi_u, F_{u,l}]$ via the splitting family Φ, and sometimes write

$$[\varphi, F] \longrightarrow [\Phi_u, F_{u,1}] + \cdots + [\Phi_u, F_{u,l}]$$

symbolically. If $l \geq 2$, then we call Φ an analytic *proper* splitting family of φ. If $l = 1$, then we call Φ an *equisingular deformation* of φ.

PROPOSITION 1.2. *Let $\Phi : M \to \Delta_\epsilon \times \Delta_\delta$ be an analytic splitting family of a degeneration $\varphi : S \to \Delta_\epsilon$. Then there exist real numbers ϵ', δ' ($0 < \epsilon' < \epsilon, 0 < \delta' < \delta$) such that the restriction $\overline{\Phi} : \overline{M} \to \overline{\Delta}_{\epsilon'} \times \overline{\Delta}_{\delta'}$ of Φ to $\overline{M} = \Phi^{-1}(\overline{\Delta}_{\epsilon'} \times \overline{\Delta}_{\delta'})$ has the following properties:*

(a) *Let $\mathcal{D}$ be the discriminant locus of Φ in $\Delta_\epsilon \times \Delta_\delta$. Then $\mathcal{D}$ does not intersect $\partial\overline{\Delta}_{\epsilon'} \times \overline{\Delta}_{\delta'}$.*

(b) *Put $\overline{M}_u = \overline{\Phi}^{-1}(\overline{\Delta}_{\epsilon'} \times \{u\})$ and $\partial\overline{M}_u = \overline{\Phi}^{-1}(\partial\overline{\Delta}_{\epsilon'} \times \{u\})$ for any $u \in \overline{\Delta}_{\delta'}$. Let $\overline{\Phi}_u : \overline{M}_u \to \overline{\Delta}_{\epsilon'} \times \{u\}$ be the restriction of $\overline{\Phi}$ over $u \in \overline{\Delta}_{\delta'}$. Then there exists a diffeomorphism*

$$f_u : \overline{M}_0 \longrightarrow \overline{M}_u$$

with $f_u(\partial \overline{M}_0) = \partial \overline{M}_u$ so that the composition map $\mathbf{i}_u \circ \overline{\Phi}_u \circ f_u|_{\partial \overline{M}_0}$ coincides with the restriction map $\Phi_0|_{\partial \overline{M}_0}$, where $\mathbf{i}_u : \overline{\Delta}_{\epsilon'} \times \{u\} \to \overline{\Delta}_{\epsilon'} \times \{0\}$ is the natural map induced by the identity of the first factor.

Namely, an analytic splitting family Φ induces a C^∞ splitting of $\overline{M}_0$ to $\overline{M}_u$ in the sense of [7, Definition 3.4].

Proof. We fix complex coordinates t and u on Δ_ϵ and Δ_δ, respectively. Denote by $pr_1 : \Delta_\epsilon \times \Delta_\delta \to \Delta_\epsilon$ and $pr_2 : \Delta_\epsilon \times \Delta_\delta \to \Delta_\delta$ the natural projections. There exist real numbers ϵ' and δ' ($0 < \epsilon' < \epsilon, 0 < \delta' < \delta$), an open covering $M = (\cup_{i \in \mathbf{I}} U_i) \cup (\cup_{j \in \mathbf{J}} U_j)$ (for $\mathbf{I} \cap \mathbf{J} = \emptyset$) and a system of coordinates $(z_k^{(1)}, z_k^{(2)}, u_k)$ on U_k ($k \in \mathbf{I} \cup \mathbf{J}$) which satisfy the following:

(i) $\mathbf{I}$ is a finite set and $\Phi^{-1}(\partial \overline{\Delta}_{\epsilon'} \times \{0\})$ is contained in $\cup_{i \in \mathbf{I}} U_i$. Moreover the set $\Phi(\cup_{i \in \mathbf{I}} U_i) \cap \mathcal{D}$ is empty.

(ii) For sufficiently small real numbers ϵ'' and δ'' ($0 < \epsilon'' < \epsilon', 0 < \delta'' < \delta'$), we put the region $\mathcal{N}$ on $\Delta_\epsilon \times \Delta_\delta$ by

$$\mathcal{N} = \{t \in \Delta_\epsilon \mid \epsilon' - \epsilon'' \leq |t| \leq \epsilon' + \epsilon''\} \times \{u \in \Delta_\delta \mid |u| \leq \delta''\}.$$

Then the set $\Phi(\cup_{j \in \mathbf{J}} U_j) \cap \mathcal{N}$ is empty.

(iii) For $k \in \mathbf{I} \cup \mathbf{J}$, we have $u_k = u$. The map $pr_2 \circ \Phi$ on U_k coincides with the third projection $(z_k^{(1)}, z_k^{(2)}, u_k) \to u_k$.

(iv) For $i \in \mathbf{I}$, we have $z_i^{(2)} = t$. The map $pr_1 \circ \Phi$ on U_i coincides with the second projection $(z_i^{(1)}, z_i^{(2)}, u_i) \to z_i^{(2)}$.

The transition function on $U_k \cap U_l$ ($k, l \in \mathbf{I} \cup \mathbf{J}$) is given by $z_k^{(\alpha)} = f_{kl}^{(\alpha)}(z_l^{(1)}, z_l^{(2)}, u_l)$ ($\alpha = 1, 2$) and $u_k = u_l$, where $f_{kl}^{(\alpha)}(z_l^{(1)}, z_l^{(2)}, u_l)$ is a holomorphic function of variables $z_l^{(1)}, z_l^{(2)}$ and u_l. Moreover, if $k, l \in \mathbf{I}$, then we have $f_{kl}^{(2)}(z_l^{(1)}, z_l^{(2)}, u_l) = z_l^{(2)}$. Now let $\{\mu_k\}_{k \in \mathbf{I} \cup \mathbf{J}}$ be the partition of unity subordinate to the covering $\{U_k\}_{k \in \mathbf{I} \cup \mathbf{J}}$. We define a C^∞ vector field X on M by

$$\sum_{k \in \mathbf{I} \cup \mathbf{J}} \mu_k \frac{\partial}{\partial u_k} \ .$$

Then X is written on U_k ($k \in \mathbf{I} \cup \mathbf{J}$) as

(1.2.1) $$\sum_{\alpha=1}^{2} \left(\sum_l \mu_l \frac{\partial f_{kl}^{(\alpha)}}{\partial u_k}\right) \frac{\partial}{\partial z_k^{(\alpha)}} + \frac{\partial}{\partial u_k} \ .$$

Moreover, on $U_i \cap \Phi^{-1}(N)$ $(i \in \mathbf{I})$, X is written as

$$(1.2.2) \qquad (\sum_l \mu_l \frac{\partial f_{kl}^{(1)}}{\partial u_k}) \frac{\partial}{\partial z_k^{(1)}} + \frac{\partial}{\partial u_k} \quad .$$

By changing δ sufficiently small if necessary, it follows from the fundamental theorem of ordinary differential equation that there exists the integral curve with respect to X. Namely, there exists a diffeomorphism

$$f : M_0 \times \Delta_\delta \longrightarrow M.$$

By (1.2.1) and (1.2.2), f naturally induces the desired diffeomorphism.

