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ON A GENERALIZATION OF TEST IDEALS

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Abstract. The test ideal $\tau(R)$ of a ring R of prime characteristic is an important object in the theory of tight closure. In this paper, we study a generalization of the test ideal, which is the ideal $\tau(\mathfrak{a}^t)$ associated to a given ideal $\mathfrak{a}$ with rational exponent $t \geq 0$. We first prove a key lemma of this paper (Lemma 2.1), which gives a characterization of the ideal $\tau(\mathfrak{a}^t)$. As applications of this key lemma, we generalize the preceding results on the behavior of the test ideal $\tau(R)$. Moreover, we prove an analogue of so-called Skoda's theorem, which is formulated algebraically via adjoint ideals by Lipman in his proof of the "modified Briançon–Skoda theorem."

Introduction

Let R be a Noetherian commutative ring of characteristic $p > 0$. The test ideal $\tau(R)$ of R, introduced by Hochster and Huneke [HH2], is defined to be the annihilator ideal of all tight closure relations in R and plays an important role in the theory of tight closure. In [HY], the first-named author and Yoshida introduced a generalization of tight closure, which we call $\mathfrak{a}$-tight closure associated to a given ideal $\mathfrak{a}$, and defined the ideal $\tau(\mathfrak{a})$ to be the annihilator ideal of all $\mathfrak{a}$-tight closure relations in R. We can also consider $\mathfrak{a}^t$-tight closure and the ideal $\tau(\mathfrak{a}^t)$ with rational exponent $t \geq 0$ (or, rational coefficient in a sense), and even more, those with several rational exponents.

The ideals $\tau(\mathfrak{a}^t)$ have several nice properties similar to those of multiplier ideals $\mathcal{J}(\mathfrak{a}^t)$ defined via resolution of singularities in characteristic zero; see [La] for a systematic study of multiplier ideals. Among them, we have an analogue of Lipman's "modified Briançon–Skoda theorem" ([HY, Theorem 2.1], cf. [Li]) and the subadditivity theorem in regular local rings ([HY, Theorem 4.5], cf. [DEL]). It is notable that the above properties of

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the ideals $\tau(\mathfrak{a}^t)$ are proved quite algebraically via characteristic p methods. On the other hand, we can prove that the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ in a normal $\mathbb{Q}$-Gorenstein ring of characteristic zero coincides, after reduction to characteristic $p \gg 0$, with the ideal $\tau(\mathfrak{a}^t)$; see [HY] and also [Ta].

In this paper, we study further properties of $\mathfrak{a}^t$-tight closure and the ideal $\tau(\mathfrak{a}^t)$ in characteristic $p > 0$, improve results obtained in [HY], and give some applications. To do this, we first prove a key lemma of this paper (Lemma 2.1), which gives a characterization of (elements of) the ideal $\tau(\mathfrak{a}^t)$. Precisely speaking, Lemma 2.1 characterizes the ideal $\tilde{\tau}(\mathfrak{a}^t)$ given in Definition 1.4, which is contained in the ideal $\tau(\mathfrak{a}^t)$ and expected to coincide with $\tau(\mathfrak{a}^t)$. The identification of the ideals $\tau(\mathfrak{a}^t)$ and $\tilde{\tau}(\mathfrak{a}^t)$ holds true in some reasonable situations, e.g., in normal $\mathbb{Q}$-Gorenstein rings; see [AM], [HY] and [LS2].

Although the description of Lemma 2.1 is somewhat complicated, it turns out to be very useful in studying various properties of the ideals $\tau(\mathfrak{a}^t)$. As applications of this key lemma, we answer a question raised in [HY] (Corollary 2.3) and consider the relationship of test elements and $\mathfrak{a}^t$-test elements (Corollary 2.4). We also apply Lemma 2.1 to the study of the behavior of the ideals $\tau(\mathfrak{a}^t)$ under localization (Proposition 3.1), completion (Proposition 3.2) and finite morphisms which are étale in codimension one (Theorem 3.3). These results generalize the preceding results [LS2] and [BSm] on the behavior of the test ideal $\tau(R)$, and we hope that the proofs become simpler with the use of Lemma 2.1.

Other ingredients of this paper are Theorems 4.1 and 4.2, which assert that if $\mathfrak{a}$ is an ideal with a reduction generated by l elements, then $\tau(\mathfrak{a}^l) = \tau(\mathfrak{a}^{l-1})\mathfrak{a}$. This is an analogue of so-called Skoda's theorem [La], which is formulated algebraically via adjoint ideals (or multiplier ideals) by Lipman [Li] in his proof of the "modified Briançon–Skoda theorem." Theorem 4.1 is a refinement of [HY, Theorem 2.1] and is proved by an easy observation on the relationship of regular powers and Frobenius powers of ideals (cf. [AH]). Theorem 4.2 is based on the same idea, but is proved under a slightly different assumption with the aid of Lemma 2.1. We can find an advantage of the ideal $\tau(\mathfrak{a})$ in the simplicity of the proofs of Theorems 4.1 and 4.2, because the proof of Skoda's theorem for multiplier ideals needs a deep vanishing theorem which is proved only in characteristic zero [La], [Li].

§1. Preliminaries

In this paper, all rings are excellent reduced commutative rings with unity. For a ring R, we denote by R° the set of elements of R which are not in any minimal prime ideal. Let R be a ring of prime characteristic $p > 0$ and $F\colon R \to R$ the Frobenius map which sends $x \in R$ to $x^p \in R$. The ring R viewed as an R-module via the e-times iterated Frobenius map $F^e\colon R \to R$ is denoted by ${}^eR$. Since R is assumed to be reduced, we can identify $F\colon R \to {}^eR$ with the natural inclusion map $R \hookrightarrow R^{1/p^e}$. We say that R is *F-finite* if ${}^1R$ (or $R^{1/p}$) is a finitely generated R-module.