<div align="right">Q.E.D.</div>

§2. Special classes of fiber germs and relative canonical system

2.1   Let $\varphi : S \to \Delta_\epsilon$ be a hyperelliptic degeneration of curves of genus g with a unique singular fiber $F = \varphi^{-1}(0)$. In [1] §1, we defined special classes type 0_0, class I and class II of singular fiber germs of hyperelliptic degenerations. However the definition of class II were incomplete there. Here we correct it in the following way.

Let B be a branch curve on $W = \mathbf{P}^1 \times \Delta_\epsilon$ corresponding to the double cover of W which is birationally equivalent to the relative canonical map of φ. B is determined up to elementary transformations of the ambient $\mathbf{P}^1$-bundle W. Let $\psi : W \to \Delta_\epsilon$ be the natural projection and Γ_0 the fiber of ψ over $0 \in \Delta_\epsilon$. If B contains Γ_0, then we set $B_{hor} := B - \Gamma_0$ and otherwise we set $B_{hor} = B$, which we call the horizontal part of B.

The branch curve B is said to be trivial if one of the following two conditions is satisfied:

(a) B_{hor} intersects Γ_0 at just one point Q_1 and B_{hor} is not tangential to Γ_0 at Q_1, i.e. for the blow-up at Q_1, the proper transform of Γ_0 and that of B_{hor} do not intersect.

(b) B_{hor} intersects Γ_0 at just two points Q_1 and Q_2 such that B_{hor} is smooth at Q_1 meeting transversally to Γ_0 and B_{hor} is not tangential to Γ_0 at Q_2.

A branch curve which is not trivial is called *nontrivial*. Note that any trivial branch curve is modified to nontrivial one via a finite times of elementary transformations and vice versa.

Now we say the $[\varphi, F]$ belongs to class II if at least one nontrivial branch curve B contains Γ_0 and moreover any nontrivial branch curve B with $B \supset \Gamma_0$ have the five properties (a) through (e) in §1.3.3 of [1].

Note that the definition in [1] lacks the non-triviality condition.

2.2   Let us recall the classification [1] of the fiber germs of hyperelliptic degenerations of genus three which belong to these special classes (See Figure 1):

(1) $[\varphi, F]$ is of type 0_0 if and only if we can choose a branch curve B which is smooth and meets Γ_0 transversally except at one point P where the order of contact is two. In this class, the singular fiber F is an irreducible stable curve with one node.

(2) $[\varphi, F]$ belongs to class I if and only if we can choose a branch curve B which does not contain Γ_0 and meets Γ_0 transversally except at one point P which is an ordinary singularity of B of multiplicity 4. In this class, the singular fiber F is a stable curve with two nodes consisting of two smooth elliptic components.

(3) Class II consists of three subclasses II(i), II(ii) and II(iii). In any cases we can choose a branch curve B such that B contains Γ_0 and B_{hor} has the following properties, respectively:

(II(i)) B_{hor} meets Γ_0 at two points P and Q, which are ordinary singularities of B of multiplicity 4 and 6, respectively. Then F is a stable curve with one node consisting of two smooth components of genera one and two respectively.

(II(ii)) B_{hor} meets Γ_0 at two points P and Q and both of them are as follows: B has three local irreducible components which are smooth and mutually tangent of order two. (Note that one of them is Γ_0.) Then F has an irreducible decomposition $2F' + 2F'' + 2F'''$ where F', F'' and F''' are nonsingular curves of genera one, zero and one respectively, and the intersection numbers are $(F')^2 = (F''')^2 = -1$, $(F'')^2 = -2$, $F'F'' = F''F''' = 1$ and $F'F''' = 0$.

(II(iii)) B_{hor} meets Γ_0 at one point P. B has five local irreducible components at P which are smooth and mutually tangent of order two. (Note that one of them is Γ_0.) Then F is written as $F = 2F'$ where F' is a nonsingular curve of genus two.

$$0_0 \qquad \mathrm{I} \qquad \mathrm{II(i)} \qquad \mathrm{II(ii)} \qquad \mathrm{II(iii)}$$
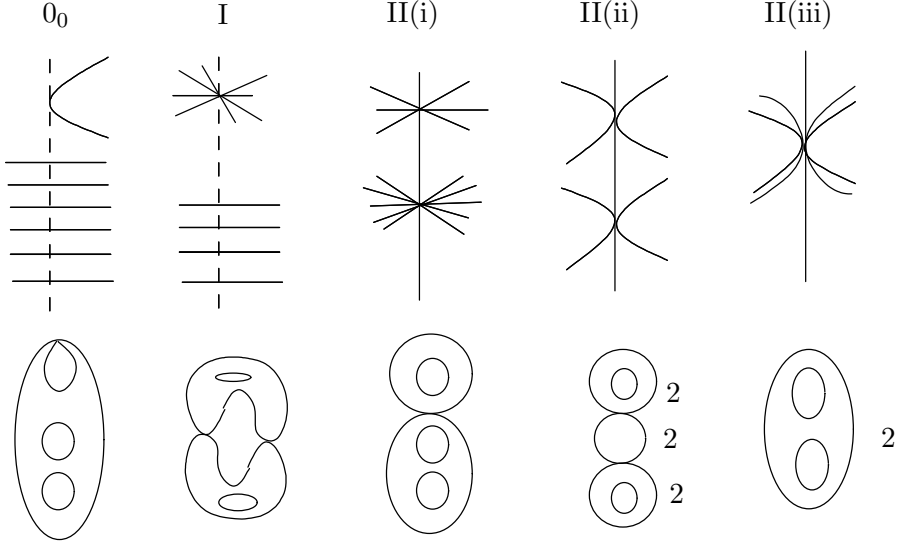
(Figure 1)

2.3   Horikawa [6] gives a method to determine relative (pluri)canonical systems of fiber germs of hyperelliptic degenerations explicitly. Here we will give a brief sketch of this method for class II (i) fiber germs of hyperelliptic degenerations of genus three.

Let B be a branch curve on W with the properties in 2.2 (II(i)). Then a canonical resolution of B is given by the minimal embedded resolution $\sigma = \sigma_2\sigma_1 : W_2 \to W$ of B where σ_i is a blow-up ($i = 1, 2$). The exceptional set of σ is written as $D_1 + D_2$ where D_1, D_2 are nonsingular rational curves on W_2. Let B_2 denote the proper transform of B on W_2 and S_2 the double cover of W_2 whose branch curve is B_2. Then a holomorphic 2-form on S_2 is the pull back of a section of the bundle $K_{W_2} + L_2$ on W_2 where L_2 is a square-root bundle of B_2. Let H denote a section of $\psi : W \to \Delta_\epsilon$. Since $B_2 = \sigma^* B - 4D_1 - 6D_2$, $K_{W_2} \cong \sigma^* K_W + D_1 + D_2$, $B \cong 8H$ and $K_W \cong -2H$, we have $K_{W_2} + L_2 \cong \sigma^*(2H) - D_1 - 2D_2$. Consequently we get the following identification:

$$(2.3.1) \qquad \Gamma(K_{S_2}) \cong \{\psi \in \Gamma(2H); \sigma^*(\psi) \geq -D_1 - 2D_2\}$$

Let $(x : y, t)$ be coordinates of $W = \mathbf{P}^1 \times \Delta_\epsilon$ such that $\sigma^*(x) \geq D_1$ and $\sigma^*(y) \geq D_2$. Since $\sigma^*(t) \geq D_1 + D_2$, we conclude that the right hand side of (2.3.1) is generated by t^2x^2, txy and ty^2 over $\mathbf{C}\{t\}$.

Moreover, since the identification (2.3.1) is given by $(\psi) \mapsto \sigma^*(\psi) - D_1 - 2D_2$, we also have $\mathrm{Bs}|K_{S_2}| = 2E$ where E is the reduced part of the pull back of the proper transform of $\Gamma_0 \subset W$.

Since S_2 is given by a blow-up of S whose exceptional curve is E, we finally conclude that the center of this blow-up, which is nothing but the node of the singular fiber, is the unique base point of the (relative) canonical system.

Applying the same arguments for fiber germs in other classes, we have:

LEMMA 2.4.   *With an appropriate choice of the coordinates $(x : y, t)$ of W, the relative canonical system of a fiber germs $[\varphi, F]$ of a hyperelliptic degeneration of genus three is generated by $\varphi_0, \varphi_1, \varphi_2$ over $\mathbf{C}\{t\}$ where*

$$(\varphi_0, \varphi_1, \varphi_2)$$
$$= \begin{cases} (\quad x^2, \quad xy, \quad y^2 \ ) & \textit{if } [\varphi, F] \textit{ belongs to type } 0_0 \\ (\quad x^2, \quad xy, \quad ty^2 \ ) & \textit{if } [\varphi, F] \textit{ belongs to class I} \\ (\ t^2x^2, \quad txy, \quad ty^2 \ ) & \textit{if } [\varphi, F] \textit{ belongs to class II(i) or II(ii)} \\ (\quad tx^2, \quad xy, \quad ty^2 \ ) & \textit{if } [\varphi, F] \textit{ belongs to class II(iii)} \end{cases}$$

COROLLARY 2.5.   *The relative canonical system of a singular fiber germ of genus three is fixed point free if the germ belongs to type 0_0, class I or class II. If the germ belongs to type 0_0 or class I, the relative canonical system has no base point either.*

Note that the first half of Corollary 2.5 is well known by the general theory (cf. [10]).

By [1], any germ of a hyperelliptic degeneration of genus 3 splits into the set of germs in 0_0, I, $II(i)$, $II(ii)$ and $II(iii)$ via a finite times of hyperelliptic splitting families. Hence we have the following:

PROPOSITION 2.6.   *Any fixed component of the relative canonical system of a germ of a hyperelliptic degeneration of genus three disappears via a finite times of local deformations.*

Remark 2.7.   If we choose alternative branch curves for a fiber germ, we may obtain other generators of the canonical system. In fact, if we apply the above method to the proper transform B' of the branch curve B in §2.3 by the elementary transformation $W \to W$ whose center is the 6-ple point of B, we get generators (x^2, xy, ty^2) instead of (t^2x^2, txy, ty^2) in Lemma 2.4 (with an appropriate choice of coordinate systems) .

§3. A monodromy invariant

In this section, we define a certain monodromy invariant of a fiber germ of a hyperelliptic degeneration and calculate it for the fiber germs of genus three which belong to the special classes defined in §2.2.

3.1   We use the same notation as in §2.1. Let $\gamma \subset \Delta_\epsilon$ be a simple closed path winding around the origin $0 \in \Delta_\epsilon$ once. Then the restriction of ψ on Supp B induces an unramified covering of γ. Moreover $\psi^{-1}(\gamma) \cap$ Supp B is contained in $\Delta_N \times \gamma \subset W$, where Δ_N is a disk in $\mathbf{P}^1$ of sufficiently large radius. Since $\Delta_N \times \gamma$ is canonically embedded in a three-dimensional sphere, we may consider $\psi^{-1}(\gamma) \cap$ Supp B as a classical link.

Let $\mathcal{N}(B)$ denote the number of components of the link $\psi^{-1}(\gamma) \cap$ Supp B. Note that $\mathcal{N}(B)$ does not depend on the choice of the closed path γ. Furthermore:

PROPOSITION-DEFINITION 3.2. $\mathcal{N}(B)$ *does not depend on the choice of the branch curve B. We denote it by $\mathcal{N}([\varphi, F])$, and call it the N-number* (*Namba number*) *of the germ $[\varphi, F]$.*

First proof of Proposition 3.2.   Let B and B' be two branch curves on W corresponding to φ. If we restrict φ over a neighborhood of γ, then the relative canonical map of φ becomes a morphism. Let R be the ramification divisor of this morphism. Then both Supp B and Supp B' are isomorphic to Supp R over there. Therefore $\psi^{-1}(\gamma) \cap$ Supp B is homeomorphic to $\psi^{-1}(\gamma) \cap$ Supp B'. Hence $\mathcal{N}(B) = \mathcal{N}(B')$. (Remark that $\psi^{-1}(\gamma) \cap$ Supp B and $\psi^{-1}(\gamma) \cap$ Supp B' are not necessarily equivalent to each other as links.)

Q.E.D

3.3   In order to give an alternative proof of Proposition 3.2, let us recall the notion of the braid monodromy of branched coverings (cf. [12]). Let $U \subset \Delta_\epsilon$ be an open neighborhood of γ such that the branch curve B restricted over U is an unbranched covering of U of degree $n = 2g + 2$. Hence we have a natural homomorphism

$$\Psi_B : \pi_1(U, p) \to \mathbf{B}_n$$

where $p \in \gamma$ is the base point and $\mathbf{B}_n$ is the braid group of n-strings. We call Ψ_B the *braid monodromy* of the branch curve B. Note that $\Psi_B(\gamma)$ gives a braid representation of the link $\psi^{-1}(\gamma) \cap$ Supp B.

Now let $\tau : W \to W$ be an elementary transformation and let B_1 be the transformation of B by τ as the branch curve (cf. [5], [6]). Then we have:

LEMMA 3.4. *The braid $\Psi_{B_1}(\gamma) \in \mathbf{B}_n$ is given as a transform of $\Psi_B(\gamma)$ $\in \mathbf{B}_n$ by a full Dehn twist, and hence $\mathcal{N}(B) = \mathcal{N}(B_1)$ $(=: \mathcal{N}([\varphi, F]))$.*

Proof. Let x and t be a local coordinate of the fiber and of Δ_ϵ, respectively. Then the elementary transformation $\tau : W \to W$ is given by $(x, t) \mapsto (tx, t)$. On the other hand, suppose that γ is given by $\theta \mapsto t = \exp(2\pi\sqrt{-1}\theta)$ where $\theta \in [0, 1]$ is a parameter. Then τ is given by $x \to \exp(2\pi\sqrt{-1}\theta) \cdot x$ over a point $t(\theta)$ of γ. This implies the assertion.