Let R be a ring of characteristic $p > 0$ and let M be an R-module. For each $e \in \mathbb{N}$, we denote $\mathbb{F}^e(M) = \mathbb{F}^e_R(M) := {}^eR \otimes_R M$ and regard it as an R-module by the action of $R = {}^eR$ from the left. Then we have the induced e-times iterated Frobenius map $F^e\colon M \to \mathbb{F}^e(M)$. The image of $z \in M$ via this map is denoted by $z^q := F^e(z) \in \mathbb{F}^e(M)$. For an R-submodule N of M, we denote by $N^{[q]}_M$ the image of the induced map $\mathbb{F}^e(N) \to \mathbb{F}^e(M)$.

Now we recall the definition of $\mathfrak{a}^t$-tight closure. See [HY] for details.

Definition 1.1. Let $\mathfrak{a}$ be an ideal of a ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $N \subseteq M$ be R-modules. Given a rational number $t \geq 0$, the *$\mathfrak{a}^t$-tight closure* $N^{*\mathfrak{a}^t}_M$ of N in M is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists $c \in R^\circ$ such that

$$cz^q\mathfrak{a}^{\lceil tq \rceil} \subseteq N^{[q]}_M$$

for all large $q = p^e$, where $\lceil tq \rceil$ is the least integer which is greater than or equal to tq. The $\mathfrak{a}^t$-tight closure $I^{*\mathfrak{a}^t}$ of an ideal $I \subseteq R$ is just defined by $I^{*\mathfrak{a}^t} = I^{*\mathfrak{a}^t}_R$.

Remark 1.2. The rational exponent t for $\mathfrak{a}^t$-tight closure in Definition 1.1 is just a formal notation, but it is compatible with "real" powers of the ideal. Namely, if $\mathfrak{b} = \mathfrak{a}^n$ for $n \in \mathbb{N}$, then $\mathfrak{a}^t$-tight closure is the same as $\mathfrak{b}^{t/n}$-tight closure. This allows us to extend the definition to several rational exponents: Given ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r \subseteq R$ with $\mathfrak{a}_i \cap R^\circ \neq \emptyset$ and rational numbers $t_1, \dots, t_r \geq 0$, if $t_i = tn_i$ for nonnegative $t \in \mathbb{Q}$ and $n_i \in \mathbb{N}$ with $i = 1, \dots, r$, we can define $\mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}$-tight closure to be $(\mathfrak{a}_1^{n_1} \cdots \mathfrak{a}_r^{n_r})^t$-tight closure. If N is a submodule of an R-module M, then an element $z \in M$ is in the $\mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}$-tight closure $N^{*\mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}}_M$ of N in M if and only if there exists $c \in R^\circ$ such that $cz^q\mathfrak{a}_1^{\lceil t_1 q \rceil} \cdots \mathfrak{a}_r^{\lceil t_r q \rceil} \subseteq N^{[q]}_M$ for all $q = p^e \gg 0$.

Since $N_M^{*\mathfrak{a}^t}/N \cong 0_{M/N}^{*\mathfrak{a}^t}$ for R-modules $N \subseteq M$ ([HY, Proposition 1.3 (1)]), the case where $N = 0$ is essential. Using the $\mathfrak{a}^t$-tight closure of the zero submodule, we can define two ideals $\tau(\mathfrak{a}^t)$ and $\tilde{\tau}(\mathfrak{a}^t)$.

Proposition-Definition 1.3. ([HY, Definition-Theorem 6.5]) *Let R be an excellent reduced ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and $t \geq 0$ a rational number. Let $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum, taken over all maximal ideals $\mathfrak{m}$ of R, of the injective envelopes of the residue fields $R/\mathfrak{m}$. Then the following ideals are equal to each other and we denote it by $\tau(\mathfrak{a}^t)$.*

(i) $\bigcap_M \mathrm{Ann}_R(0_M^{*\mathfrak{a}^t})$, *where M runs through all finitely generated R-modules.*

(ii) $\bigcap_{M \subseteq E} \mathrm{Ann}_R(0_M^{*\mathfrak{a}^t})$, *where M runs through all finitely generated submodules of E.*

(iii) $\bigcap_{J \subseteq R} (J : J^{*\mathfrak{a}^t})$, *where J runs through all ideals of R.*

The description (ii) of $\tau(\mathfrak{a}^t)$ in Proposition-Definition 1.3 means that $\tau(\mathfrak{a}^t)$ is the annihilator ideal of the "finitistic $\mathfrak{a}^t$-tight closure" of zero in E, that is, the union of $0_M^{*\mathfrak{a}^t}$ in E taken over all finitely generated submodules M of E. It would be natural to consider the $\mathfrak{a}^t$-tight closure of zero in E instead of the finitistic tight closure.

Definition 1.4. Let $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be as in Proposition-Definition 1.3. Then we define the ideal $\tilde{\tau}(\mathfrak{a}^t)$ by

$$\tilde{\tau}(\mathfrak{a}^t) = \mathrm{Ann}_R(0_E^{*\mathfrak{a}^t}).$$

We have $\tilde{\tau}(\mathfrak{a}^t) \subseteq \tau(\mathfrak{a}^t)$ in general, since $0_M^{*\mathfrak{a}^t} \subseteq 0_E^{*\mathfrak{a}^t}$ for all (finitely generated) submodules M of E. We do not know any example in which the $\mathfrak{a}^t$-tight closure of zero and the finitistic $\mathfrak{a}^t$-tight closure of zero disagree in E, and it seems reasonable to assume the following condition.

(1.4.1) We say that *condition* $(*)$ *is satisfied for* $\mathfrak{a}^t$ if $\tau(\mathfrak{a}^t) = \tilde{\tau}(\mathfrak{a}^t)$.