Proposition 3.2 follows from Lemma 3.4, since every two branch curves B and B' on W corresponding to the same degeneration φ are transformed to each other via a finite number of elementary transformations.

The following characterizations are useful to calculate N-numbers of given hyperelliptic degenerations:

LEMMA 3.5. (i) *$\mathcal{N}([\varphi, F])$ coincides with the number of local irreducible components of B which are not vertical.*
(ii) *$\mathcal{N}([\varphi, F])$ coincides with the number of cyclic permutations in the minimal decomposition of the image of $\gamma \in \pi_1(U, p)$ in the symmetric group S_n.*

Proof. The assertion (i) is clear by the definition, while (ii) follows from (i) since each of the local irreducible components of B which are not vertical induces one of the cyclic permutations appearing in the minimal decomposition of the image of $\gamma \in \pi_1(U, p)$.

The following is a direct consequence of Lemma 3.5 (i):

LEMMA 3.6. *Let $[\varphi, F]$ be a fiber germ of hyperelliptic degenerations of genus three which belongs to either Type 0_0, Class I or Class II. Then*

$$\mathcal{N}([\varphi, F]) = \begin{cases} 7 & if \quad [\varphi, F] \quad belongs\ to \quad 0_0 \\ 8 & if \quad [\varphi, F] \quad belongs\ to \quad \text{I } or\ \text{II(i)} \\ 4 & if \quad [\varphi, F] \quad belongs\ to \quad \text{II(ii) } or\ \text{II(iii)} \end{cases}$$

§4. Conservations of invariants

Let $\varphi : S \to \Delta_\epsilon$ be a hyperelliptic degeneration of curves of genus g with a unique singular fiber $F = \varphi^{-1}(0)$. Let $\mathcal{E}(F)$, $\mathcal{H}([\varphi, F])$ and $\mathcal{N}([\varphi, F])$ be the Euler contribution, the Horikawa index (cf. [1] §4) and the N-number respectively. Let $\Phi : M \to \Delta_\epsilon \times \Delta_\delta$ be a *hyperelliptic* splitting family of φ, i.e. Φ is an analytic splitting family of φ such that the general fiber of Φ is a hyperelliptic curve. In this section, we discuss the conservation properties of the above three invariants under Φ.

PROPOSITION 4.1. *Assume that the germ $[\varphi, F]$ splits into the germs $[\Phi_u, F_{u,1}], \cdots, [\Phi_u, F_{u,l}]$ via a hyperelliptic splitting family Φ. Then :*

$$\mathcal{E}(F) = \sum_{i=1}^{l} \mathcal{E}(F_{u,i}), \quad \mathcal{H}([\varphi, F]) = \sum_{i=1}^{l} \mathcal{H}([\Phi_u, F_{u,i}])$$

for any $u \in \Delta_\delta$.

Proof. By the argument of Terasoma [14], there exist a compact complex surface V and a hyperelliptic holomorphic family $\mu : V \to \mathbf{P}^1$ of genus g which satisfy the followings: Let $\{Q_0, Q_1, \ldots, Q_n\} \subset \mathbf{P}^1$ be the set of all critical values of μ. Let Δ be a small disk neighborhood around Q_0 and let $\tilde{\mu} := \mu|_{\tilde{V}} : \tilde{V} = \mu^{-1}(\Delta) \to \Delta$ be the restriction map. Then:

(i) There exist orientation preserving diffeomorphisms $h : S \to \tilde{V}$ and $\alpha : \Delta_\epsilon \to \Delta$ such that $\tilde{\mu} \circ h = \alpha \circ \varphi$.

(ii) We have
$$\sigma^{top}([\varphi, F]) = \sigma^{top}([\mu, \mu^{-1}(Q_0)])$$
where σ^{top} is the topological local signature of the germ defined by Endo [4].

(iii) For any $1 \leq i \leq n$, $\mu^{-1}(Q_i)$ is a Lefschetz fiber.

Now we fix $u \in \Delta_\delta \setminus \{0\}$. After shrinking Δ_ϵ if necessary, $\Phi_u : M_u \to \Delta_\epsilon \times \{u\}$ is naturally diffeomorphic to $\varphi : S \to \Delta_\epsilon$ by Proposition 1.2. Therefore from the condition (i), we can patch the two pieces $\mu|_{V \setminus \tilde{V}} : V \setminus \tilde{V} \to \mathbf{P}^1 \setminus \Delta$ and $\Phi_u : M_u \to \Delta_\epsilon \times \{u\}$ along their boundaries, and obtain a locally analytic fibration (in the sense of [7] and [4]) $\mu' : V' \to \mathbf{P}^1$ such that:

(a) μ' is a hyperelliptic (differentiable) family which has just $l + n$ singular fibers. Each of them is isomorphic to each of $[\Phi_u, F_{u,1}], \ldots, [\Phi_u, F_{u,l}]$, $[\mu, \mu^{-1}(Q_1)], \ldots, [\mu, \mu^{-1}(Q_n)]$ as holomorphic fiber germs.
(b) V' is naturally diffeomorphic to V.

Since the topological Euler number $\chi^{top}(V)$ of V coincides with $\chi^{top}(V')$, the first assertion follows from the equalities

$$\chi^{top}(V) = \mathcal{E}(F) + n - 2(2g - 2), \quad \chi^{top}(V') = \sum_{i=1}^{l} \mathcal{E}(F_{u,i}) + n - 2(2g - 2).$$

Now Endo's theorem [4] says that

$$\text{Sign}(V) \quad = \quad \sigma^{top}([\varphi, F]) + \sum_{i=1}^{n} \sigma^{top}([\mu, \mu^{-1}(Q_i)]),$$

$$\text{Sign}(V') \quad = \quad \sum_{i=1}^{l} \sigma^{top}([\Phi_u, F_{u,i}]) + \sum_{i=1}^{n} \sigma^{top}([\mu, \mu^{-1}(Q_i)])$$

where $\text{Sign}(V)$ is the signature of the intersection form on $\text{H}^2(V, \mathbf{R})$ and so on. Therefore we have

$$\sigma^{top}([\varphi, F]) = \sum_{i=1}^{l} \sigma^{top}([\Phi_u, F_{u,i}]).$$

On the other hand, let $\sigma := 1/(2g + 1) \cdot (g\mathcal{H} - (g + 1)\mathcal{E})$ be the algebro-geometric local signature defined in [1, Definition 4.6]. Then σ coincides with σ^{top} by Terasoma's theorem [14]. Hence we have

$$\sigma([\varphi, F]) = \sum_{i=1}^{l} \sigma([\Phi_u, F_{u,i}]).$$

Therefore from the first assertion and the definition of σ, the second assertion also follows.

$$\text{Q.E.D.}$$

4.2.   Let $\gamma \subset \Delta_\epsilon$ be a closed path around the origin such that $\gamma \times \Delta_\delta$ does not intersect the discriminant locus on $\Delta_\epsilon \times \Delta_\delta$ with respect to Φ. Then, there does not exist any singular fiber of Φ_u on the outside-region of γ in $\Delta_\epsilon \times \{u\}$ for each $u \in \Delta_\delta$. Let B_u be a branch curve on $W_u = \mathbf{P}^1 \times \Delta_\epsilon$ with

respect to the hyperelliptic degeneration Φ_u with several singular fibers, and let $\Psi_u : W_u \to \Delta_\epsilon$ be the natural projection. By the same way as in §2, the number $\mathcal{N}_u$ of components of the link $\Psi_u^{-1}(\gamma) \cap \operatorname{Supp}B_u$ is well-defined.

PROPOSITION 4.3. *Let $\Phi : M \to \Delta_\epsilon \times \Delta_\delta$ be a hyperelliptic splitting family of φ. Then we have $\mathcal{N}([\varphi, F]) = \mathcal{N}_u$ for any $u \in \Delta_\delta$.*

Proof. Let $\Phi_{K_{M/\Delta_\epsilon \times \Delta_\delta}} : M \to \mathcal{W}$ be the relative canonical rational map of Φ. If we restrict Φ over a neighborhood of $\gamma \times \Delta_\delta$, then the ramification divisor $\mathcal{R}$ on M with respect to $\Phi_{K_{M/\Delta_\epsilon \times \Delta_\delta}}$ is well-defined. Moreover $\Psi_u^{-1}(\gamma) \cap \operatorname{Supp}B_u$ is homeomorphic to $\Phi_u^{-1}(\gamma) \cap \operatorname{Supp}\mathcal{R}$. Since the restriction of $\operatorname{Supp}\mathcal{R}$ over γ is unbranched covering of γ, $\operatorname{Supp}\mathcal{R}$ is homeomorphic to the direct product $(\Phi_0^{-1}(\gamma) \cap \operatorname{Supp}\mathcal{R}) \times (\gamma \times \Delta_\delta)$ over $\gamma \times \Delta_\delta$. Therefore, for each $u \in \Delta_\delta$, $\Phi_u^{-1}(\gamma) \cap \operatorname{Supp}\mathcal{R}$ is homeomorphic to $\Phi_0^{-1}(\gamma) \cap \operatorname{Supp}B$, which we identify with $\varphi^{-1}(\gamma) \cap \operatorname{Supp}B$. Hence the assertion follows.

Q.E.D

4.4 Let $F_{u,j} = \Phi_u^{-1}(P_{u,j})$ $(j = 1, 2, \ldots, l)$ be all the singular fibers of Φ_u for $u \neq 0$. Then the braid monodromy $\Phi : \pi_1(\Delta_\epsilon - \{P_{u,1}, \ldots P_{u,l}\}) \to \mathbf{B}_n$ is defined. In this situation, with an appropriate choice of branch curve, the image of the simple closed path γ given in 4.2 by the braid monodromy of $[\varphi, F]$ is a product of the braid monodromies of $[\varphi_1, F_{u.1}]$, $\ldots$, $[\varphi_l, F_{u.l}]$ in a suitable order. Therefore we have:

LEMMA 4.5.

$$2g + 2 - \mathcal{N}_u \leq \sum_{j=1}^{l} \left(2g + 2 - \mathcal{N}([\varphi_j, F_{u,j}]) \right).$$

§5. Existence and non-existence of certain splittings

Let $\mathcal{J}$ be the set of fiber germs of hyperelliptic degenerations of genus 3 of types 0_0, I, $II(i)$, $II(ii)$ and $II(iii)$ defined in 2.2. By Example 4.5 of [1] and easy calculations, the Horikawa indices and the Euler contributions of these types of germs are as follows:

	0_0	I	II (i)	II (ii)	II (iii)
H-index	0	2/3	5/3	10/3	8/3
E-cont.	1	2	1	4	2

Let Φ be a hyperelliptic proper splitting family $[\varphi, F] \to [\Phi_u, F_{u,1}] + \cdots + [\Phi_u, F_{u,l}]$ of a degeneration φ such that all the germs $[\varphi, F], [\Phi_u, F_{u,1}], \cdots, [\Phi_u, F_{u,l}]$ belong to $\mathcal{J}$. Proposition 4.1 implies that there are only two possibilities of this splitting as follows:

  (a) $II(ii) \to 0_0 + 0_0 + II(i) + II(i)$,
  (b) $II(ii) \to I + II(iii)$.

We prove the non-existence of the family (a) and the existence of the family (b).

PROPOSITION 5.1.   *There exists no hyperelliptic splitting family $[\varphi, F]$ $\to [\Phi_u, F_{u,1}] + [\Phi_u, F_{u,2}] + [\Phi_u, F_{u,3}] + [\Phi_u, F_{u,4}]$ such that the germs $[\varphi, F]$, $[\Phi_u, F_{u,1}], [\Phi_u, F_{u,2}], [\Phi_u, F_{u,3}]$ and $[\Phi_u, F_{u,4}]$ are of types $II(ii), 0_0, 0_0, II(i)$ and $II(i)$, respectively.*

Proof.   Assume that such a splitting family Φ exists. Since $\mathcal{N}([\Phi_u, F_{u,1}])$ $= \mathcal{N}([\Phi_u, F_{u,2}]) = 7$ and $\mathcal{N}([\Phi_u, F_{u,3}]) = \mathcal{N}([\Phi_u, F_{u,4}]) = 8$ by Lemma 3.6, it follows that $\mathcal{N}_u \geq 6$ by Lemma 4.5. Since $\mathcal{N}([\varphi, F]) = 4$, this contradicts Proposition 4.3.

$$\text{Q.E.D.}$$

COROLLARY 5.2.   *The conservation properties of the Horikawa indices and the Euler contributions described in Proposition 4.1 are not sufficient for the existence of hyperelliptic splitting families.*

Remark 5.3.   By a slight modification of the argument in Proposition 5.1, we can show that a fiber germ of type $II(ii)$ does not split into four germs as in (a) not only via one splitting family but also via stepwise several splitting families.

5.4   For the discussion of the family (b), we prepare some arguments of the monodromy. In general, let $\varphi\colon S \to \Delta_\epsilon$ be a degeneration of curves of genus $g \geq 2$ with a unique singular fiber $F = f^{-1}(0)$. If we fix $t_0 \in \Delta^*$, then the monodromy action of $\pi_1(\Delta^*, t_0) \simeq \mathbf{Z}$ defines modulo isotopy an orientation-preserving homeomorphism $f_\varphi : S_{t_0} \to S_{t_0}$ of the smooth fiber S_{t_0}. Since the change of the base point t_0 corresponds to the conjugation in the mapping class group $\mathcal{M}_g$ of genus g, φ induces a unique element $[f_\varphi] \in \widehat{\mathcal{M}}_g$ where $\widehat{\mathcal{M}}_g$ is the set of conjugacy classes of $\mathcal{M}_g$. We call $[f_\varphi]$ the *topological monodromy* of φ. Matsumoto-Montesinos' theorem [8] says that

the element of $\widehat{\mathcal{M}}_g$ is realizable as a topological monodromy of a certain degeneration if and only if it is a conjugacy class of a pseudo-periodic map of negative twist. Moreover this class is completely determined by the invariants which are called valencies, screw numbers and the action of the extended partition graph ([13], [9]).