Condition $(*)$ is satisfied in many situations. For example, if R is a graded ring, then condition $(*)$ is satisfied for the unit ideal R (see [LS1]), and if R is an excellent $\mathbb{Q}$-Gorenstein normal local ring, then condition $(*)$ is satisfied for every rational number $t \geq 0$ and every ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ ([HY, Theorem 1.13]; see also [AM]).

In most part of this paper, we will assume condition $(*)$ to deduce various properties of the ideal $\tau(\mathfrak{a}^t)$, because what we actually do is to prove those properties for the ideal $\tilde{\tau}(\mathfrak{a}^t)$, which are translated into those for $\tau(\mathfrak{a}^t)$ under condition $(*)$. So, we remark that most of our results do hold true for $\tilde{\tau}(\mathfrak{a}^t)$ without assuming condition $(*)$.

Finally we recall the notion of an $\mathfrak{a}^t$-test element, which is quite useful to handle the $\mathfrak{a}^t$-tight closure operation. See [HY, Theorems 1.7 and 6.4] for the existence of $\mathfrak{a}^t$-test elements.

Definition 1.5. Let $\mathfrak{a}$ be an ideal of a ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a rational number. An element $d \in R^\circ$ is called an *$\mathfrak{a}^t$-test element* if for every finitely generated R-module M and every element $z \in M$, the following holds: $z \in 0_M^{*\mathfrak{a}^t}$ if and only if $dz^q\mathfrak{a}^{\lceil tq \rceil} = 0$ for all powers $q = p^e$ of p.

Remark 1.6. In the case where $\mathfrak{a} = R$ is the unit ideal, the ideal $\tau(\mathfrak{a}) = \tau(R)$ is called the test ideal of R and an R-test element is nothing but a test element as defined in [HH2]. In this case, $\tau(R) \cap R^\circ$ is exactly equal to the set of test elements. However, $\tau(\mathfrak{a}^t) \cap R^\circ$ is not equal to the set of $\mathfrak{a}^t$-test elements in general. The relationship between the ideal $\tau(\mathfrak{a}^t)$ and $\mathfrak{a}^t$-test elements is not a priori clear, but we will see later in Corollary 2.4 that an element of $\tau(\mathfrak{a}^t) \cap R^\circ$ is always an $\mathfrak{a}^t$-test element.

The reader is referred to [HY] for basic properties of $\mathfrak{a}^t$-tight closure and the ideal $\tau(\mathfrak{a}^t)$. Among them, see especially Propositions 1.3 and 1.11 and Theorem 1.7, together with Section 6, of [HY].

All results in this paper are proved in characteristic $p > 0$. But it may help better understanding of the results to keep in mind the correspondence of the ideals $\tau(\mathfrak{a}^t)$ and the multiplier ideals defined via resolution of singularities in characteristic zero ([HY]). Namely, given a rational number $t \geq 0$ and an ideal $\mathfrak{a}$ of a normal $\mathbb{Q}$-Gorenstein ring essentially of finite type over a field of characteristic zero, the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ coincides, after reduction to characteristic $p \gg 0$, with the ideal $\tau(a^t)$.

§2. A characterization of the ideal $\tau(\mathfrak{a}^t)$

We first show a key lemma of this paper, which gives a characterization of the ideal $\tilde{\tau}(\mathfrak{a}^t)$ and reveals the relationship between the ideal $\tau(\mathfrak{a}^t)$ and $\mathfrak{a}^t$-test elements under condition $(*)$. Its essential idea is found in [Ta, Theorem 3.13].

LEMMA 2.1. *Let $(R, \mathfrak{m})$ be an F-finite local ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and $t \geq 0$ a rational number. Fix a system of generators $x_1^{(e)}, \ldots, x_{r_e}^{(e)}$ of $\mathfrak{a}^{\lceil tq \rceil}$ for each $q = p^e$. Then for any element $c \in R$, the following four conditions are equivalent to each other.*

(i) $c \in \tilde{\tau}(\mathfrak{a}^t)$.

(ii) *For any element $d \in R^\circ$ and any integer $e_0 \geq 0$, there exist an integer $e_1 \geq e_0$ and R-homomorphisms $\phi_i^{(e)} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ for $e_0 \leq e \leq e_1$ and $1 \leq i \leq r_e$ such that*

$$c = \sum_{e=e_0}^{e_1} \sum_{i=1}^{r_e} \phi_i^{(e)}((dx_i^{(e)})^{1/p^e}).$$

(iii) *For an $\mathfrak{a}^t$-test element $d \in R^\circ$, there exist a positive integer e_1 and R-linear maps $\phi_i^{(e)} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ for $0 \leq e \leq e_1$ and $1 \leq i \leq r_e$ such that*

$$c = \sum_{e=0}^{e_1} \sum_{i=1}^{r_e} \phi_i^{(e)}((dx_i^{(e)})^{1/p^e}).$$

(iii)$'$ *For an $\mathfrak{a}^t$-test element $d \in R^\circ$, there exist a positive integer e_1 and R-linear maps $\phi_i^{(e)} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ for $0 \leq e \leq e_1$ and $1 \leq i \leq r_e$ such that*

$$c \in \sum_{e=0}^{e_1} \sum_{i=1}^{r_e} \phi_i^{(e)}((d\mathfrak{a}^{\lceil tp^e \rceil})^{1/p^e}).$$

Proof. We prove the equivalence of conditions (i) and (ii). Let $F^e \colon E_R \to {}^eR \otimes_R E_R$ be the e-times iterated Frobenius map induced on the injective envelope $E_R := E_R(R/\mathfrak{m})$ of the residue field $R/\mathfrak{m}$. For an element $d \in R^\circ$ and an integer $e \geq 0$, we write

$$d\mathbf{x}^{(e)} F^e := {}^t(dx_1^{(e)} F^e, \ldots, dx_{r_e}^{(e)} F^e) \colon E_R \to ({}^eR \otimes_R E_R)^{\oplus r_e}.$$