Now we consider fiber germs $[\varphi, F]$ of types $I, II(ii)$ and $II(iii)$. We determine the topological monodromy of these fiber germs, and at the same time describe their semi-stable reductions explicitly.

If $[\varphi, F]$ is of type I, then F itself is a stable curve and the topological monodromy is a full Dehn twist along the vanishing cycle which is a disjoint union of two simple closed curves.

Suppose $[\varphi, F]$ is of type $II(ii)$. The branch curve B on $W = \mathbf{P}^1 \times \Delta_\epsilon$ corresponds to φ is written by $\Gamma + \sum_{i=1}^4 B_i$ where Γ is a fiber of $\pi : W \to \Delta_\epsilon$, Γ and B_i $(i = 1, 2)$ are mutually two-tangential at a point P and Γ and B_i $(i = 3, 4)$ are mutually two-tangential at a point Q $(Q \neq P)$ (see Figure 2).
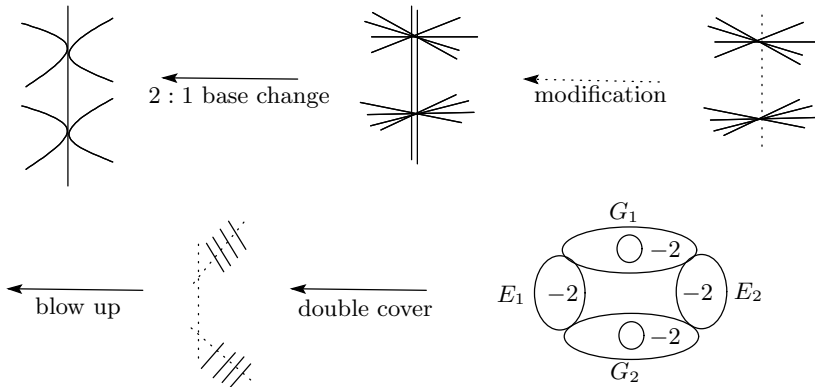
(Figure 2)

Let $\mu : \Delta_{\sqrt{\epsilon}} \to \Delta_\epsilon$ be the double cover with the branch point at the origin. Put $\widetilde{\mu} = id_{\mathbf{P}^1} \times \mu : \widetilde{W} = \mathbf{P}^1 \times \Delta_{\sqrt{\epsilon}} \to W$ where $id_{\mathbf{P}^1}$ is the identity map of $\mathbf{P}^1$. The pull back $\widetilde{\mu}^* B$ is written by $2\Gamma' + \sum_{i=1}^4 \sum_{j=1}^2 B'_{ij}$ where Γ' is a fiber of $\widetilde{\pi} : \widetilde{W} \to \Delta_{\sqrt{\epsilon}}$, Γ' and B'_{ij} $(1 \leq i \leq 2, 1 \leq j \leq 2)$ intersect transversally at a point P' and Γ' and B'_{ij} $(3 \leq i \leq 4, 1 \leq j \leq 2)$ intersect transversally at a point Q' $(Q' \neq P')$. Therefore the normalization of the

fiber product of S and $\Delta_{\sqrt{\epsilon}}$ over Δ_ϵ coincides with the double cover $\widetilde{S}$ of $\widetilde{W}$ branched along $\sum_{i=1}^{4} \sum_{j=1}^{2} B'_{ij}$. Let S^* be the nonsingular model of $\widetilde{S}$ obtained by the canonical resolution ([5]). Then the natural map

$$\widetilde{\varphi} : S^* \to \Delta_{\sqrt{\epsilon}}$$

has the singular fiber $\widetilde{F} = G_1 + G_2 + E_1 + E_2$ where G_i $(i = 1, 2)$ are nonsingular elliptic curves, E_i $(i = 1, 2)$ are nonsingular rational curves, $G_1^2 = G_2^2 = E_1^2 = E_2^2 = -2$, $G_i E_j = 1$ $(1 \leq i, j \leq 2)$ and $G_1 G_2 = E_1 E_2 = 0$ (see Figure 2). $\widetilde{\varphi}$ is the semi-stable reduction of φ.

On the other hand, there exists an automorphism $\rho : S^* \to S^*$ of order 2 compatible with the canonical involution of $\Delta_{\sqrt{\epsilon}}$ so that ρ acts on $\widetilde{F}$ as follows: ρ acts on G_i $(i = 1, 2)$ as an automorphism of order 2 without fixed points, $\rho(E_1) = E_2$ and $\rho(E_2) = E_1$. The quotient space of the fiber space $\widetilde{\varphi}$ by ρ coincides with the original fiber space φ.

Therefore the topological monodromy of the degeneration φ is isotopic to the pseudo-periodic map $f : \Sigma_3 \to \Sigma_3$ of negative twist of a real two dimensional orientable closed manifold Σ_3 of genus 3 as follows: The admissible system $\mathcal{C}$ of cut curves consists of two disjoint simple closed (real) curves $C_1 + C_2$ so that $\Sigma_3 - \mathcal{C}$ consists of two connected components $\mathcal{B}_1, \mathcal{B}_2$ of genus 1 (see Figure 3). f acts on $\mathcal{B}_i$ $(i = 1, 2)$ as a periodic map of order 2 without fixed points, $f(C_1) = C_2$, $f(C_2) = C_1$ and f^2 acts on Σ_3 as the two-times Dehn twist along each of C_i $(i = 1, 2)$.
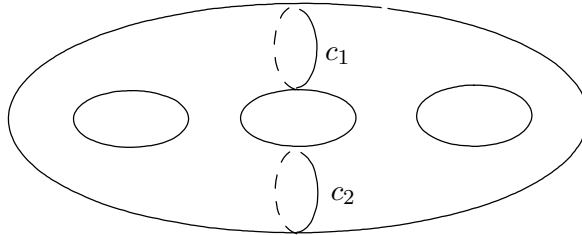
(Figure 3)

Suppose $[\varphi, F]$ is of type $II(iii)$. We get the stable family after a two-to-one base change. This is nothing but a smooth family of genus 3. The topological monodromy is isotopic to a periodic map of order 2 without fixed points.