By definition and condition $(*)$, $c \in \tau(\mathfrak{a}^t)$ if and only if $c \cdot \bigcap_{e \geq e_0} \mathrm{Ker}(d\mathbf{x}^{(e)}F^e) = 0$ holds for every element $d \in R^\circ$ and every integer $e_0 \geq 0$. Since E_R is an Artinian R-module, there exists an integer $e_1 \geq e_0$ (depending on $d \in R^\circ$ and $e_0 \geq 0$) such that

$$\bigcap_{e \geq e_0} \mathrm{Ker}(d\mathbf{x}^{(e)}F^e) = \bigcap_{e=e_0}^{e_1} \mathrm{Ker}(d\mathbf{x}^{(e)}F^e).$$

Therefore denoting

$$\Phi = \Phi_d^{(e_0,e_1)} := {}^t(d\mathbf{x}^{(e_0)}F^{e_0}, \dots, d\mathbf{x}^{(e_1)}F^{e_1}) \colon E_R \to \bigoplus_{e=e_0}^{e_1} ({}^eR \otimes_R E_R)^{\oplus r_e},$$

we see that $c \in \tau(\mathfrak{a}^t)$ if and only if for every $d \in R^\circ$ and $e_0 \geq 0$, there exists $e_1 \geq e_0$ such that $c \cdot \mathrm{Ker}(\Phi_d^{(e_0,e_1)}) = 0$.

Since R is F-finite, the map $\Phi = \Phi_d^{(e_0,e_1)} \colon E_R \to \bigoplus_{e=e_0}^{e_1} ({}^eR \otimes_R E_R)^{\oplus r_e}$ is the Matlis dual of the map

$$\Psi = (\psi^{(e_0)}, \dots, \psi^{(e_1)}) \colon \bigoplus_{e=e_0}^{e_1} \mathrm{Hom}_R(R^{1/p^e}, R)^{\oplus r_e} \to R,$$

where the map $\psi^{(e)} \colon \mathrm{Hom}_R(R^{1/p^e}, R)^{\oplus r_e} \to \mathrm{Hom}_R(R, R) = R$ is induced by the R-linear map $R \to (R^{1/p^e})^{\oplus r_e}$ sending 1 to $((dx_1^{(e)})^{1/p^e}, \dots, (dx_{r_e}^{(e)})^{1/p^e})$. It then follows that $c \cdot \mathrm{Ker}(\Phi) = 0$ if and only if $c \in \mathrm{Im}(\Psi)$. By the definition of Ψ, this is equivalent to saying that there exist R-linear maps $\phi_i^{(e)} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ for $e_0 \leq e \leq e_1$ and $1 \leq i \leq r_e$ such that

$$c = \sum_{e=e_0}^{e_1} \sum_{i=1}^{r_e} \phi_i^{(e)}((dx_i^{(e)})^{1/p^e}).$$

The equivalence of conditions (i) and (iii) (resp. (iii)$'$) is obtained in the same way.

Remark 2.2. An advantage of Lemma 2.1 is that it is applicable even in the absense of the completeness of R. For example, compare the hypotheses in Theorem 4.1 and Theorem 4.2.

As immediate consequences of Lemma 2.1, we have the following corollaries. First Lemma 2.1 gives an affirmative answer to a question raised in [HY, Discussion 5.18].

Corollary 2.3. *Let $(R,\mathfrak{m})$ be an F-finite local ring of characteristic $p>0$, and suppose that condition $(*)$ is satisfied for the maximal ideal $\mathfrak{m}$. Then $\tau(\mathfrak{m})=R$ if and only if for every $\mathfrak{m}$-primary ideal $\mathfrak{a}\subset R$, we have a strict containment $\tau(\mathfrak{a})\supsetneq\mathfrak{a}$. In particular, if R is a regular local ring with $\dim R\ge 2$, then $\tau(\mathfrak{a})\supsetneq\mathfrak{a}$ for every $\mathfrak{m}$-primary ideal $\mathfrak{a}\subset R$.*

Proof. Suppose that $\tau(\mathfrak{m})=R$. We will show that $\tau(\mathfrak{a})\supseteq(\mathfrak{a}:\mathfrak{m})$ for every ideal $\mathfrak{a}\subseteq R$ such that $\mathfrak{a}\cap R^\circ\neq\emptyset$. By Lemma 2.1, for any element $d\in R^\circ$ and any integer $e_0\ge 0$, there exist an integer $e\ge e_0$, an element $x\in\mathfrak{m}^{p^e}$ and an R-module homomorphism $\phi\colon R^{1/p^e}\to R$ such that $\phi((dx)^{1/p^e})=1$. Now fix any element $y\in(\mathfrak{a}:\mathfrak{m})$. Since $\phi((dxy^{p^e})^{1/p^e})=\phi((dx)^{1/p^e})y=y$ and $xy^{p^e}\in\mathfrak{a}^{p^e}$, we have $y\in\tau(\mathfrak{a})$ by using Lemma 2.1 again. The converse implication is trivial. The latter assertion follows from [HY, Theorem 2.15].