PROPOSITION 5.5. *There exists a hyperelliptic splitting family* $[\varphi, F] \to$ $[\Phi_u, F_{u,1}] + [\Phi_u, F_{u,2}]$ *such that the germs* $[\varphi, F]$, $[\Phi_u, F_{u,1}]$ *and* $[\Phi_u, F_{u,2}]$ *are of types* $II(ii), I$ *and* $II(iii)$, *respectively.*

Proof. We fix the coordinates t and u on Δ_ϵ and Δ_δ, respectively. Put $\boldsymbol{\Delta} := \Delta_\epsilon \times \Delta_\delta$. We denote by $(x : y : z)$ a system of homogeneous coordinates of $\mathbf{P}^2$. Let $\mathcal{W}$ be the divisor on $\Delta_\epsilon \times \Delta_\delta \times \mathbf{P}^2$ defined by the equation

(5.5.1) $(t^2 - u^2)(x^2 + y^2 + z^2) + xy = 0.$

Let $\pi : \mathcal{W} \to \boldsymbol{\Delta}$ be the natural map and put $pr_1 : \boldsymbol{\Delta} \to \Delta_\epsilon$ and $pr_2 : \boldsymbol{\Delta} \to \Delta_\delta$ the first and the second projection, respectively. We denote by $\mathcal{D}$ the discriminant locus of π. Since δ is sufficiently small, $\mathcal{D}$ is written by $D_1 + D_2$ where $D_1 = \{t + u = 0\}$ and $D_2 = \{t - u = 0\}$. The fiber space π has a conic bundle structure so that $\pi^{-1}(Q)$ $(Q \in \mathcal{D})$ consists of two lines, which we denote by $L_{Q,1} + L_{Q,2}$.

We fix mutually distinct non-zero complex numbers α and β. Let H_i $(1 \le i \le 4)$ be the divisors on $\boldsymbol{\Delta} \times \mathbf{P}^2$ defined by the equations $x + y + \alpha z = 0$, $x + y + \beta z = 0$, $x + y - \alpha z = 0$ and $x + y - \beta z = 0$, respectively. We set $B_i = H_i \cap \mathcal{W}$ $(1 \le i \le 4)$ and put $B = \sum_{i=1}^4 B_i$. Then $\pi^{-1}(Q) \cap B$ consists of mutually distinct eight points for any $Q \in \boldsymbol{\Delta}$. Moreover if $Q \in \mathcal{D}$, then four points of $\pi^{-1}(Q) \cap B$ are on $L_{Q,1}$ and the other four points are on $L_{Q,2}$.

Let $\rho' : \mathcal{V} \to \mathcal{W}$ be the double cover branched along B. $\mathcal{V}$ is smooth and is realized on $\boldsymbol{\Delta} \times \mathbf{P}^2 \times \mathbf{C}$ by the equations (5.5.1) and

$$\xi^2 + (x + y + \alpha z)(x + y + \beta z)(x + y - \alpha z)(x + y - \beta z) = 0,$$

where ξ is the coordinate of $\mathbf{C}$. We set

$$\rho = \pi \circ \rho' : \mathcal{V} \to \boldsymbol{\Delta}.$$

Then the general fiber $\rho^{-1}(Q)$ $(Q \in \boldsymbol{\Delta} \setminus \mathcal{D})$ is a smooth hyperelliptic curve of genus 3, while the singular fiber $\rho^{-1}(Q)$ $(Q \in \mathcal{D})$ consists of two nonsingular elliptic curves $G_1 + G_2$ so that G_1 and G_2 meet transversally at two points P_1 and P_2.

Now we define the action $\overline{\tau}$ on $\boldsymbol{\Delta} \times \mathbf{P}^2 \times \mathbf{C}$ by

$$t \longmapsto -t, \ u \longmapsto u; \ x \longmapsto -x, \ y \longmapsto -y, \ z \longmapsto z; \ \xi \longmapsto -\xi.$$

Then $\overline{\tau}$ preserves the divisor $\mathcal{V}$ and the restriction $\tau := \overline{\tau}|_\mathcal{V}$ induces an automorphism of $\mathcal{V}$ of order 2. The action τ is free from fixed points and has the following properties:

(i) We denote by $Q_{u,j}$ ($u \in \Delta_\delta$, $1 \leq j \leq 2$) the points on $\mathbf{\Delta}$ defined by $pr_2^{-1}(u) \cap D_j$. Then τ maps the fiber $\pi^{-1}(Q_{u,1})$ isomorphically onto $\pi^{-1}(Q_{u,2})$.

(ii) If the point Q on $\mathbf{\Delta}$ is on the locus $pr_1^{-1}(0)$, then the restriction of τ to $\pi^{-1}(Q)$ induces an automorphism.

(iii) The restriction map of τ on the fiber $\pi^{-1}(\mathbf{0})$ over the origin $\mathbf{0} \in \mathbf{\Delta}$ induces the involution on each components G_1 and G_2, so that the map interchanges P_1 and P_2.

Let M' be the quotient smooth threefold of $\mathcal{V}$ by the group generated by τ. We denote by Δ_{ϵ^2} the quotient of Δ_ϵ by the involution $t \longmapsto -t$. Then the natural map

$$\Phi' : M' \longrightarrow \Delta_{\epsilon^2} \times \Delta_\delta$$

induces a fiber space of hyperelliptic curves of genus 3. We denote by $\Phi'_u : M'_u \to \Delta_{\epsilon^2} \times \{u\}$ the restriction of Φ' over $u \in \Delta_\delta$. Then:

(a) For $u \neq 0$, M'_u is smooth and the degeneration Φ'_u has just two singular fibers so that the topological structures of these germs coincide with the topological structures of germs of type I and $II(iii)$, respectively.

(b) The fiber of Φ'_0 is smooth except over 0. The fiber $(\Phi'_0)^{-1}(0)$ is the union of two multiple of two nonsingular elliptic curves $2\widetilde{G_1} + 2\widetilde{G_2}$ so that $\widetilde{G_1}$ and $\widetilde{G_2}$ intersect at a point $\overline{P}$. Moreover M_0 has an A_1-singularity at $\overline{P}$.

After the two-to-one base change $\Delta_{\sqrt{\delta}} \to \Delta_\delta$, the A_1-singularity $\overline{P}$ is resolved by the method of Atiyah [3]. Considering the observation in §4.4, the resolution space becomes a desired splitting family.

<div align="right">Q.E.D.</div>

Note that the germ of type $II(ii)$ constructed in Proposition 5.5 has a very special analytic structure.

5.6   Here we give another construction of a splitting of a fiber germ in $II(ii)$. This construction is simpler than the former, but the former seems more useful in the study of the monodromy types of singular fibers of the general members of the splitting families.

Let t denote a coordinate on Δ_ϵ and $(x : y)$ a homogeneous coordinate of $\mathbf{P}^1$. Let B be the divisor on $W = \mathbf{P}^1 \times \Delta_\epsilon$ defined by the equation

(5.6.1) $t(x^4 - t^2 y^4)(t^2 x^4 - y^4) = 0.$

Let $\varphi : S \to \Delta_\epsilon$ be a hyperelliptic degeneration of genus three whose branch curve is B. Then the corresponding fiber germ $[\varphi, F]$ is of type $II(ii)$ (cf. 2.2). Let $\sigma : W \to W$ be the action given by

$$x \longmapsto -x, \; y \longmapsto y, \; t \longmapsto t.$$

Then σ preserves the divisor B and is lifted to S as a fiber-preserving holomorphic involution in two ways, namely if one is $\tilde{\sigma} : S \to S$, then another is $\tilde{\sigma}h : S \to S$ where $h : S \to S$ is the hyperelliptic involution. It is easy to see that the fixed point locus of one of them, say $\tilde{\sigma}$, is $F' + F'''$ while that of the other, say $\tilde{\sigma}h$, is F'' where F', F'' and F''' are the same as in 2.2 (II (ii)).