Corollary 2.4. *Let $(R,\mathfrak{m})$ be an F-finite local ring of characteristic $p>0$. If condition $(*)$ is satisfied for the unit ideal R, then a test element of R is an $\mathfrak{a}^t$-test element for all ideals $\mathfrak{a}\subseteq R$ such that $\mathfrak{a}\cap R^\circ\neq\emptyset$ and all rational numbers $t\ge 0$.*

Proof. Let c be a test element of R, that is, an element of $\tau(R)\cap R^\circ$. Given any ideals $\mathfrak{a},J$ of R, any rational number $t\ge 0$, any element $z\in J^{*\mathfrak{a}^t}$ and any power q of p, it is enough to show that $cz^q\mathfrak{a}^{\lceil tq\rceil}\subseteq J^{[q]}$. Since $z\in J^{*\mathfrak{a}^t}$, there exist $d\in R^\circ$ and $e_0\in\mathbb{N}$ such that $dz^Q\mathfrak{a}^{\lceil tQ\rceil}\subseteq J^{[Q]}$ for every power $Q\ge p^{e_0}$ of p. Then by Lemma 2.1, there exist $e_1\in\mathbb{N}$ and $\phi_e\in\mathrm{Hom}_R(R^{1/p^e},R)$ for $e_0\le e\le e_1$ such that $c=\sum_{e=e_0}^{e_1}\phi_e(d^{1/p^e})$. Since $dz^{qp^e}(\mathfrak{a}^{\lceil tq\rceil})^{[p^e]}\subseteq dz^{qp^e}\mathfrak{a}^{\lceil tqp^e\rceil}\subseteq J^{[qp^e]}$ for every $e_0\le e\le e_1$, we have

$$d^{1/p^e}z^q\mathfrak{a}^{\lceil tq\rceil}R^{1/p^e}\subseteq J^{[q]}R^{1/p^e}.$$

Applying ϕ_e and summing up, we obtain

$$cz^q\mathfrak{a}^{\lceil tq\rceil}=\sum_{e=e_0}^{e_1}\phi_e(d^{1/p^e})z^q\mathfrak{a}^{\lceil tq\rceil}\subseteq\sum_{e=e_0}^{e_1}\phi_e(J^{[q]}R^{1/p^e})\subseteq J^{[q]},$$

as required.

Remark 2.5. Since $\tilde\tau(\mathfrak{a}^t)\subseteq\tilde\tau(R)$, we see from Corollary 2.4 that any element of $\tau(\mathfrak{a}^t)\cap R^\circ$ is an $\mathfrak{a}^t$-test element as long as condition $(*)$ is satisfied for $\mathfrak{a}^t$. Also, Corollary 2.4 is considered a refinement of [HY, Theorem 1.7], which asserts that, if the localized ring R_c at an element $c\in R^\circ$ is strongly

F-regular, then some power c^n of c is an $\mathfrak{a}^t$-test element for all ideals $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and all rational numbers $t \geq 0$. Indeed, if R_c is strongly F-regular, then some power c^n of c is a test element (precisely speaking, an element of $\tilde{\tau}(R) \cap R^\circ$) by [HH1] (see also [HH3]), so that c^n is an $\mathfrak{a}^t$-test element for all $\mathfrak{a}$ and t by Corollary 2.4.

§3. Behavior of $\tau(\mathfrak{a})$ under localization, completion and finite homomorphisms

PROPOSITION 3.1. (cf. [LS2]) *Let $(R, \mathfrak{m})$ be an F-finite local ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and $t \geq 0$ a rational number. Let W be a multiplicatively closed subset of R and suppose that condition $(*)$ is satisfied for $\mathfrak{a}^t$ and $(\mathfrak{a}R_W)^t$. Then*

$$\tau((\mathfrak{a}R_W)^t) = \tau(\mathfrak{a}^t)R_W.$$

Proof. Fix a system of generators $x_1^{(e)}, \ldots, x_{r_e}^{(e)}$ of $\mathfrak{a}^{\lceil tq \rceil}$ for each $q = p^e$. If an element $c \in R_W$ is contained in $\tau((\mathfrak{a}R_W)^t)$, then by Lemma 2.1, for any element $d \in R^\circ$ and any nonnegative integer e_0, there exist an integer $e_1 \geq e_0$ and R_W-homomorphisms $\phi_i^{(e)} \in \mathrm{Hom}_{R_W}(R_W^{1/p^e}, R_W)$ for $e_0 \leq e \leq e_1$ and $1 \leq i \leq r_e$ such that

$$c = \sum_{e=e_0}^{e_1} \sum_{i=1}^{r_e} \phi_i^{(e)}((dx_i^{(e)})^{1/p^e}).$$

Since R is F-finite, there exists an element $y \in W$ such that we can regard $y\phi_i^{(e)}$ as an element of $\mathrm{Hom}_R(R^{1/p^e}, R)$ for all $e_0 \leq e \leq e_1$ and $1 \leq i \leq r_e$. Therefore, thanks to Lemma 2.1 again, we have $cy \in \tau(\mathfrak{a}^t)$, that is, $c \in \tau(\mathfrak{a}^t)R_W$. The converse argument just reverses this. The proposition is proved.

PROPOSITION 3.2. (cf. [LS2]) *Let $(R, \mathfrak{m})$ be an F-finite local ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and $t \geq 0$ a rational number. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R and suppose that condition $(*)$ is satisfied for $\mathfrak{a}^t$ and $(a\widehat{R})^t$. Then*

$$\tau((\mathfrak{a}\widehat{R})^t) = \tau(\mathfrak{a}^t)\widehat{R}.$$

Proof. Fix a system of generators $x_1^{(e)}, \ldots, x_{r_e}^{(e)}$ of $\mathfrak{a}^{\lceil tq \rceil}$ for each $q = p^e$. Since R is F-finite, we can take an element $d \in R^\circ$ which is an $\mathfrak{a}^t$- and