Let $S \to T_1$ be the double covering induced by the involution $\tilde{\sigma}$. Then T_1 is a smooth surface and has a pencil of curves of genus two whose only singular fiber is of the form $\Gamma_1 = \Gamma' + 2\Gamma'' + \Gamma'''$, where Γ', Γ'' and Γ''' are nonsingular curves of respective genera one, zero, one with intersection numbers $\Gamma'\Gamma'' = \Gamma''\Gamma''' = 1$, $\Gamma'\Gamma''' = 0$, $(\Gamma')^2 = (\Gamma''')^2 = -2$ and $(\Gamma'')^2 = -1$. Moreover the branch locus on T_1 is $B_{T_1} = \Gamma' + \Gamma'''$. Let $T_1 \to T$ be the contraction of the (-1) curve Γ''. Then naturally T also has a pencil of curves of genus two and the branch locus B_T on T coincides with the unique singular fiber Γ on T.

Let $B_{T,u}$ be the family of divisors on T where $u \in \Delta_\delta$ is a parameter and $B_{T,u}$ is defined as the fiber of $T \to \Delta_\epsilon$ over the point $t = u \in \Delta_\epsilon$. Then since $B_{T,u}$'s are smooth curves of genus two for $u \neq 0$ and $B_{T,0} = B_T$ has an ordinary double point as the unique bad point, we get a splitting family $\Phi : M \to \Delta_\epsilon \times \Delta_\delta$ of genus 3 which satisfies (i) the degeneration $\Phi_0 : M_0 \to \Delta_\epsilon \times \{0\}$ coincides with $\varphi : S \to \Delta_\epsilon$ and (ii) the degeneration $\Phi_u : M_u \to \Delta_\epsilon \times \{u\}$ ($u \neq 0$) has two singular fibers $F_{u,1} = \Phi_u^{-1}(0)$ and $F_{u,2} = \Phi_u^{-1}(u)$.

Accola's result ([2] Lemma 5.10) says that a compact Riemann surface of genus three which is a double covering of a compact Riemann surface of genus two is hyperelliptic. Therefore the general fiber of $\Phi : M \to \Delta_\epsilon \times \Delta_\delta$ is hyperelliptic.

§6. Hyperelliptic atomic fibers of genus 3

DEFINITION 6.1. The set of germs of fibers $\mathcal{A} = \{[\varphi_i, F_i]\}_{i \in I}$ of hyperelliptic degenerations of genus g is said to be the complete system of hyperelliptic atomic fibers of genus g if the following conditions are satisfied:

(i) There exists no hyperelliptic proper splitting family of $[\varphi_i, F_i]$ for any $i \in I$.

(ii) The germ $[\varphi, F]$ of any hyperelliptic degeneration of genus g splits into the members of $\mathcal{A}$ via a finite number of hyperelliptic splitting families.

More precisely, $[\varphi, F]$ has an equisingular deformation to $[\varphi', F']$ so that $[\varphi', F']$ splits into $[\Phi_u, F_{u,1}], \cdots, [\Phi_u, F_{u,l}]$ via a certain hyperelliptic splitting family $\Phi : M \to \Delta_\epsilon \times \Delta_\delta$. If $[\Phi_u, F_{u,i}]$ for some i $(1 \le i \le l)$ is not contained in $\mathcal{A}$, then $[\Phi_u, F_{u,i}]$ has a hyperelliptic splitting family after an equisingular deformation (if it is necessary). We continue this process. Then after a finite steps, all the germs of singular fibers of resulting degenerations are contained in $\mathcal{A}$.

THEOREM 2.   *Let $\mathcal{A}$ be the set of germs of types $0_0, I, II(i)$ and $II(iii)$. Then $\mathcal{A}$ is the complete system of hyperelliptic atomic fibers of genus 3.*

Proof.   By [1], any germ of a hyperelliptic degeneration of genus 3 splits into the set of germs of types 0_0, I, $II(i)$, $II(ii)$ and $II(iii)$ via a finite number of hyperelliptic proper splitting families (without using equisingular deformations).

Let $[\varphi, F]$ be any germ of type $II(ii)$. Then $[\varphi, F]$ has an equisingular deformation to the special germ of type $II(ii)$ which is constructed in Proposition 5.5. Indeed, the equation of a representative of the branch curve B of $[\varphi, F]$ on $W = \mathbf{P}^1 \times \Delta_\epsilon$ is written by

$$ t \prod_{i=1}^{2} \prod_{j=1}^{4} \{ t + (x - a_i)^2 (b_{ij} + g_{ij}(t)) + (x - a_i)^3 f_{ij}(t, x) \} = 0 $$

where t is the coordinate of Δ, x is the inhomogeneous coordinate of $\mathbf{P}^1$, a_1 and a_2 are mutually distinct complex numbers, $b_{i1}, \cdots, b_{i4}$ $(i = 1, 2)$ are mutually distinct non-zero complex numbers, $g_{ij}(t)$ and $f_{ij}(t, x)$ are holomorphic functions of t and (t, x) respectively. We define the divisor $\mathbf{B}$ on $\mathbf{W} = W \times \Delta_\delta$ by

$$ t \prod_{i=1}^{2} \prod_{j=1}^{4} \{ t + (x + (-1)^i u + a_i u - a_i)^2 ((k_{ij} - b_{ij})u + b_{ij} + (1 - u)g_{ij}(t)) $$
$$ + (1 - u)(x + (-1)^i u + a_i u - a_i)^3 f_{ij}(t, x) \} = 0 $$

where u is the coordinate of Δ_δ and k_{ij} $(1 \le i \le 2, 1 \le j \le 4)$ are complex numbers which satisfy $k_{11} = -k_{12} = k_{21} = -k_{22}$, $k_{13} = -k_{14} = k_{23} = -k_{24}$ and $k_{11} \ne k_{13}$. Then the resolution space of the double cover of $\mathbf{W}$ branched along $\mathbf{B}$ has the desired germ over $u = 1$.

This germ splits into germs of types I and $II(iii)$ by Proposition 5.5. Since any germ of type 0_0, I, $II(i)$ or $II(iii)$ does not have proper splitting family by Proposition 4.1, the assertion follows.

<div align="right">Q.E.D.</div>

Remark 6.3. By similar arguments as those in the proof of Theorem 6.2, we can show that any singular fiber germ of type $II(ii)$ has an equisingular deformation to the special germ which is constructed in 5.6.

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Tatsuya Arakawa
Department of Information and Computer Engineering
Gunma NationalCollege of Technology
580 Toriba, Maebashi, Gunma, 371-8530
JAPAN
`arakawa@ice.gunma-ct.ac.jp`

Tadashi Ashikaga
Department of Applied Physics and Information
Faculty of Engineering, Tôhoku-Gakuin University
Chuo 1-13-1, Tagajo, Miyagi, 985-8537
JAPAN
`tashikaga@tjcc.tohoku-gakuin.ac.jp`