$(\mathfrak{a}\widehat{R})^t$-test element by [HY, Theorem 6.4]. If an element $c \in \widehat{R}$ is contained in $\tau((\mathfrak{a}\widehat{R})^t)$, then by Lemma 2.1, there exist an integer $e_1 > 0$ and $\widehat{R}$-homomorphisms $\phi_i^{(e)} \in \mathrm{Hom}_{\widehat{R}}(\widehat{R}^{1/p^e}, \widehat{R})$ for $0 \le e \le e_1$ and $1 \le i \le r_e$ such that

$$c = \sum_{e=0}^{e_1} \sum_{i=1}^{r_e} \phi_i^{(e)}((dx_i^{(e)})^{1/p^e}).$$

Since R is F-finite, $\mathrm{Hom}_{\widehat{R}}(\widehat{R}^{1/p^e}, \widehat{R}) \cong \widehat{R} \otimes_R \mathrm{Hom}_R(R^{1/p^e}, R)$, so that there exist $y_{i,1}^{(e)}, \dots, y_{i,s_{e,i}}^{(e)} \in \widehat{R}$ and $\psi_{i,1}^{(e)}, \dots, \psi_{i,s_{e,i}}^{(e)} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ with $\phi_i^{(e)} = \sum_{j=1}^{s_{e,i}} y_{i,j}^{(e)} \otimes \psi_{i,j}^{(e)}$ for all $0 \le e \le e_1$ and $1 \le i \le r_e$. Then

$$c = \sum_{e=0}^{e_1} \sum_{i=1}^{r_e} \sum_{j=1}^{s_{e,i}} y_{i,j}^{(e)} \psi_{i,j}^{(e)}((dx_i^{(e)})^{1/p^e}).$$

Therefore, thanks to Lemma 2.1 again, we have $c \in \tau(\mathfrak{a}^t)\widehat{R}$. The converse argument just reverses this. The proposition is proved.

Our characterization is also applicable to the following situation; see also [BSm].

THEOREM 3.3. *Let $(R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ be a pure finite local homomorphism of F-finite normal local rings of characteristic $p > 0$ which is étale in codimension one. Let $\mathfrak{a}$ be a nonzero ideal of R and let t be a nonnegative rational number. Assume that condition $(*)$ is satisfied for $\mathfrak{a}^t$ and $(\mathfrak{a}S)^t$. Then*

$$\tau((\mathfrak{a}S)^t) \cap R = \tau(\mathfrak{a}^t).$$

Moreover if $R \hookrightarrow S$ is flat, then

$$\tau((\mathfrak{a}S)^t) = \tau(\mathfrak{a}^t)S.$$

Proof. Since $R \hookrightarrow S$ is pure, we have $\tau((\mathfrak{a}S)^t) \cap R \subseteq \tau(\mathfrak{a}^t)$ by [HY, Proposition 1.12]. We will prove the reverse inclusion. Fix a system of generators $x_1^{(e)}, \dots, x_{r_e}^{(e)}$ of $\mathfrak{a}^{\lceil tq \rceil}$ for every $q = p^e$. Take an element $d \in R^\circ$ which is not only an $\mathfrak{a}^t$-test element but also an $(\mathfrak{a}S)^t$-test element (cf. Remark 2.5, [BSm, Remark 6.5]). If an element c belongs to $\tau(\mathfrak{a}^t)$, then by Lemma 2.1, there exist an integer $e_1 > 0$ and R-linear maps $\phi_i^{(e)} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ for $0 \le e \le e_1$ and $1 \le i \le r_e$ such that

$$c = \sum_{e=0}^{e_1} \sum_{i=1}^{r_e} \phi_i^{(e)}((dx_i^{(e)})^{1/p^e}).$$

Let $c_{e,i} = \phi_i^{(e)}((dx_i^{(e)})^{1/p^e})$ and let $E_S = E_S(S/\mathfrak{n})$ be the injective envelope of the residue field of S. Since $R \hookrightarrow S$ is étale in codimension one, by tensoring $\phi_i^{(e)}$ with E_S over R, we have the commutative diagram

$$\begin{tikzcd} E_S \arrow[r, "dx_i^{(e)} F_S^e"] \arrow[dr, "c_{e,i}"'] & {}^eS \otimes_S E_S \arrow[d] \\ & E_S, \end{tikzcd}$$

where $F_S^e\colon E_S \to {}^eS \otimes_S E_S$ is the induced e-times iterated Frobenius map on E_S (see the proof of [W, Theorem 2.7]). Hence $\mathrm{Ker}(dx_i^{(e)} F_S^e) \subseteq (0 : c_{e,i})_{E_S}$ for every $0 \le e \le e_1$ and $1 \le i \le r_e$. Since $\sum_{e=0}^{e_1} \sum_{i=1}^{r_e} c_{e,i} = c$, we have $c \cdot \bigcap_{e=0}^{e_1} \bigcap_{i=1}^{r_e} \mathrm{Ker}(dx_i^{(e)} F_S^e) = 0$. By condition $(*)$, this implies that $c \in \tau((\mathfrak{a}S)^t)$.

Now we will show the latter assertion assuming that $R \hookrightarrow S$ is flat. By the above argument, we already know that $\tau((\mathfrak{a}S)^t) \supseteq \tau(\mathfrak{a}^t)S$. Suppose that $c \in \tau((\mathfrak{a}S)^t)$. Since S is a free R-module, we can choose a basis $s_1, \ldots, s_k$ for S over R and write $c = \sum_{j=1}^k c_j s_j$ for $c_j \in R$. For any ideal $I \subseteq R$ and any $z \in I^{*\mathfrak{a}^t}$, clearly $z \in (IS)^{*(\mathfrak{a}S)^t}$. By the definition of $\tau((\mathfrak{a}S)^t)$, we have $\sum_{j=1}^k c_j z s_j = cz \in IS = \bigoplus_{j=1}^k I s_j$. It follows that $c_j z \in I$, therefore $c_j \in \tau(\mathfrak{a}^t)$ for every $j = 1, \ldots, k$. Thus $c \in \tau(\mathfrak{a}^t)S$.

The following example shows that the last equality in Theorem 3.3 breaks down in the absence of the flatness, even for the case $\mathfrak{a} = R$.

EXAMPLE 3.4. Let $S = k[[x, y, z]]/(x^n + y^n + z^n)$ be the Fermat hypersurface of degree n over a field of characteristic $p \ge n + 1$, $R = S^{(r)}$ the rth Veronese subring of S, and assume that $n \ge 3$ and $r \ge 2$ are not divisible by p. Then $\tau(R) = R_{\ge \lfloor 1+(n-3)/r \rfloor} = R_{\ge \lceil (n-2)/r \rceil}$ (note that $\lfloor 1 + (n-3)/r \rfloor = \lfloor (n-2)/r + (r-1)/r \rfloor = \lceil (n-2)/r \rceil$) and $\tau(S) = S_{\ge n-2}$. Hence $\tau(S) \cap R = R_{\ge \lceil (n-2)/r \rceil} = \tau(R)$, but $\tau(R)S = S_{\ge \lceil (n-2)/r \rceil r} \subsetneq S_{\ge n-2} = \tau(S)$ if $n - 2$ is not divisible by r.

§4. Lipman-Skoda's theorem

In [Li], Lipman proves under the Grauert–Riemenschneider vanishing theorem that, if $\mathfrak{a}$ is an ideal of a regular local ring with a reduction generated by l elements, then $\mathcal{J}(\mathfrak{a}^l) = \mathcal{J}(\mathfrak{a}^{l-1})\mathfrak{a}$, where $\mathcal{J}(\mathfrak{b})$ denotes the multiplier ideal (or adjoint ideal in the sense of [Li]) associated to an ideal $\mathfrak{b}$.

This result is called Skoda's theorem [La] and formulated algebraically by Lipman in his proof of "modified Briançon–Skoda theorem." We give a simple proof of the corresponding equality for the ideal $\tau(\mathfrak{a}^l)$. The following version is just a refinement of [HY, Theorem 2.1]; see also Remark 1.2 for the definition of the ideal $\tau(\mathfrak{a}^l\mathfrak{b}^t)$ with "bi-exponents."

THEOREM 4.1. *Let $(R,\mathfrak{m})$ be a complete local ring of characteristic $p>0$ and let $\mathfrak{a}\subseteq R$ be an ideal of positive height with a reduction generated by l elements. Let $\mathfrak{b}$ be an ideal of R such that $\mathfrak{b}\cap R^\circ\neq\emptyset$ and $t\geq 0$ a rational number. Then*

$$\tau(\mathfrak{a}^l\mathfrak{b}^t)=\tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a}.$$

Proof. We will see that

$$0_M^{*\mathfrak{a}^l\mathfrak{b}^t}=0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}:\mathfrak{a}\text{ in }M$$

for any R-module M. By fundamental properties of $\mathfrak{a}^t$-tight closure [HY, Proposition 1.3], the inclusion $0_M^{*\mathfrak{a}^l\mathfrak{b}^t}\subseteq 0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}:\mathfrak{a}$ is immediate, and to prove the reverse inclusion, we may assume without loss of generality that $\mathfrak{a}$ itself is generated by l elements. Let $z\in 0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}:\mathfrak{a}$, i.e., $z\mathfrak{a}\subseteq 0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}$. Then there exists $c\in R^\circ$ such that $cz^q\mathfrak{a}^{[q]}(\mathfrak{a}^{l-1})^q\mathfrak{b}^{\lceil tq\rceil}=0$ in $\mathbb{F}^e(M)$ for all $q=p^e\gg 0$. Since $\mathfrak{a}$ is generated by l elements, one has $\mathfrak{a}^{ql}=\mathfrak{a}^{q(l-1)}\mathfrak{a}^{[q]}$, so that $cz^q\mathfrak{a}^{lq}\mathfrak{b}^{\lceil tq\rceil}=0$ for all $q=p^e\gg 0$, that is, $z\in 0_M^{*\mathfrak{a}^l\mathfrak{b}^t}$. Thus we have $0_M^{*\mathfrak{a}^l\mathfrak{b}^t}=0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}:\mathfrak{a}$.

Now assume that $(R,\mathfrak{m})$ is a complete local ring and let $E=E_R(R/\mathfrak{m})$, the injective envelope of the R-module $R/\mathfrak{m}$. Then by the Matlis duality, $\operatorname{Ann}_E(\tau(\mathfrak{a}^{l-1}\mathfrak{b}^t))$ is equal to the union of $0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}$ taken over all finitely generated R-submodules M of E. Hence, if $z\in\operatorname{Ann}_E(\tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a})$, then there exists a finitely generated submodule $M\subset E$ such that $z\in(0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}:\mathfrak{a})_E$. Replacing M by $M+Rz\subset E$, one has $z\in(0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}:\mathfrak{a})_M=0_M^{*\mathfrak{a}^l\mathfrak{b}^t}$. Consequently, $\operatorname{Ann}_E(\tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a})$ is equal to the union of $(0_M^{*\mathfrak{a}^{l-1}\mathfrak{b}^t}:\mathfrak{a})_M=0_M^{*\mathfrak{a}^l\mathfrak{b}^t}$ taken over all finitely generated submodules $M\subset E$. Therefore

$$\tau(\mathfrak{a}^l\mathfrak{b}^t)=\bigcap_{M\subset E}\operatorname{Ann}_R(0_M^{*\mathfrak{a}^l\mathfrak{b}^t})=\operatorname{Ann}_R(\operatorname{Ann}_E(\tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a}))=\tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a}.$$

The characterization of the ideal $\tau(\mathfrak{a}^t)$ given in Lemma 2.1 enables us to replace the completeness assumption in the above theorem by condition $(*)$.

THEOREM 4.2. *Let* $(R, \mathfrak{m})$ *be an F-finite local ring of characteristic* $p > 0$ *and let* $\mathfrak{a} \subseteq R$ *be an ideal of positive height with a reduction generated by* l *elements. Let* $\mathfrak{b}$ *be an ideal of* R *such that* $\mathfrak{b} \cap R^\circ \neq \emptyset$ *and* t *a nonnegative rational number. If condition* $(*)$ *is satisfied for* $\mathfrak{a}^{l-1}\mathfrak{b}^t$ *and* $\mathfrak{a}^l\mathfrak{b}^t$, *then*

$$\tau(\mathfrak{a}^l\mathfrak{b}^t) = \tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a}.$$

Proof. We may assume without loss of generality that $\mathfrak{a}$ is generated by l elements, so that $\mathfrak{a}^{ql} = \mathfrak{a}^{q(l-1)}\mathfrak{a}^{[q]}$ for all $q = p^e$. We fix an element $d \in R^\circ$ such that R_d is regular. Then, thanks to Remark 2.5, some power d^n is an $\mathfrak{a}^k\mathfrak{b}^t$-test element for all $k \geq 0$.

By Lemma 2.1, an element $c \in R$ is in $\tau(\mathfrak{a}^l\mathfrak{b}^t)$ if and only if there exist finitely many R-homomorphisms $\phi_i^{(e)} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ for $0 \leq e \leq e_1$ and $1 \leq i \leq r_e$ such that

$$c \in \sum_{e=0}^{e_1}\sum_{i=1}^{r_e} \phi_i^{(e)}((d^n\mathfrak{a}^{p^e l}\mathfrak{b}^{\lceil tp^e\rceil})^{1/p^e}).$$

Since

$$\begin{aligned}\phi_i^{(e)}((d^n\mathfrak{a}^{p^e l}\mathfrak{b}^{\lceil tp^e\rceil})^{1/p^e}) &= \phi_i^{(e)}((d^n\mathfrak{a}^{p^e(l-1)}\mathfrak{a}^{[p^e]}\mathfrak{b}^{\lceil tp^e\rceil})^{1/p^e}) \\ &= \phi_i^{(e)}((d^n\mathfrak{a}^{p^e(l-1)}\mathfrak{b}^{\lceil tp^e\rceil})^{1/p^e})\mathfrak{a} \\ &\subseteq \tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a}\end{aligned}$$

again by Lemma 2.1, this is equivalent to saying that $c \in \tau(\mathfrak{a}^{l-1}\mathfrak{b}^t)\mathfrak{a}$.

COROLLARY 4.3. (Modified Briançon–Skoda, cf. [BSk], [HY], [HH2], [Li]) *Let* $(R, \mathfrak{m})$ *and* $\mathfrak{a} \subseteq R$ *be as in Theorem* 4.1 *or* 4.2. *Then*

$$\tau(\mathfrak{a}^{n+l-1}) \subseteq \mathfrak{a}^n$$

for all $n \geq 0$. *In particular, if* R *is weakly F-regular, then* $\overline{\mathfrak{a}^{n+l-1}} \subseteq \mathfrak{a}^n$ *for all* $n \geq 0$.

COROLLARY 4.4. *Let* $(R, \mathfrak{m})$ *be a* d*-dimensional local ring of characteristic* $p > 0$ *with infinite residue field* $R/\mathfrak{m}$ *and* $\mathfrak{a} \subseteq R$ *an ideal of positive height. Let* $\mathfrak{b}$ *be an ideal of* R *such that* $\mathfrak{b} \cap R^\circ \neq \emptyset$ *and* t *a nonnegative rational number. We assume that* $(R, \mathfrak{m})$ *is complete or it is F-finite and* *condition* $(*)$ *is satisfied for* $\mathfrak{a}^{n+d-1}\mathfrak{b}^t$ *for every* $n \geq 0$*. Then for any* $n \geq 0$ *one has*

$$\tau(\mathfrak{a}^{n+d-1}\mathfrak{b}^t) = \tau(\mathfrak{a}^{d-1}\mathfrak{b}^t)\mathfrak{a}^n.$$

Proof. We can assume that $\mathfrak{a}$ is a proper ideal of R. Since the residue field $R/\mathfrak{m}$ is infinite, by [NR], $\mathfrak{a}$ has a reduction ideal generated by at most d elements. Therefore the assertion immediately follows from Theorems 4.1 and 4.2.

EXAMPLE 4.5. (cf. [HY, Theorem 2.15]) Let R be a d-dimensional regular local ring of characteristic $p > 0$ with the maximal ideal $\mathfrak{m}$. Then

$$\tau(\mathfrak{m}^n) = \begin{cases} R & \text{if } n < d, \\ \mathfrak{m}^{n-d+1} & \text{if } n \geq d. \end{cases}$$

In particular, $\tau(\mathfrak{m}^{d-1}) \supsetneq \tau(\mathfrak{m}^{d-2})\mathfrak{m}$. This shows that $l \geq d$ is the best possible bound for the equality $\tau(\mathfrak{m}^l) = \tau(\mathfrak{m}^{l-1})\mathfrak{m}$ in this case.

Remark 4.6. Let $(R, \mathfrak{m})$ and $\mathfrak{a}$ be as in Theorem 4.1 or 4.2 and let $\mathfrak{q}$ be any reduction of $\mathfrak{a}$. Then $\tau(\mathfrak{a}^{l-1})\mathfrak{a} = \tau(\mathfrak{a}^{l-1})\mathfrak{q}$. In particular, $\tau(\mathfrak{a}^{l-1}) = \tau(\mathfrak{q}^{l-1})$ is contained in the coefficient ideal $\mathfrak{c}(\mathfrak{a}, \mathfrak{q})$ of $\mathfrak{a}$ with respect to $\mathfrak{q}$; cf. [AH].

If R is Gorenstein, $\mathfrak{a}$ is $\mathfrak{m}$-primary with minimal reduction $\mathfrak{q}$ and if the Rees algebra $R[\mathfrak{a}t]$ is F-rational, then the equality $\tau(\mathfrak{a}^{d-1}) = \mathfrak{c}(\mathfrak{a}, \mathfrak{q})$ holds; see [HY] and Hyry [Hy].

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